Weak-strong stability and phase-field approximation of interface evolution problems in fluid mechanics and in material sciences

by

Alice Marveggio

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Abstract

This thesis concerns the application of variational methods to the study of evolution problems arising in fluid mechanics and in material sciences. The main focus is on weak-strong stability properties of some curvature driven interface evolution problems, such as the two-phase Navier–Stokes flow with surface tension and multiphase mean curvature flow, and on the phase-field approximation of the latter. Furthermore, we discuss a variational approach to the study of a class of doubly nonlinear wave equations.

First, we consider the two-phase Navier–Stokes flow with surface tension within a bounded domain. The two fluids are immiscible and separated by a sharp interface, which intersects the boundary of the domain at a constant contact angle of ninety degree. We devise a suitable concept of varifolds solutions for the associated interface evolution problem and we establish a weak-strong uniqueness principle in case of a two dimensional ambient space. In order to focus on the boundary effects and on the singular geometry of the evolving domains, we work for simplicity in the regime of same viscosities for the two fluids.

The core of the thesis consists in the rigorous proof of the convergence of the vectorial Allen-Cahn equation towards multiphase mean curvature flow for a suitable class of multi-well potentials and for well-prepared initial data. We even establish a rate of convergence. Our relative energy approach relies on the concept of gradient-flow calibration for branching singularities in multiphase mean curvature flow and thus enables us to overcome the limitations of other approaches. To the best of the author's knowledge, our result is the first quantitative and unconditional one available in the literature for the vectorial/multiphase setting.

This thesis also contains a first study of weak-strong stability for planar multiphase mean curvature flow beyond the singularity resulting from a topology change. Previous weak-strong results are indeed limited to time horizons before the first topology change of the strong solution. We consider circular topology changes and we prove weak-strong stability for BV solutions to planar multiphase mean curvature flow beyond the associated singular times by dynamically adapting the strong solutions to the weak one by means of a space-time shift.

In the context of interface evolution problems, our proofs for the main results of this thesis are based on the relative energy technique, relying on novel suitable notions of relative energy functionals, which in particular measure the interface error. Our statements follow from the resulting stability estimates for the relative energy associated to the problem.

At last, we introduce a variational approach to the study of nonlinear evolution problems. This approach hinges on the minimization of a parameter dependent family of convex functionals over entire trajectories, known as Weighted Inertia-Dissipation-Energy (WIDE) functionals. We consider a class of doubly nonlinear wave equations and establish the convergence, up to subsequences, of the associated WIDE minimizers to a solution of the target problem as the parameter goes to zero.

Acknowledgements

I would like to express my deepest and sincere gratitude to my PhD advisor Julian Fischer for his constant guidance and support throughout the past four years. I feel grateful to him for sharing with me some of his visionary ideas, which have inspired our joint projects contained in this thesis. Our discussions, either on zoom during the pandemic or in person afterwards, are certainly part of my best working memories of my time as a PhD student. I have learned a lot from him: His brilliant and inspiring way of thinking and approaching math challenges has been enlightening. I really hope to continue jointly working with him on such ambitious research projects originating from his visions.

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I am grateful to all my collaborators, as a great part of this thesis has been the result of our joint work. A special and sincere thanks goes to Sebastian Hensel for being an invaluable collaborator to me. Since my first day at ISTA, he has always been available for discussions, at first to introduce me to our field, lately to investigate joint research directions.

I would like to thank all the group members of Julian's research group, all the PhD students from the 2019 cohort, as well as all the people from the third floor of Lab Building West, for both the working and the friendly moments spent together. A special mention goes to Antonio Agresti, to whom I am deeply thankful for his precious support. Throughout the last two years he has been to me an inspiring colleague, as well as a dear friend.

I warmly thank my friends from the fourth floor of the Faculty of Mathematics at the University of Vienna: You have been my happy place during hard working days and stressful times. I will truly miss you and the nice friendly atmosphere you have been creating there.

Gli anni a Vienna non sono stati dei più semplici, prima la pandemia, poi l'attentato, infine dei momenti di difficoltà personale che la mia famiglia e i miei amici più cari mi hanno vista affrontare. Ringrazio la mia famiglia per la comprensione e l'affetto dimostratomi mentre percorrevo questo percorso lontano da casa. Un grazie speciale va ai miei nonni Carla e Ezio, il cui ricordo porterò sempre nel cuore. Un sentito ringraziamento va ai miei amici che mi sono stati vicino, che mi hanno supportata, che mi hanno incoraggiata, che mi hanno aiutata, che mi hanno accolta, che mi hanno confortata, che mi hanno ospitata, che mi hanno regalato sorrisi e che mi hanno dimostrato tanto affetto in questi anni. Grazie di cuore.

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About the Author

Alice Marveggio completed her Bachelor in Physics at the University of Pavia in Fall 2017, under the supervision of Prof. Claudio Dappiaggi. In Summer 2019, she completed her Master in Physics with a curriculum in Mathematical and Theoretical Physics, under the supervision of Prof. Giulio Schimperna. Her Master thesis "On a non-isothermal Cahn-Hilliard model based on a microforce balance" was awarded "Premio di laurea Prof. Luigi Berzolari" (ed. 2020-2021). During her Bachelor and Master studies, she was a student at the University School for Advanced Studies IUSS Pavia. She joined the research group of Prof. Julian Fischer at ISTA as a PhD student in Fall 2019. Her research interests lie at the intersection of partial differential equations and calculus of variations, with a particular focus on interface evolution problems arising in continuum mechanics. During her stay at ISTA, she has also been collaborating with Prof. Ulisse Stefanelli at the University of Vienna. In October 2023, Alice will start a PostDoc position at the Hausdorff Center for Mathematics in Bonn, under the mentorship of Prof. Tim Laux and Prof. Sergio Conti.

List of Collaborators and Publications

This thesis is the result of the following scientific collaborations carried out at the Institute of the Science and Technology Austria (ISTA) during the years 2019-2023:

- "Weak-strong uniqueness for the Navier–Stokes equation for two fluids with ninety degree contact angle and same viscosities"[58], which is a joint work with Sebastian Hensel and published in *J. Math. Fluid Mech.* **24**, 93 (2022).
- "Quantitative convergence of the vectorial Allen-Cahn equation towards multiphase mean curvature flow"[49], which is a joint work with Julian Fischer and accepted for publication at *Ann. Inst. H. Poincaré Anal. Non Linéaire*. The preprint can be found on the arXiv (identifier 2203.17143).
- "Stability of planar multiphase mean curvature flow beyond a circular topology change"[47], which is work in progress together with Julian Fischer, Sebastian Hensel and Maximilian Moser.
- "Weighted Inertia-Dissipation-Energy approach to doubly nonlinear wave equations" [4], which is work in progress together with Goro Akagi, Verena Bögelein and Ulisse Stefanelli.

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CHAPTER 1

Introduction

The aim of this thesis is the application of variational methods to the study of several evolution problems arising in physical sciences. Since the problems of interest are of different nature, we provide two separate introductions.

Section 1.1 concerns interface evolution problems in fluid mechanics and material sciences, hence serves as a general introduction to Chapters 2-3-4, which inspire the title of this thesis. On the other hand, Section 1.2 introduces the variational approach to nonlinear evolution problems which will be subject of Chapter 5.

1.1 Interface evolution problems in fluid mechanics and in material sciences

An interface is a surface separating two spatial regions occupied by two different physical states, or often referred to as phases. In this sense, interfaces appear in the modeling of various physical phenomena, ranging from two-phase fluid flows, grain coarsening, image processing to crystal growth, tumor growth, and biological membranes.

The main focus of this thesis is the study of the following two models for interface evolution in fluid mechanics and in material sciences: the two-phase incompressible Navier–Stokes flow with surface tension and the multiphase mean curvature flow.

Two-phase incompressible Navier–Stokes flow with surface tension

Consider the motion of oil droplets in water. This is a standard example of flow of two immiscible, incompressible and viscous fluids with surface tension effects, which can be modeled as follows. The interface between the two fluids is transported along the fluid flow due to the immiscibility of the two fluids. The motion of each fluid is determined by the incompressible Navier-Stokes equation. The interface exerts a surface tension force on the fluids proportional to the mean curvature of the interface.

In terms of a mathematical formulation, consider a domain $\Omega \subseteq \mathbb{R}^d$ and a time interval $[0,T) \subset \mathbb{R}$. The velocity fields of the two fluids coincide on the interface due to a no-slip boundary condition at the interface $I(t)$, $t \in [0,T) \subset \mathbb{R}$, hence we consider a single velocity vector field $v : \Omega \times [0,T) \to \mathbb{R}^d$. Then, given $\chi : \Omega \times [0,T) \to \{0,1\}$ as the indicator function

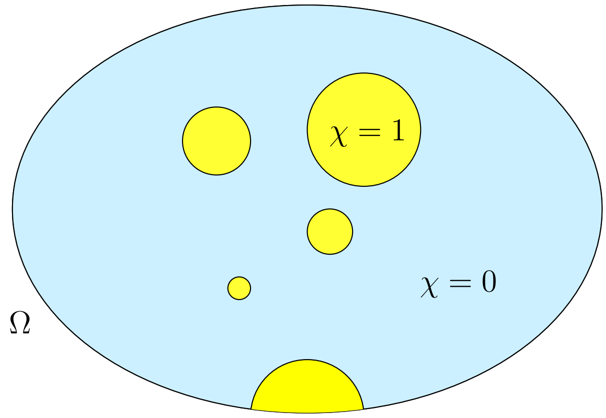


Figure 1.1: Two-phase fluid in $\Omega \subseteq \mathbb{R}^2$.

of the volume occupied by the one of the fluids so that $I = \partial\{\chi = 1\}$, the above two-phase fluid system corresponds to the following PDEs formulation

$$\partial_t \chi + (v \cdot \nabla)\chi = 0, \tag{1.1}$$
$$\partial_t(\rho(\chi)v) + \nabla \cdot (\rho(\chi)v \times v) = -\nabla p + \nabla \cdot \Big(\mu(\chi)(\nabla v + \nabla v^{\mathsf{T}})\Big) + \sigma\,\mathrm{H}\,|\nabla\chi|, \tag{1.2}$$
$$\nabla \cdot v = 0, \tag{1.3}$$

where $\rho = \rho(\chi)$ and $\mu = \mu(\chi)$ denote the density and the viscosity functions of the two-phase fluid, p is the pressure, σ is the surface tension, H is the mean curvature vector of the interface I, and $|\nabla\chi|$ the surface measure of I. In particular, denoting by n_I the normal to I, the right hand side of (1.2) encodes the Young–Laplace law along the interface:

$$[[\mathbb{T}n_I]] = \sigma\,\mathrm{H} \quad \text{along } I, \tag{1.4}$$

where $\mathbb{T} := \mu(\chi)(\nabla v + \nabla v^{\mathsf{T}}) - p\,\mathrm{Id}$ is the viscous stress tensor and $[[\cdot]]$ denotes the jump across I.

Additionally, in case of $\Omega \subset \mathbb{R}^d$ bounded, the system (1.1)-(1.3) is endowed with boundary conditions, for instance the complete slip boundary conditions:

$$v(\cdot,t) \cdot n_{\partial\Omega} = 0 \quad \text{along } \partial\Omega, \tag{1.5}$$
$$\Big(n_{\partial\Omega} \cdot \mu(\chi)(\nabla v + \nabla v^{\mathsf{T}})(\cdot,t)\tau_{\partial\Omega}\Big) = 0 \quad \text{along } \partial\Omega, \tag{1.6}$$

where $n_{\partial\Omega}$ and $\tau_{\partial\Omega}$ denote the inner pointing normal and the tangent to $\partial\Omega$, respectively. These boundary conditions not only prescribe that the fluid cannot exit from the domain and that it can move only tangentially to its boundary, but they also exclude any external contribution to the viscous stress and any friction effect with the boundary.

The energy functional for the above two-phase fluid system is given by the sum of the kinetic energy and the interface energy, namely

$$E(\chi, v) = \int_{\mathbb{R}^d} \frac{1}{2}\rho(\chi)|v|^2 \,\mathrm{d}x + \sigma\mathcal{H}^{d-1}(I),$$

and the associated energy dissipation inequality reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\chi, v) + \int_{\mathbb{R}^d} \frac{1}{2}\mu(\chi)|\nabla v + \nabla v^{\mathsf{T}}|^2 \,\mathrm{d}x \leq 0.$$

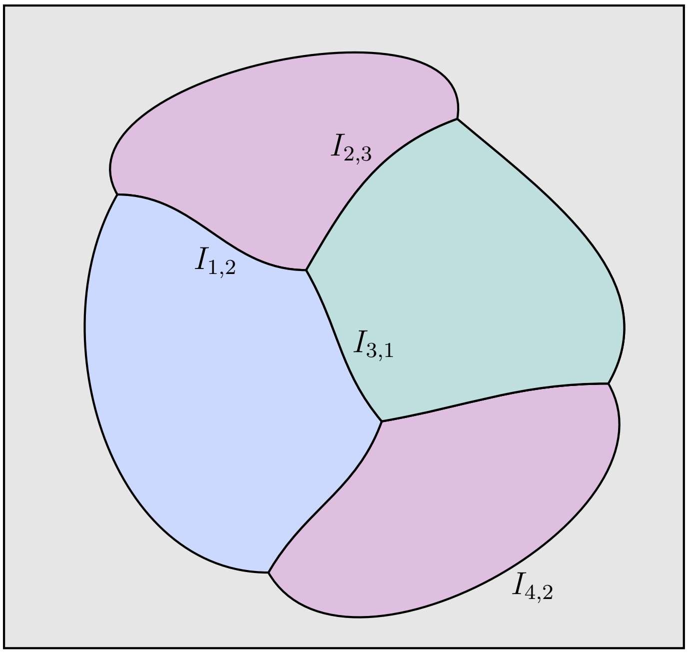


Figure 1.2: Multiphase system in which each color corresponds to a different phase (e.g., orientation of the lattice for a polycrystal).

Multiphase mean curvature flow

Consider a polycrystal, which is a solid material consisting of crystals with different orientations. A grain boundary is an interface where crystals of different orientations meet. Each orientation represents a different phase of the polycrystal as a multiphase system (cf. Figure 1.2).

A model for the evolution in time of grain boundaries in polycristals undergoing heat treatment was first introduced by Mullins in [96], and it can be mathematically formulated by means of the mean curvature flow equation

$$V_{i,j} = \sigma_{i,j} H_{i,j} \quad \text{along } I_{i,j}, \tag{1.7}$$

where $V_{i,j}$ denotes the normal velocity, $\sigma_{i,j}$ is the surface tension, and $H_{i,j}$ is the mean curvature of the interface $I_{i,j}$ separating the polycrystal regions i and j with different crystallographic orientations. The above equation has to be complemented with an equilibrium condition holding at the intersection of three grain boundaries, i.e. at a triple junction, which is known as Herring angle condition and reads as

$$\sum_{i,j} \sigma_{i,j} n_{i,j} = 0, \tag{1.8}$$

where $n_{i,j}$ denotes the unit normal pointing from phase j into phase i. Note that, in case of equal surface tensions, (1.8) corresponds to a 120° angle condition between the normals.

Multiphase mean curvature flow has a gradient-flow structure. In particular, the evolution by mean curvature can be derived as the L^2-gradient flow of the interface area functional

$$E(I) = \frac{1}{2} \sum_{i,j} \sigma_{i,j} \mathcal{H}^{d-1}(I_{i,j}), \tag{1.9}$$

which is subject to a dissipation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}E(I) = -\frac{1}{2}\sum_{i,j}\sigma_{i,j}\int_{I_{i,j}} V_{i,j}H_{i,j}\,\mathrm{d}\mathcal{H}^{d-1} = -\frac{1}{2}\sum_{i,j}\sigma_{i,j}\int_{I_{i,j}} |V_{i,j}|^2\,\mathrm{d}\mathcal{H}^{d-1}. \tag{1.10}$$

1.1.1 Solution concepts for curvature driven evolving interfaces

The main challenge in the mathematical description of curvature driven evolving sharp interface consists in the occurance of ***topology changes*** in finite time. In the literature various approaches are available for the mathematical analysis of sharp interfaces.

Classical ways for the representation of sharp interfaces range from parametrizations to measure theoretic approaches. Parametric approaches provide a detailed description of the evolution of sharp interfaces in time, however these typically lead to nonlinear partial differential equations whose study reveals to be challenging, in particular in case of topology changes. Weaker approaches involving measure theoretic methods are thus preferable in order to allow topology changes.

Weakening the concept of solution however may lead to abundance of solutions, also including unphysical non-uniqueness (e.g., a sudden vanishing of the interface at an arbitrary time), even in the absence of topology changes. Therefore, ***uniqueness*** may fail in the class of weak solutions. On the other hand, with the exception of evolution equations subject to a comparison principle (e.g., two-phase mean curvature flow), uniqueness properties of weak solutions have been mostly unexplored. In the framework of curvature driven interface evolution problems not satisfying a comparison principle, one may establish a conditional result in the form of a ***weak-strong uniqueness principle***, stating:

Prior to the first topology change, weak solutions are unique in the class of strong solutions.

In other words, the non-uniqueness of weak solutions can arise only as a consequence of topology changes, such as the pinch-off in liquid drops or the collapse of grain boundaries in a polycrystals.

Classical strong solutions and singularity formation. In the classical setting, the evolving sharp interfaces are described either by means of standard parametrization, or by means of an height function over a $(d-1)$-dimensional closed reference hypersurface U.

In the framework of the two-phase viscous fluid flow (1.1)-(1.3), short-time existence of strong solutions in the L^p-setting was established by Köhne, Prüss, and Wilke in [70], and by Wilke in [127]. In these works, the evolving interface $\gamma : U \times [0,T] \to \mathbb{R}^d$ is represented in terms of a graph of the the height function $h : U \times [0,T] \to \mathbb{R}^d$ over the initial interface $\gamma_0 : U \to \mathbb{R}^d$, namely $\gamma(s,t) = \gamma_0(s) + h(s,t)n_{\gamma_0}(s)$, where n_{γ_0} denotes the normal vector field to γ_0, whereas the evolving domain of one of the fluids is represented in terms of the associated Hanzawa transform (cf. [101]).

Short-time existence for the evolution by mean curvature flow of a smooth hypersurface was first established first by Gage and Hamilton [51], then by Huisken and Polden [61], representing the evolving hypersurfaces as graphs over the initial one in a tubular neighborhood of the latter (cf. [84]).

In the context of planar multiphase mean curvature flow, we refer to the work of Mantegazza, Novaga, Pluda and Schulze [85] for the analytical study of the evolution and of the singularity formation resulting from a topology change. Local-in-time existence and uniqueness in the

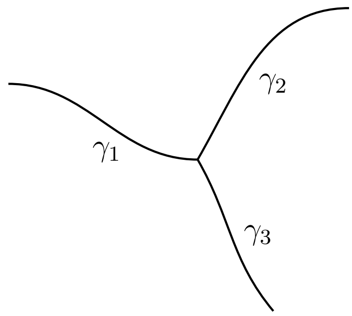


Figure 1.3: Triple junction satisfying $120°$ angle condition.

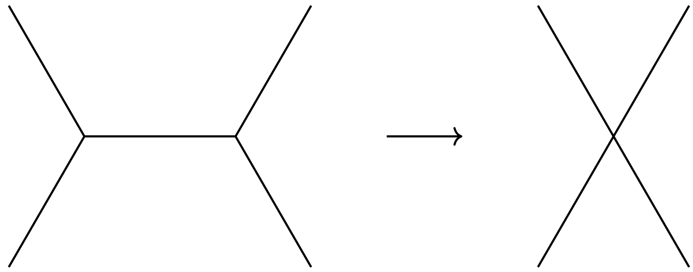


Figure 1.4: Collision of two standard triods resulting in a standard cross with angles of $60°/120°$.

case of three curves $\gamma_i : [0,1] \times [0,T] \to D \subset \mathbb{R}^2$, $i = 1,2,3$, meeting at a single triple junction with fixed end-points was proved first by Bronsard and Reitich in [24], then revised by Mantegazza, Novaga and Tortorelli in [86] (see Figure 1.3). In their works, the evolution equation (1.7) for the curves γ_i reads as

$$\partial_t \gamma_i = \frac{\partial_{ss}\gamma_i}{|\partial_s \gamma_i|^2}$$

for a specific choice of the tangential velocity (in order to allow the motion of the triple junction without affecting the motion of the curves). The authors of [85] extended the result of [86] to a network made of regular triple junctions (i.e. satisfying the Herring angle condition (1.8)). In case of non-regular initial networks (that is, with multi-points of order greater than 3 and/or non-regular 3-points), e.g., networks resulting from a collision of two triple junctions (cf. Figure 1.4), we refer to the short-time existence result of Ilmanen, Neves and Schulze [64]. In particular, their result combined together with a previous analysis of the singularities provides a restarting (and/or continuation) result for the curvature flow past a singularity formation.

As conjectured by Ilmanen on the basis of numerical simulations (cf. [85]), some self-similarly shrinking networks (i.e., shrinkers) are dynamically stable, meaning that perturbing the flow, the blow-up limit network remains the same. As a consequence, the generic singularities of the curvature flow of a network are conjectured to be (locally) asymptotically described by one of the these shrinkers. If the dynamically stable shrinker is a line or a standard triod (i.e. a $120°$ triple junction), there is no singularity, thus one may consider as dynamically stable singularities the following: the unit circle, the standard cross, the Brakke spoon, the lens, and the three-ray star (see Figures 1.4-1.5).

In the framework of classical strong solutions for bubbles clusters moving by mean curvature in higher space dimension, less is available in the literature. Local-in-time existence and uniqueness for general double bubble clusters was proved by Depner, Garcke and Kohsaka [37].

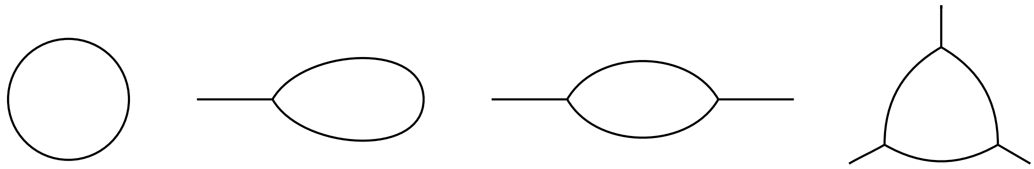


Figure 1.5: Dynamically stable shrinkers: the circle, the Brakke spoon, the lens, the three-ray star.

Level-set formulation and viscosity solutions. The notion of viscosity solutions for two-phase mean curvature flow relies on the level set approach of Osher and Sethian [99]. Instead of an evolving interface $I(t)$, we consider a function u having $I(0)$ as zero level set at the initial time $t = 0$ and evolving according to the degenerate parabolic equation

$$\partial_t u - \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) |\nabla u| = 0.$$

Then, the viscosity solution $I(t)$ is defined as the zero level set of u at time t.

The construction of a global-in-time unique viscosity solution can be found in the works by Chen, Giga and Goto [30], and Evans and Spruck [41]. These results are established due to the availability of a comparison principle, which reads as *two disjoint surfaces stay disjoint during the evolution* in the context of viscosity solution for two-phase mean curvature flow. The absence of a comparison principle for multiphase mean curvature flow prevents the applicability of these concepts in the multiphase case.

Despite the well-posedness and uniqueness of viscosity solutions, $I(t)$ may develop a non-trivial interior and thus fails to describe an interface in form of a hypersurface. This phenomenon, which is referred to as the fattening of the level sets in the literature, can be explained in terms of non-uniqueness of the evolution (cf. [118]). Nevertheless, fattening is known to not occur prior to the first topology change.

BV formulation of energy dissipating solutions. A distributional formulation of multiphase mean curvature flow in the setting of finite perimeter sets and BV functions was used in the work of Luckhaus and Sturzenhecker [83], later by Laux and Otto [73], and by Laux and Simon [74]. Their formulation of multiphase mean curvature flow relies, first on a set of BV phase indicator functions χ_i, then on the existence of velocity vector fields V_i such the following trasport equations are satisfied in a distributional sense

$$\partial_t \chi_i + (V_i \cdot \nabla)\chi_i = 0.$$

A weak formulation of the mean curvature equation (1.7) is given by

$$\sum_{i,j} \sigma_{i,j} \int_0^T \int_{I_{i,j}} V_i \cdot \varphi \, \mathrm{d}\mathcal{H}^{d-1} \, \mathrm{d}t = -\sum_{i,j} \sigma_{i,j} \int_0^T \int_{I_{i,j}} (\mathrm{Id} - n_{i,j} \otimes n_{i,j}) : \nabla \varphi \, \mathrm{d}\mathcal{H}^{d-1} \, \mathrm{d}t, \tag{1.11}$$

where φ is a smooth vectorial test function. Indeed, we note that $-\int_{I_{i,j}} (\mathrm{Id} - n_{i,j} \otimes n_{i,j}) : \nabla \varphi \, \mathrm{d}\mathcal{H}^{d-1} = \int_{I_{i,j}} H_{i,j} n_{i,j} \cdot \varphi \, \mathrm{d}\mathcal{H}^{d-1}$ for smooth surfaces $I_{i,j}$ without boundary. The energy dissipation inequality 1.10 is imposed by defining the interface normal velocities $V_{i,j}$ as restrictions of V_i to $I_{i,j}$. Conditional global-in-time existence results for BV solutions to

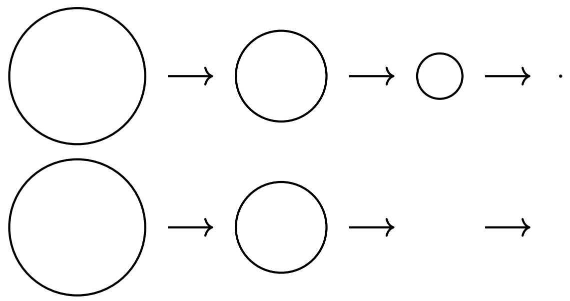


Figure 1.6: Evolution of a circular interface by mean curvature flow: The interface shrinks to a point in finite time. In the framework of Brakke solutions to mean curvature flow, the interface may suddenly disappear at any time.

multiphase mean curvature flow were established by Luckhaus and Sturzenhecker [83], by Laux and Otto [73], and by Laux and Simon [74]. Furthermore, the BV formulation allows the occurrence of topology changes, and a weak-strong uniqueness result for BV solutions to multiphase mean curvature was established prior to the first topology change [46].

Measure-valued varifold solutions. In the framework of geometric measure theory, interfaces at any time t are modeled by integral (oriented) varifolds V_t, namely non-negative measures on $\mathbb{R}^d \times G(d,d-1)$ ($\mathbb{R}^d \times \mathbb{S}^{d-1}$), where $G(d,d-1)$ ($\mathbb{S}^{d-1}$) denotes the space of all $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$ (the unit $(d-1)$-sphere). In particular, the curvature term on the right hand side of (1.11) is espressed by means of

$$-\int_{\mathbb{R}^d \times G(d,d-1)} P_Q : \nabla\varphi \, \mathrm{d}V_t$$

where P_Q denotes the orthogonal projection onto $Q \in G(d,d-1)$ and corresponds to $\mathrm{Id} - s \otimes s,\ s \in \mathbb{S}^{d-1}$, in case of oriented varifolds V_t.

In the context of two-phase incompressible Navier-Stokes flow with surface tension, Abels introduced the associated notion of varifold solutions in order to model possible oscillation of the interface on an infinitesimal scale and he proved their global-in-time existence [1]. Moreover, varifold solutions in the sense of Abels satisfying a corresponding form of the energy dissipation inequality (1.10) are unique in a suitable class of strong solutions [45].

A notion of varifold solution to mean curvature flow was given by Brakke in his pioneering work [21], in which he provided a global-in-time existence result. Brakke's notion of solutions consists of a localized version of the energy dissipation inequality (1.10), which is formulated in terms of an evolving integral varifold with locally bounded first variation. As a consequence, a sudden and arbitrary loss of surface measure at any time is admissible. In order words, in the context of Brakke solutions to mean curvature flow, the interface may suddenly disappear at any time (see Figure 1.6). In particular, Brakke solutions fail to be unique, even prior to the first topology change in the classical solution.

Kim and Tonegawa [68], and Stuvard and Tonegawa [122] introduced a notion of varifold solutions that combines the concept of Brakke solutions with an evolution equation for the different phases, proving global-in-time existence. By imposing an evolution equation for the phases, their concept of varifold solutions prevents the sudden vanishing of the interface, hence constitutes a significant improvement to Brakke's notion of solutions. Furthermore, being a natural generalization of the concept of BV solutions to varifolds, a corresponding

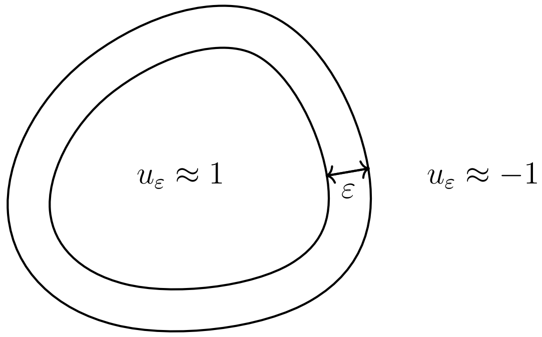


Figure 1.7: Regions in which $u_\varepsilon \approx \pm 1$ are separated by a diffuse interfacial layer of thickness proportional to ε.

weak-strong uniqueness result was established prior to the first topology change [46]. More recently, another notion of varifold solutions satisfying a global energy-dissipation inequality in the sense of De Giorgi [53] was introduced by Laux and Hensel [56].

1.1.2 Phase-field models

An alternative approach to describe the evolution of interfaces is that of phase-field models, which replace the sharp interfaces (hypersurfaces) with thin diffused interfacial regions. In other words, instead of working with a characteristic function indicating the region occupied by one of the phases, phase-field descriptions are formulated in terms of a smooth function, usually referred to in the literature as order parameter. In particular, the order parameter takes values close to given values (e.g. ± 1), and rapidly changes between these two values in a diffuse transition layer of width of order $\varepsilon < 1$ (see Figure 1.7). As a main advantage, phase-field models allow to describe the evolution of interfaces beyond topology changes and the formation geometric singularities.

The Allen-Cahn phase-field model

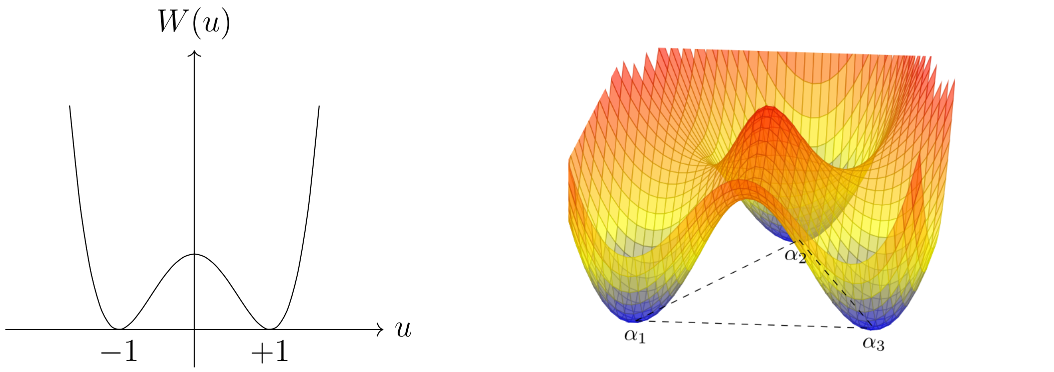
The Allen-Cahn model was introduced to the describe the dynamics of antiphase boundaries [13], namely the process of phase separation resulting from the ordering of atoms within unit cells of a lattice. Variants of the model have been introduced in the literature to describe physical processes of phase separation in multi-phase systems. The Allen-Cahn equation reads as

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2}\partial_u W(u_\varepsilon), \tag{1.12}$$

where $u_\varepsilon : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{N-1}$ is an order parameter, and $W : \mathbb{R}^{N-1} \to [0,+\infty)$ is a N-well potential vanishing in $\alpha_1, ..., \alpha_N \in \mathbb{R}^{N-1}$ (see Figure 1.8 for $N = 2$ and $N = 3$). The set of zeros $\{\alpha_1, ..., \alpha_N\}$ represent the pure phase states, which can mix together in diffuse transition layers of order $\varepsilon > 0$ where u_ε takes value along the path connecting them.

The Allen-Cahn equation is the L^2-gradient flow (accellerated by the factor $1/\varepsilon$) of the Ginzburg-Landau energy

$$E_\varepsilon(u_\varepsilon) = \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\ \mathrm{d}x, \tag{1.13}$$



(a) Double well potential $W(u) = (u^2-1)^2$ with zeros in ± 1

(b) Triple well potential with zeros in $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}^2$

Figure 1.8: N-well potentials

and formally subject to the energy dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\varepsilon(u_\varepsilon) = -\int_{\mathbb{R}^d} \varepsilon |\partial_t u_\varepsilon|^2 \,\mathrm{d}x.$$

Sharp interface limit of the Allen-Cahn model

Phase-field models can be interpreted as an approximation of sharp interface models, and thus have been widely used for numerical computations. By letting the width of the diffuse transition layers to vanish, i.e. $\varepsilon \to 0$, one expects to recover the underlying sharp interface dynamics. For this reason, the limit $\varepsilon \to 0$ is referred to in the literature as sharp interface limit.

In its sharp interface limit, the Allen-Cahn equation (1.12) describes the evolution of interfaces by mean curvature flow (1.7). Consider for simplicity the two-phase case ($N = 2$), the level-sets of u_ε are roughly parallel to the interface I and ∇u_ε points in the direction of the normal n to I. Then, since $\mathrm{div}(\nabla u_\varepsilon) = \Delta u_\varepsilon$ and $\mathrm{div}_I(n) = H$, one can already notice the analogies between (1.12) and (1.7), at least at a formal level.

At the level of the energy functionals, the analytical study of the convergence of the Ginzburg-Landau energy (1.13) towards the interface area functional (1.9) was initiated by Modica and Mortola [95]. In the two-phase case, the corresponding gamma-convergence result was established by Modica [94] and Sternberg [121]. Luckhaus and Modica [82] proved the convergence of the first variation of the energy (1.13) to mean curvature, namely the first variation of the interface area functional, again in the two-phase setting. In the framework of vectorial u_ε, a gamma-convergence result for the general multiphase case was established by Baldo [15].

The rigorous analysis of the behavior of solutions to the Allen-Cahn equation (1.12) in its sharp interface limit $\varepsilon \to 0$ has for long been available only for the scalar Allen-Cahn equation with two-well potential W, namely only for $N = 2$. Convergence for a smooth evolution was proved independently by de Mottoni and Schatzman [36] and by Chen [29] by means of rigorous asymptotic expansions. More recently, Fischer, Laux and Simon gave a short alternative proof for quantitative convergence towards a smooth solution to mean curvature flow. However, both the arguments hold only in the two-phase setting, as they do not consider the occurrence of branching singularities (i.e. triple junctions) in the limiting motion by multiphase mean curvature. More generally, convergence results for classical strong solutions towards multiphase mean curvature flow beyond the formation of dynamically stable singularities have not been available in the literature so far. To overcome the problems arising from the formation of singularities in the multiphase setting, convergence of the Allen-Cahn equation (1.12) to its sharp interface limit was established in the framework of two-phase Brakke solution by Ilmanen [62], and in that of viscosity solutions by Evans, Soner, and Souganidis [40].

To the best of our knowledge, only a formal expansion analysis for the convergence to the smooth evolution of a triple junction was carried out by Bronsard and Reitich [24]. In the multiphase setting, so far the only rigorous convergence result past singularities has been the conditional result proved by Laux and Simon [74] in the setting of BV solutions to multiphase mean curvature flow. However, their result is conditional, in the sense that they assumed that the time integral of the energy (1.13) converges to the time integral of (1.9) as $\varepsilon \to 0$.

1.1.3 The relative energy method

The relative energy (or entropy) method originates in the works of Dafermos [32] and DiPerna [38] on systems of conservation laws and it has been widely exploited in the context of fluid dynamics (cf. [126]). Recently, relative energy techiniques have been a promising tool to establish results such as

- ***weak-strong uniqueness principles for sharp interface evolution problems*** for the two-phase incompressible Navier-Stokes flow [45] and for multiphase mean curvature flow [48];
- ***quantitative convergence of phase-field models to their sharp interface limit*** for the scalar Allen-Cahn equation towards the evolution by mean curvature flow [48].

A relative energy is a nonlinear functional that measures the error between two solutions, which we denote by u and v, with v being the stronger/sharper/more regular one. In particular, for physical systems with a strictly convex energy (or entropy) functional $E[\cdot]$ subject to dissipation, the relative energy is obtained by subtracting the first order approximation around v to $E[\cdot]$, namely by

$$E[u|v] := E[u] - DE[v](u - v) - E[v].$$

In order to compute the time evolution of $E[u|v]$, one needs the energy (or entropy) dissipation inequality for u, and the weak formulation of the evolution equation for u tested by some nonlinear functional of v. Then one may exploit the properties of $E[u|v]$ in order to deduce a relative energy inequality of the form

$$\frac{d}{dt}E[u|v] \leq CE[u|v],$$

for some constant $C = C(T, v) > 0$. At last, a Gronwall-type argument allows to obtain a stability estimate

$$E[u|v](t) \leq e^{Ct}E[u|v](0), \quad \text{for any } t \in [0, T], \tag{1.14}$$

and thus to conclude about either weak-strong uniqueness properties or convergence rates for well-prepared initial data.

Depending on the problem under study, the relative energy may consist of several terms, each controlling different quantities. Below we introduce the relevant ones in the framework of the two-phase incompressible Navier-Stokes flow and of mean curvature flow (as well as of its phase-field Allen-Cahn approximation).

Velocity error. In the framework of the incompressible Navier-Stokes flow of two fluids with same viscosities, the relative energy functional has a term controlling the error between the velocities u and v in the L^2-norm

$$E_{\mathsf{vel}}[u|v] = \int_{\mathbb{R}^d} \frac{1}{2}\rho(\chi_u)|u - v|^2 \,\mathrm{d}x, \tag{1.15}$$

where ρ denotes the density function. In the case of different viscosities of the two fluids, we refer the reader to [45] for a detailed discussion.

Interface error. From the interface area functional, one can deduce a contribution $E_{\mathsf{int}}[I_u|I_v]$ in the relative entropy providing control the interface error between a measure-theoretic interface $I_u = \partial^*\{\chi_u = 1\}$ and a strong interface $I_v = \partial\{\chi_v = 1\}$. Formally, the ansatz for $E_{\mathsf{int}}[I_u|I_v]$ is of the form

$$E_{\mathsf{int}}[I_u|I_v] = \sigma \int_{\mathbb{R}^d} 1 - \xi \cdot n_u \,\mathrm{d}|\nabla\chi_u|, \tag{1.16}$$

where σ is the surface tension, n_u is the measure-theoretic unit normal along I_u, whereas ξ is a smooth extension of the unit normal vector field to I_v in its space-time neighborhood. In particular, ξ coincides along I_v with its unit normal vector field. Additionally, away from I_v, the length of ξ decreases quadratically in the distance to I_v. Note that $E_{\mathsf{int}}[I_u|I_v]$ controls the squared error of the interface normals

$$E_{\mathsf{int}}[I_u|I_v] \geq \frac{\sigma}{2} \int_{\mathbb{R}^d} |n_u - \xi|^2 \,\mathrm{d}|\nabla\chi_u|.$$

Furthermore, $E_{\mathsf{int}}[I_u|I_v]$ controls the total area of the part of the interface I_u which is not locally a graph over I_v.

In the framework of oriented varifold solutions, the ansatz for interfacial contribution in the relative energy functional reads as

$$E_{\mathsf{int}}[I_u, V|I_v] = \sigma \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} 1 - \xi \cdot s \,\mathrm{d}V_t \geq \frac{\sigma}{2} \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |s - \xi|^2 \,\mathrm{d}V_t. \tag{1.17}$$

Then, having a compatibility condition of the form

$$\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \varphi s \,\mathrm{d}V_t = \int_{\mathbb{R}^d} \varphi \,\mathrm{d}\nabla\chi_u \tag{1.18}$$

for every smooth test function φ with compact support, one can define the Radon–Nikodym derivative

$$\theta_t := \frac{\mathrm{d}|\nabla\chi_u(t)|}{\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}}.$$

The quantity $1/\theta_t$ corresponds to the multiplicity of the varifold V_t. In particular, one can rewrite $E_{\mathsf{int}}[I_u, V|I_v]$ as

$$E_{\mathsf{int}}[I_u, V|I_v] = \sigma \int_{\mathbb{R}^d} 1 - \xi \cdot n_u \,\mathrm{d}|\nabla\chi_u| + \sigma \int_{\mathbb{R}^d} 1 - \theta_t \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}},$$

where the last term controls the multiplicity error of the varifold. In turn, one can also obtain a control of the squared error in the normal of the varifold $\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |s-n_u|^2 \,\mathrm{d}V_t$.

In the context of the convergence of phase-field approximations, formally, one has to replace I_u with the diffuse interfacial region, in which the order parameter u_ε takes values between two given values, for example ± 1. Being its diffuse interface approximation [94], the Ginzburg-Landau energy $E_\varepsilon(u_\varepsilon)$ from (1.13) plays the role of the interface area $\int_{\mathbb{R}^d} 1 \,\mathrm{d}|\nabla\chi_u|$. On the other hand, the phase-field approximation of $n_u \mathrm{d}|\nabla\chi_u|$ is given by $\nabla\psi(u_\varepsilon)\,\mathrm{d}x = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}|\nabla\psi(u_\varepsilon)|\,\mathrm{d}x$ with $\psi(u_\varepsilon) = \int_0^{u_\varepsilon}\sqrt{2W(s)}\,\mathrm{d}s$ from the Modica-Mortola trick for a two-well potential W. As a result, the relative energy ansatz for the proof of the convergence of the scalar Allen-Cahn equation reads as [48]

$$E_{\mathrm{int}}[u_\varepsilon|I_v] = \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) - \xi\cdot\nabla\psi(u_\varepsilon)\,\mathrm{d}x. \tag{1.19}$$

In particular, by adding a zero, one can rewrite $E_{\mathrm{int}}[u_\varepsilon|I_v]$ as

$$E_{\mathrm{int}}[u_\varepsilon|I_v] = \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) - |\nabla\psi(u_\varepsilon)|\,\mathrm{d}x + \int_{\mathbb{R}^d}\Big(1-\xi\cdot\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\Big)|\nabla\psi(u_\varepsilon)|\,\mathrm{d}x,$$

thus obtaining the control of the local lack of equipartition of energy between the terms $\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2$ and $\frac{1}{\varepsilon}W(u_\varepsilon)$

$$E_{\mathrm{int}}[u_\varepsilon|I_v] \geq \int_{\mathbb{R}^d} \frac{1}{2}\Big|\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\Big|^2\,\mathrm{d}x,$$

as well as of the error between the normals

$$E_{\mathrm{int}}[u_\varepsilon|I_v] \geq \int_{\mathbb{R}^d} \frac{1}{2}\Big|\xi - \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\Big|^2|\nabla\psi(u_\varepsilon)|\,\mathrm{d}x.$$

Weighted volume error. In the limit of vanishing length of the weak interface I_u, the interfacial contribution $E_{\mathrm{int}}[I_u|I_v]$ in the relative energy lacks of coercivity. For this reason, one introduces in the relative energy functional an additional term $E_{\mathrm{vol}}[\chi_u|\chi_v]$ which controls the error between the volumes with weight ϑ. In particular, ϑ behaves approximately like a sign distance to I_v in the proximity of I_v, and $E_{\mathrm{vol}}[\chi_u|\chi_v]$ is of the form

$$E_{\mathrm{vol}}[\chi_u|\chi_v] = \int_{\mathbb{R}^d} (\chi_u - \chi_v)\vartheta\,\mathrm{d}x, \tag{1.20}$$

which is nonnegative for $\vartheta > 0$ in $\{\chi_u = 1\}$ and $\vartheta < 0$ in $\{\chi_u = 0\}$.

1.1.4 Contributions of this thesis

The main interest of this thesis resides in the application of the relative energy techniques in order to establish

- weak-strong stability for the planar two-phase incompressible Navier-Stokes flow with 90 degree contact angle condition (cf. Chapter 2);
- quantitative convergence of the vectorial Allen-Cahn equation towards multiphase mean curvature flow (cf. Chapter 3);

- weak-strong stability for planar multiphase mean curvature flow beyond a circular topology change (cf. Chapter 4).

These results are quantitative, as are derived as a consequence of an associated stability estimate of the form of (1.14). In particular, qualitative weak-strong uniqueness principles and qualitative convergence can be deduced as a consequence.

A weak-strong uniqueness principle for the two-phase incompressible Navier-Stokes flow with 90 degree contact angle condition

In a recent work [45], a weak-strong uniqueness principle for varifold solutions (in the sense of Abels [1]) was established for the flow of two immiscible, viscous and incompressible fluids with surface tension. In Chapter 2, in the planar case and for matched shear viscosities, we extend the weak-strong uniqueness result of [45] to the setting of a bounded domain $\Omega \subset \mathbb{R}^2$, imposing a 90° contact angle condition for the interface I_v if it intersects with the boundary of the domain $\partial\Omega$ at a contact point $c(t)$, namely

$$n_{\partial\Omega}(c(t)) \cdot n_{I_v}(c(t),t) = 0 \quad \text{for any } t \in [0,T],$$

where $n_{\partial\Omega}$ and n_{I_v} denote the normal to $\partial\Omega$ and I_v, respectively. This result has been obtained in collaboration with Sebastian Hensel [58] and reads as

Energy dissipating varifold solutions (χ_u, u, V) *in the sense of Abels [1] to the incompressible two-phase Navier-Stokes flow for two fluids* (1.1)-(1.3) *with* 90° *contact angle and same viscosities satisfy a weak-strong stability estimate. In particular, as long as a strong solution* (χ_v, v) *exists, any energy dissipating varifold solution starting from the same initial data has to coincide with the unique strong solution.*

We establish this result by exploiting the relative energy method, namely by studying the time evolution of a relative energy functional $E[\chi_u, u, V|\chi_v, v]$ of the form devised in [45], which consists of the sum of the terms (1.15) and (1.20) with integrals on a bounded domain $\Omega \subset \mathbb{R}^2$, and of (1.17) with its integral on the closure of Ω, thus encoding the 90° contact angle condition. As a result, we obtain a weak-strong stability estimate of the form of

$$E[\chi_u, u, V|\chi_v, v](t) \leq Ce^{Ct}E[\chi_u, u, V|\chi_v, v](t), \quad \text{for any } t \in [0,T],$$

with $C = C(T, \chi_v, v) > 0$.

Our proof requires a nontrivial further development of ideas from [45] in order to incorporate the contact angle condition. The main challenges of our work are twofold. First, we need to introduce a varifold solution concept in the sense of Abels [1] in the setting of bounded domains with 90° contact angle condition at any moving contact point $c(t)$. Indeed, the work of Abels [1] only deals with the full-space setting in the presence of surface tension. Second, we contruct a suitable boundary adapted extension ξ of the normal vector field to the strong solution interface I_v, i.e. n_{I_v}, in a space-time neighborhood of any moving contact point $c(t)$ along $\partial\Omega$. In particular, our smooth boundary adapted extension ξ of n_{I_v} is subject to the boundary condition

$$\xi \cdot n_{\partial\Omega} = 0 \quad \text{on } \partial\Omega \times [0,T]. \tag{1.21}$$

Including the boundary condition (1.21) in the vicinity of a space-time trajectory of a moving contact point $c(t)$ requires a perturbation of the rather trivial standard bulk construction, thus additional nontrivial technical work.

In the spirit of the two-step strategy of [46], we additionally establish a conditional weak-strong uniqueness result in the three dimensional setting: The missing ingredient is a contruction for the boundary adapted extension ξ in the vicinity of any moving contact line $I_v \cap \partial\Omega$ satisfying a 90° contact angle condition.

At last, we observe that our result holds in the regime of the same shear viscosity for the two fluids. However, for our construction for the boundary adapted extension ξ we do not rely on this assumption. Hence, we expect that one could generalize our result in case of different viscosities of the two fluids by adapting the kinetic energy contribution of the relative energy (1.15) to this regime, thus implementing the highly technical technique developed in [45].

Quantitative convergence of the vectorial Allen-Cahn equation towards multiphase mean curvature flow

The rigorous analysis of the behavior of solutions to the Allen-Cahn equation (1.12) in the limit of vanishing interface width $\varepsilon \to 0$ has long been available only for the scalar Allen-Cahn equation with two-well potential W [29] [36] [48] [40] [62]. As for the vectorial Allen-Cahn equation (1.12) with a potential W with $N \geq 3$ distinct minima, the only previous results on the sharp interface limit have been a formal expansion analysis [24] and a conditional convergence result [74]. To the best of our knowledge, not even an unconditional proof of qualitative convergence for well-prepared initial data has been available so far. One of the main challenges has been the occurance of branching singularities (i.e. triple junctions) in the conjectured limit of multiphase mean curvature flow.

In Chapter 3, we give rigorous proof for the sharp interface limit of the vectorial Allen-Cahn equation (1.12) in a *multiphase* setting by means of the relative energy approach. This is joint work with Julian Fischer [49] and our main result states

As long as a strong solution to multiphase mean curvature flow exists, solutions to the vectorial Allen-Cahn equation for a suitable class of potentials and with well-prepared initial data converge towards multiphase mean curvature flow in the limit of vanishing interface width parameter $\varepsilon \to 0$.

We introduce a notion of relative energy which generalizes (1.19) to the multiphase setting, and thus relies on the concept of gradient-flow calibration ξ_i for branching singularities in multiphase mean curvature flow (cf. [46] for triple junctions in $\mathbb{R}^2$, [57] for double-bubbles in $\mathbb{R}^3$). In other words, $\xi_i - \xi_j$ can be interpreted as a suitable extension of the unit normal vector field of the smooth interfaces between phases i and j. Having this ingredient at our disposal, our relative energy ansatz is

$$E[u_\varepsilon|\xi] = \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) + \sum_{i=1}^{N} \xi_i \cdot \nabla\psi_i(u_\varepsilon)\,\mathrm{d}x, \tag{1.22}$$

where $\psi_i : \mathbb{R}^{N-1} \to [0,1]$ are suitable phase indicator functions of class $C^{1,1}$ such that $\psi_i(\alpha_j) = \delta_{i,j}$. The functions $\psi_j - \psi_i$ play a role that is somewhat similar to that of $\psi(u) = \int_0^u \sqrt{2W(s)}\,ds$ in the Modica-Mortola trick for a two-well potential W.

The properties of the gradient-flow calibration ξ_i and the assumptions on the functions ψ_i ensure the coercivity properties of $E[u_\varepsilon|\xi]$ necessary to prove the stability estimate

$$E[u_\varepsilon|\xi](t) \leq Ce^{Ct}E[u_\varepsilon|\xi](0), \quad \text{for any } t \in [0,T],$$

where $C = C(T, \bar{\chi}_i) > 0$ and $\bar{\chi}_i$ are the phase indicator functions from the strong solution to multiphase mean curvature flow. Then, for suitable initial data $u_\varepsilon(\cdot, 0)$ satisfying $E[u_\varepsilon|\xi](0) \leq C(\bar{\chi}_i(0))\varepsilon$, one can deduce

$$||\psi_i(u_\varepsilon(\cdot, t)) - \bar{\chi}_i(\cdot, t)||_{L^1(\mathbb{R}^d)} \leq C\varepsilon^{1/2}, \quad \text{for any } t \in [0, T],$$

establishing $O(\varepsilon^{1/2})$ as a rate of convergence.

Our result holds for a suitable class of multi-well potentials W, e.g. the triple wells one in 1.8. Indeed, a challenging task has been the construction of the functions ψ_i, which relies on the properties of W. We expect that our strategy can be generalised in order to work also for a larger class of multi-well potentials, making our result even stronger. However, this task is not trivial at all, as it requires additional technical work in order to compensate the relaxation of the assumptions on W, thus we leave it for future work.

Weak-strong stability for planar multiphase mean curvature flow beyond a circular topology change

In a recent work [46], Fischer, Hensel, Laux and Simon have established a weak-strong uniqueness principle for BV solutions to planar multi-phase mean curvature flow. Their result was proved by means of the relative energy technique, obtaining a weak-strong stability estimate holding prior to the first topology change.

In Chapter 4, following the relative energy approach of [46], we prove a weak-strong stability estimate holding up past the formation of the simplest dynamically stable singularity [85], a shrinking circle. This implies a weak-strong uniqueness principle for weak BV solutions to planar multiphase mean curvature flow beyond circular topology changes. This result has been obtained in collaboration with Julian Fischer, Sebastian Hensel and Maximilian Moser and reads as

Energy dissipating BV solutions in the sense of Laux and Otto [73] resp. Laux and Simon [74] to planar multiphase mean curvature flow (1.7) *satisfy a weak-strong stability estimate past a circular topology change. In particular, any energy dissipating BV solution starting from the same initial data has to coincide with the unique two-phase circular strong solution beyond the singular time at which it shrinks to a point.*

More precisely, in our statement, a two-phase circular strong solution is any smooth, closed and simple curve in the plane which is close to a circle and evolves by mean curvature flow (for an exact circle, see Figure 1.9). Our notion of strong solution is justified by the works of Gage and Hamilton [51] and of Grayson [55], stating that: Given some smooth, bounded, open and simply connected initial set $\mathcal{A}_0 \subset \mathbb{R}^2$ with boundary $\partial\mathcal{A}_0$ evolving in time into $\partial\mathcal{A}(t)$ by mean curvature flow, then $\partial\mathcal{A}(t)$ becomes circular in the process, in the sense that $\partial\mathcal{A}(t)$ gets asymptotically close to a circle of radius $r(t) = \sqrt{2(T_{ext} - t)}$, and it shrinks to a point at the extinction time $T_{ext} = \frac{\mathrm{vol}(\mathcal{A}_0)}{\pi}$.

The previous weak-strong stability result of [46] is limited to time horizons before the first topology change of the strong solution. The reason is that the time-dependent constant $C(t)$ in the associated relative energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E[\chi|\bar{\chi}](t) \leq C(t) E[\chi|\bar{\chi}](t), \quad \text{for any } t \in [0, T_{ext}],$$

is non-integrable, in particular $C(t) \sim (T_{ext} - t)^{-1}$. This leads to the study of the stability of the leading order non-integrable tems near the singularity.

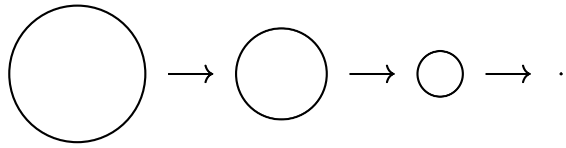


Figure 1.9: Evolution of a circle by mean curvature flow, i.e., $r'(t) = -\frac{1}{r(t)}$ for any $t \in (0, T_{ext})$, $r(0) = r_0$ and $T_{ext} = \frac{1}{2}r_0^2$.

We overcome the issue of the blowing-up contant $C(t) \sim (T_{ext} - t)^{-1}$ at the singular time T_{ext} by developing a weak-strong stability theory for circular topology change up to dynamic space-time shift $(z(t), T(t)) \in \mathbb{R}^2 \times \mathbb{R}$. The role of the space-time shift is that of dynamically adapting the strong solution to the weak BV solution so that the leading-order non-integrable contributions in the relative energy inequality are compensated. As a result, denoting by $\bar{\chi}^{z,T} = z + \bar{\chi}(T(\cdot))$ the space-time shifted strong solutuion, we obtain a Gronwall inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} E[\chi|\bar{\chi}^{z,T}](t) \leq -\frac{\alpha}{r^2(T(t))} E[\chi|\bar{\chi}^{z,T}](t) \quad \text{for } \alpha \in (0,5),$$

whence one can deduce the weak-strong stability estimate

$$E[\chi|\bar{\chi}^{z,T}](t) \leq \left(\frac{r(T(t))}{r_0}\right)^{\alpha} E[\chi|\bar{\chi}^{z,T}](0),$$

for any time $t \in (0, t_\chi)$, where $t_\chi = \sup\{t : T(t) < T_{ext}\}$.

In order to prove weak-strong stability for circular topology change up to dynamic space-time shift, we work in two different regimes for the weak BV solution. The non-regular regime correspond to the times $t \in (0, t_\chi)$ at which the dissipation term (cf. (1.10)) satisfies

$$\frac{1}{2}\sum_{i,j} \sigma_{i,j} \int_{I_{i,j}} |V_{i,j}|^2 \, \mathrm{d}\mathcal{H}^{d-1} \geq \frac{\Lambda}{r(T(t))} \quad \text{for some large } \Lambda > 0,$$

hence the dynamic space-time shift is not strictly needed in order to compensate the non-integrability of the leading oder terms. On the other hand, the regular regime corresponds to the times at which the dissipation term is strictly bounded by $\frac{\Lambda}{r(T(t))}$. In particular, in the regular regime our proof relies on the fact that the weak BV solution reduces to a graph over the strong solution at regular times.

At last, we note that we expect our method to have further applications to other types of dynamically stable shrinkers, as well as to prove quantitative convergence of diffuse interface (Allen-Cahn) approximations for planar mean curvature flow beyond the associated singularities.

1.2 A variational approach to nonlinear evolution problems

We introduce a variational approach to nonlinear evolution problems, whose interest resides in the possibility of connecting a nonlinear PDE problem with the constrained minimization of a convex functional. As a result, this approach turns out to be helpful in proving global-in-time existence of weak solutions as limits of a subsequence of minimizers, solving an elliptic-in-time regularization of the target problem.

This idea has to be traced back at least to Ilmanen [63], whose proof of existence and partial regularity of the Brakke mean curvature flow of varifolds is based on this variational technique. Remarkably, this approach can be applied to certain classes of hyperbolic problems as well. Indeed, De Giorgi conjectured this possibility in the setting of semilinear wave equations [54], which was rigorously proved later in [109, 120].

1.2.1 The Weighted Inertia-Dissipation-Energy approach

A conjecture by De Giorgi [54] states:

Minimizers of suitable convex functionals with initial data as boundary conditions converge, up to subsequences, to global weak solutions to the semilinear wave equation

$$\partial_t^2 u - \Delta u + |u|^{p-2}u = 0, \quad p \geq 2,$$

as the paramter ε goes to zero.

The functionals $I_\varepsilon(u)$ are integrals in space-time of a convex Lagrangian exponentially weighted by a parameter $\varepsilon > 0$, where the initial data of the wave equation serve as boundary conditions. Their minimizers thus solve the corresponding Euler-Lagrange equation parametrized by $\varepsilon > 0$, which is an elliptic-in-time regularization of the target problem. As ε tends to zero, the minimizers converge, up to subsequences, to a solution of the nonlinear wave equation. De Giorgi's conjecture was proved by Stefanelli [120] (on bounded time intervals $(0, T)$ with $T > 0$) and by Serra and Tilli [109] (on the original, unbounded time interval $(0, \infty)$).

The variational approach proposed by De Giorgi for the semilinear wave equation can be extended to some mixed hyperbolic-parabolic equations. In this more general setting, the approach relies on the minimization of different classes of the Weighted Inertia-Dissipation-Energy (WIDE) functionals, namely

$$I_\varepsilon(u) = \int_0^T \mathrm{e}^{-t/\varepsilon}\Big(\int_\Omega \frac{\varepsilon^2\rho}{2}|\partial_t^2 u|^2 \,\mathrm{d}x + \varepsilon\nu\psi(\partial_t u) + \phi(u)\Big)\,\mathrm{d}t, \quad \rho, \nu > 0, \tag{1.23}$$

where $T \in (0, \infty]$, $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain, $\frac{\varepsilon^2\rho}{2}|\partial_{tt}u|^2$ corresponds to the inertial term, while the functionals ψ and ϕ represent the dissipative potential and the energy of the system, respectively. The semilinear wave equation studied in [109, 120] is obtained by the choices $\nu = 0$ and

$$\phi(u) := \int_\Omega (\frac{1}{2}|\nabla u|^2 + |u|^p)\,\mathrm{d}x, \quad p \geq 2,$$

in (1.23). More in general, the hyperbolic setting corresponds to the choice $\nu = 0$, whereas the parabolic one to $\rho = 0$. In the parabolic case ($\rho = 0$), we refer to the method as Weighted Energy-Dissipation (WED) approach.

Mielke and Stefanelli [93] exploited the WED approach to gradient flows in Hilbert spaces H for $T < \infty$, by setting $\rho = 0$ and by considering general λ-convex energy functionals ϕ on H in (1.23). In particular, the class of PDE problems considered in [93] includes the scalar Allen-Cahn equation (1.12) by working with the Hilbert space $X = H_0^1(\Omega)$ and with the WED functionals

$$I_\varepsilon(u_\varepsilon) = \int_0^T \mathrm{e}^{-t/\varepsilon}\Big(\int_\Omega \frac{1}{2}|\partial_t u_\varepsilon|^2 + \frac{1}{2\varepsilon}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\,\mathrm{d}x\Big)\,\mathrm{d}t.$$

Doubly nonlinear problems corresponding to the choice $\rho = 0$ in (1.23) were first investigated by Akagi and Stefanelli in [7] assuming

$$\psi(\partial_t u) = \int_\Omega \frac{1}{p}|\partial_t u|^p \,\mathrm{d}x, \quad \phi(u) = \int_\Omega \frac{1}{q}|\nabla u|^q + F(u)\,\mathrm{d}x, \quad 2 \le p < q^*,$$

where F is as smooth and convex function and q^* is the Sobolev conjugate of q, then by Akagi, Melchionna and Stefanelli in [5, 6, 8] considering abstract functionals ψ, ϕ in the general setting of Banach spaces. The mixed hyperbolic and parabolic case for $T < \infty$ was studied by Liero and Stefanelli [78], where the techniques were combined in order to deal with both ν and $\rho > 0$, a quadratic dissipation and a specific choice of ϕ, namely

$$\psi(\partial_t u) = \int_\Omega \frac{1}{2}|\partial_t u|^2 \,\mathrm{d}x, \quad \phi(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u)\,\mathrm{d}x,$$

where F is λ-convex with derivative of polynomial growth. A more general class of nonlinear hyperbolic problems was later considered in [110] for a quadratic dissipation functional ψ and for $T = \infty$.

The WIDE approach has been used also to investigate parabolic problems which have been introduced in Section 1.1 and are of interest for this thesis. Ortiz, Schmidt and Stefanelli [98] exploited the method to prove the existence of a classical Leray-Hopf weak solution to the incompressible Navier-Stokes system

$$\partial_t v + v \cdot \nabla v - \nu \Delta v + \nabla p = 0, \quad \operatorname{div} u = 0,$$

by means of the stabilized WED functionals

$$I_\varepsilon(u) = \int_0^\infty \mathrm{e}^{-t/\varepsilon}\Big(\int_\Omega \frac{1}{2}|\partial_t v + v\cdot\nabla v|^2 + \frac{\sigma}{2}|v\cdot\nabla v|^2 + \frac{\nu}{2\varepsilon}|\nabla v|^2\,\mathrm{d}x\Big)\,\mathrm{d}t, \quad \sigma, \nu > 0,$$

under the incompressibility constraint $\operatorname{div} v = 0$ and for given initial and (homogeneous Dirichlet) boundary conditions. Recently, Bathory and Stefanelli [17] extended the result of [98] by considering non-Newtonian fluids, by allowing for in- and outlets, and by assuming general, nonhomogeneous boundary conditions. In the context of viscosity solutions to mean curvature flow (cf. Subsec. 1.1.1), Spadaro and Stefanelli [119] proved that minimizers of the relaxed WED functional

$$I_\varepsilon(u) = \int_0^T \mathrm{e}^{-t/\varepsilon}\Big(\int_\Omega \frac{1}{2}|\partial_t u|^2\,\mathrm{d}x + \frac{1}{\varepsilon}A(u)\Big)\,\mathrm{d}t, \quad T < \infty,$$

converge to the gradient-flow trajectories of the relaxed area functional

$$A(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\,\mathrm{d}x + |D^s u|(\Omega) + \int_{\partial\Omega} |\varphi - u|\,\mathrm{d}\mathcal{H}^{d-1},$$

along $D(A) = BV(\Omega)$, where $D^s u$ denotes the singular part of the Radon measure Du, whereas the function φ encodes the initial and boundary condition.

The WIDE variational approach gives the possibility to connect a difficult nonlinear PDE problem with a the constrained minimization of a uniformly convex functional of the form of (1.23). Furthermore, the application of the tools of the calculus of variations may provide a variational insight to a differential problem. For instance, the functional (1.23) may admit a unique minimizer, whereas no uniqueness may be known for the corresponding PDE problem under general nonlinearities. In this regard, the variational approach may serve as a selection criterion in some nonuniqueness situation. This possibility has been already checked for a specific ODE case in [77], but also in [6] whenever ψ or ϕ is strictly convex.

1.2.2 Contribution of this thesis

In this thesis we discuss the Weighted Inertia-Dissipation-Energy (WIDE) approach to a class of doubly nonlinear wave equations (cf. Chapter 5).

Weighted Inertia-Dissipation-Energy approach to doubly nonlinear wave equations

The aim of our work in collaboration with Goro Akagi, Verena Bögelein and Ulisse Stefanelli [4] is to generalize the result of [78] to the case of non-quadratic dissipation. We consider the WIDE functionals I_ε (1.23) with ψ which is convex, of p-growth, and twice differentiable on $L^p(\Omega)$ for $p \in [2, 4)$, and ϕ of the form

$$\phi(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u)\ \mathrm{d}x,$$

where F is convex and of r-growth with $r \in [0, p-1]$. In the following, we denote by $\mathrm{d}_V\psi$ the Gâteaux differential of ψ.

Our main result states as follows

The unique minimizers of the WIDE functional I_ε converge, up to subsequences, to a strong solution to the doubly nonlinear wave equation

$$\rho\partial_t^2 u + \nu \mathrm{d}_V\psi(\partial_t u) - \Delta u + f(u) = 0 \quad \textit{in } H^{-1}(\Omega), \quad \textit{for almost all } t \in (0,T),$$

as the paramter ε goes to zero.

Our analysis relies on specific estimates on the WIDE minimizers, which are obtained by readapting the ideas of [78, 120]. The main novelties of our work reside in the argument for the decomposition and the representation of the subdifferential of the WIDE functional I_ε and in the strategy adopted in order to identify the nonlinearity $\mathrm{d}_V\psi(u)$ in the limit. Moreover, we investigate the viscous limit $\rho \to 0$, both at the level of the functionals and at the level of their minimizers. In particular, we prove that for $(\rho, \varepsilon) \to (0, 0)$ the minimizers of I_ε converge to the unique solution of the doubly nonlinear problem $\nu \mathrm{d}_V\psi(\partial_t u) - \Delta u + f(u) = 0$ in $H^{-1}(\Omega)$, for almost all $t \in (0, T)$.

CHAPTER 2

Weak-strong uniqueness for the Navier–Stokes equation for two fluids with ninety degree contact angle and same viscosities

This chapter contains the paper "Weak-strong uniqueness for the Navier–Stokes equation for two fluids with ninety degree contact angle and same viscosities"[58], which is a joint work with Sebastian Hensel and published in *J. Math. Fluid Mech.* **24**, 93 (2022).

Abstract. We consider the flow of two viscous and incompressible fluids within a bounded domain modeled by means of a two-phase Navier–Stokes system. The two fluids are assumed to be immiscible, meaning that they are separated by an interface. With respect to the motion of the interface, we consider pure transport by the fluid flow. Along the boundary of the domain, a complete slip boundary condition for the fluid velocities and a constant ninety degree contact angle condition for the interface are assumed. In the present work, we devise for the resulting evolution problem a suitable weak solution concept based on the framework of varifolds and establish as the main result a weak-strong uniqueness principle in 2D. The proof is based on a relative entropy argument and requires a non-trivial further development of ideas from the recent work of Fischer and the first author (Arch. Ration. Mech. Anal. 236, 2020) to incorporate the contact angle condition. To focus on the effects of the necessarily singular geometry of the evolving fluid domains, we work for simplicity in the regime of same viscosities for the two fluids.

2.1 Introduction

2.1.1 Context

The question of uniqueness or non-uniqueness of weak solution concepts in the context of classical fluid mechanics models has seen a series of intriguing breakthroughs throughout the last three decades. In case of the Euler equations, the journey started with the seminal works of Scheffer [106] and Shnirelman [114] providing the construction of compactly supported nonzero weak solutions. The first example of an energy dissipating weak solution to the Euler equations is again due to Shnirelman [115]. Later, De Lellis and Székelyhidi Jr. not only strengthened these results in their groundbreaking works (see, e.g., [34] and [35]), but in retrospect even more importantly introduced a novel perspective on the problem: their proofs are based on a nontrivial transfer of convex integration techniques from typically geometric PDEs to the framework of the Euler equations. Indeed, their ideas eventually culminated in the resolution of Onsager's conjecture by Isett [65]; see also the work of Buckmaster, De Lellis, Székelyhidi Jr. and Vicol [26].

By now, these developments also generated spectacular results for the Navier–Stokes equations. For instance, Buckmaster and Vicol [27] as well as Buckmaster, Colombo and Vicol [25] establish that mild solutions in the energy class are non-unique. The constructed solutions are not Leray–Hopf solutions, i.e., it is not proven that they are subject to the energy dissipation inequality. However, Albritton, Brué and Colombo [11] even show in a very recent preprint that one can construct an external force such that there exists a finite time horizon so that one may construct at least two distinct Leray–Hopf solutions for the associated forced full-space Navier–Stokes equations in 3D (both starting from zero initial data).

Hence, in terms of uniqueness of weak solutions the best one can expect in general is essentially a weak-strong uniqueness principle. Roughly speaking, this refers to uniqueness of weak solutions within a class of sufficiently regular solutions. In the context of the incompressible Navier–Stokes equations, such results are classical and can be traced back to the works of Leray [75], Prodi [100] and Serrin [111]. In the case of the compressible Navier–Stokes equations, we mention the works of Germain [52], Feireisl, Jin and Novotný [43], as well as Feireisl and Novotný [44]. The usual strategy to establish these results is based on a by now widely used method which infers weak-strong uniqueness from a quantitative stability estimate for a suitable distance measure between two solutions, the so-called relative entropy (or relative energy). We refer to the survey article by Wiedemann [126] for an overview on the relative entropy method in the context of mathematical fluid mechanics.

In the present work, we are concerned with the question of weak-strong uniqueness with respect to a two-phase free boundary fluid problem within a physical domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$. More precisely, we study this question in terms of a suitably devised concept of varifold solutions for the evolution problem of the flow of two incompressible Navier–Stokes fluids separated by a sharp interface. Along the boundary of the domain, a complete slip boundary condition for the fluid velocities as well as a constant ninety degree contact angle condition for the interface are assumed. For the precise PDE formulation of the model, we refer to Subsection 2.1.2. For a discussion of the weak solution concept and its precise definition, we instead refer to Subsection 2.1.3 and Definition 11, respectively. The main result of the present work establishes in 2D a weak-strong uniqueness principle for the above introduced two-phase free boundary fluid problem. We refer to Theorem 1 for the precise mathematical formulation of our result. In the spirit of [46], we also derive a conditional weak-strong uniqueness result in the three-dimensional setting; cf. Proposition 4 for the precise statement. To the best of the authors' knowledge, the present work is the first to establish weak-strong uniqueness in the context of an interface evolution problem incorporating contact point dynamics in combination with a fluid mechanical coupling.

Even when neglecting the fluid mechanics, uniqueness of weak solutions in form of a weak-strong uniqueness principle is in general the best one can expect also for interface evolution problems. In this context, this is due to the formation of singularities and topology changes; see already, for instance, the work of Brakke [21] for mean curvature flow of networks of interfaces in $\mathbb{R}^2$ or the work of Angenent, Ilmanen and Chopp [14] for mean curvature flow of surfaces in $\mathbb{R}^3$. When restricting to the full-space setting $\Omega = \mathbb{R}^d$ and thus neglecting non-trivial boundary effects, Fischer and the first author [45] recently established a weak-strong uniqueness principle up to the first singularity formation for the corresponding two-phase free boundary fluid problem considered in this work. Their approach relies on a suitable extension of the relative entropy method to get control on the difference in the underlying geometries of two solutions; cf. Subsection 2.1.4 for a discussion in this direction. Their ideas were later generalized by Fischer, Laux, Simon and the first author [46] to derive a weak-strong uniqueness principle for BV solutions of Laux and Otto [73] to mean curvature flow of networks of interfaces in $\mathbb{R}^2$, or even for canonical multiphase Brakke flows of Stuvard and Tonegawa [122] (cf. also [56]).

The main challenges of the present work are twofold. First, we need to devise a weak solution concept for the above introduced two-phase free boundary fluid problem. We emphasize that this is not already contained in the work of Abels [1] which in the presence of surface tension only deals with the full-space setting. Even though our notion of varifold solutions is clearly directly inspired by Abels' formulation, some additional thoughts are necessary in the present setting of a bounded domain with contact point dynamics (cf. again Subsection 2.1.3 for a discussion). Indeed, the point is to formulate a solution concept which on one side is weak enough to allow for a satisfactory global-in-time existence theory (cf. Section 2.8 for a sketch of an existence proof along the lines of the argument of Abels [1]), but on the other side is still strong enough to support a weak-strong uniqueness principle. To obtain the latter, the second challenge of the present work is to further develop parts of the analysis of Fischer and the first author [45] to deal with the non-trivial boundary effects and the necessarily singular geometry of the evolving fluid domains. Due to the latter two, it turns out to be beneficial to implement the relative entropy argument based on a two-step procedure rather in the spirit of [46] than the more direct approach from [45] (cf. Subsection 2.2.2 for further discussion).

2.1.2 Strong PDE formulation of the two-phase fluid model

We start with a description of the underlying evolving geometry. Denoting by Ω a bounded domain in $\mathbb{R}^d$ with smooth and orientable boundary $\partial\Omega$, $d \in \{2,3\}$, each of the two fluids is contained within a time-evolving domain $\Omega^+(t) \subset \Omega$ resp. $\Omega^-(t) \subset \Omega$, $t \in [0,T)$. The interface separating both fluids is given as the common boundary between the two fluid domains. Denoting it at time $t \in [0,T)$ by $I(t) \subset \overline{\Omega}$, we then have a disjoint decomposition of $\overline{\Omega}$ in form of $\overline{\Omega} = \Omega^+(t) \cup \Omega^-(t) \cup (I(t) \cap \Omega) \cup \partial\Omega$ for every $t \in [0,T)$. We write $n_{\partial\Omega}$ to refer to the inner pointing unit normal vector field of $\partial\Omega$, as well as $n_I(\cdot,t)$ to denote the unit normal vector field along $I(t)$ pointing towards $\Omega^+(t)$, $t \in [0,T)$.

With respect to internal boundary conditions along the separating interface, first, a no-slip boundary condition is assumed. This in fact allows to represent the two fluid velocity fields by a single continuous vector field v. We also consider a single scalar field p as the pressure, which in contrast may jump across the interface. Second, along the interface the internal forces of the fluids have to match a surface tension force. Denoting by $\chi(\cdot,t)$ the characteristic function associated with the domain $\Omega^+(t)$, $t \in [0,T)$, and defining $\mu(\chi) := \mu^+\chi + \mu^-(1-\chi)$ with μ^+ and μ^- being the viscosities of the two fluids, the stress tensor $\mathbb{T} := \mu(\chi)(\nabla v + \nabla v^\mathsf{T}) - p\,\mathrm{Id}$ is required to satisfy

$$[[\mathbb{T}n_I]](\cdot,t) = \sigma\,\mathrm{H}_I(\cdot,t) \quad \text{along } I(t) \tag{2.1}$$

for all $t \in [0,T)$, where moreover $[[\cdot]]$ denotes the jump in normal direction, $\sigma > 0$ is the fixed surface tension coefficient of the interface, and $\mathrm{H}_I(\cdot,t)$ represents the mean curvature vector field along the interface $I(t)$, $t \in [0,T)$.

With respect to boundary conditions along $\partial\Omega$, we assume in terms of the two fluids a complete slip boundary conditions. In terms of the evolving geometry, a ninety degree contact angle condition at the contact set of the fluid-fluid interface with the boundary of the domain is imposed. Mathematically, this amounts to

$$v(\cdot,t)\cdot n_{\partial\Omega} = 0 \qquad \text{along } \partial\Omega, \tag{2.2}$$

$$\Big(n_{\partial\Omega}\cdot \mu(\chi)(\nabla v + \nabla v^{\mathsf{T}})(\cdot,t)B\Big) = 0 \qquad \text{along } \partial\Omega \tag{2.3}$$

for all $t \in [0,T)$ and all tangential vector fields B along $\partial\Omega$, as well as

$$n_I(\cdot,t)\cdot n_{\partial\Omega} = 0 \qquad \text{along } I(t)\cap\partial\Omega \tag{2.4}$$

for all $t \in [0,T)$. These boundary conditions not only prescribe that the fluid cannot exit from the domain and that it can move only tangentially to its boundary, but they also exclude any external contribution to the viscous stress and any friction effect with the boundary. Observe also that the ninety degree contact angle condition is consistent with the complete slip boundary conditions (2.2) and (2.3), in the sense that (2.4) together with (2.2) implies (2.3). Furthermore, the ninety degree contact angle needs to be imposed only as an initial condition: for later times it can be deduced using (2.2) and (2.3) and a Gronwall-type argument. For details, see the remark after Definition 10.

Now, defining $\rho(\chi) := \rho^+\chi + \rho^-(1-\chi)$ with ρ^+ and ρ^- representing the densities of the two fluids, the fluid motion is given by the incompressible Navier–Stokes equation, which by (2.1) and (2.3) can be formulated as

$$\partial_t\Big(\rho(\chi)v\Big) + \nabla\cdot\Big(\rho(\chi)v\otimes v\Big) = -\nabla p + \nabla\cdot\Big(\mu(\chi)(\nabla v + \nabla v^{\mathsf{T}})\Big) + \sigma\,\mathrm{H}_I\,|\nabla\chi|\llcorner\Omega, \tag{2.5}$$

$$\nabla\cdot v = 0, \tag{2.6}$$

where $|\nabla\chi|(\cdot,t)\llcorner\Omega$ represents the surface measure $\mathcal{H}^{d-1}\llcorner(I(t)\cap\Omega)$, $t\in[0,T)$. Second, the interface is assumed to be transported along the fluid flow. In other words, the associated normal velocity of the interface is given by the normal component of the fluid velocity v. Thanks to (2.2), (2.4) and (2.6), this is formally equivalent to

$$\partial_t\chi + (v\cdot\nabla)\chi = 0. \tag{2.7}$$

Finally, from a modeling perspective, the total energy of the PDE system (2.5)–(2.7) is given by the sum of kinetic and surface tension energies

$$E[\chi,v] := \int_\Omega \frac{1}{2}\rho(\chi)|v|^2\,\mathrm{d}x + \sigma\int_\Omega 1\,\mathrm{d}|\nabla\chi| + \sigma^+\int_{\partial\Omega}\chi\,\mathrm{d}S + \sigma^-\int_{\partial\Omega}(1-\chi)\,\mathrm{d}S, \tag{2.8}$$

where σ^+ and σ^- are the surface tension coefficients of $\partial\Omega\cap\overline{\Omega_t^+}$ and $\partial\Omega\cap\overline{\Omega_t^-}$, respectively. Note that the ninety degree contact angle condition (2.4) corresponds to $\sigma^- = \sigma^+$. Indeed, a general constant contact angle $\alpha\in(0,\pi)$ is prescribed by Young's equation which in our notation reads as follows

$$\sigma\cos\alpha = \sigma^+ - \sigma^-.$$

In particular, by subtracting the constant $\int_{\partial\Omega} 1\,\mathrm{d}S$ from (2.8) we see that the relevant part of the total energy does not contain a surface energy contribution along $\partial\Omega$ in our special case of a constant ninety degree contact angle. By formal computations, one finally observes that this energy satisfies an energy dissipation inequality

$$E[\chi, v](T') + \int_0^{T'} \int_\Omega \frac{\mu(\chi)}{2} |\nabla v + \nabla v^T|^2 \,\mathrm{d}x \,\mathrm{d}t \leq E[\chi, v](0), \quad T' \in [0, T). \tag{2.9}$$

2.1.3 Varifold solutions for two-phase fluid flow with 90° contact angle

In terms of weak solution theories for the evolution problem (2.5)–(2.7), the energy dissipation inequality suggests to consider velocity fields in the space $L^\infty(0,T; L^2(\Omega;\mathbb{R}^d)) \cap L^2(0,T; H^1(\Omega;\mathbb{R}^d))$, and the evolving geometry may be modeled based on a time-evolving set of finite perimeter so that the associated characteristic function χ is an element of $L^\infty(0,T; BV(\Omega;\{0,1\}))$.

However, a well-known problem arises when considering limit points of a sequence of pairs $(\chi_k, v_k)_{k\in\mathbb{N}}$ representing solutions originating from an approximation scheme for (2.5)–(2.7). Ignoring the time variable for the sake of the discussion, the main point is that a uniform bound of the form $\sup_{k\in\mathbb{N}} \|\chi_k\|_{BV(\Omega)} < \infty$ in general does not suffice to pass to the limit (not even subsequentially) in the surface tension force $\sigma\, \mathrm{H}_{I_k} |\nabla\chi_k| \llcorner \Omega$. Recalling that we work in a setting with a ninety degree angle condition, this term is represented in distributional form by

$$\int_\Omega \mathrm{H}_{I_k} \cdot B \,\mathrm{d}|\nabla\chi_k| = -\int_\Omega (\mathrm{Id} - n_k \otimes n_k) : \nabla B \,\mathrm{d}|\nabla\chi_k| \tag{2.10}$$

for all smooth vector fields B which are tangential along $\partial\Omega$, where $n_k = \frac{\nabla\chi_k}{|\nabla\chi_k|}$ denotes the measure-theoretic interface unit normal. One may pass to the limit on the right hand side of the previous display provided $|\nabla\chi_k|(\Omega) \to |\nabla\chi|(\Omega)$. However, for standard approximation schemes there is in general no reason why this should be true. For instance, hidden boundaries may be generated within Ω in the limit. Furthermore, but now specific to the setting of a bounded domain, nontrivial parts of the approximating interfaces may converge towards the boundary $\partial\Omega$.

The upshot is that one has to pass to an even weaker representation of the surface tension force than (2.10). A popular workaround is based on the concept of (oriented) varifolds. In the setting of the present work and in view of the preceding discussion, this in fact amounts to consider the space of finite Radon measures on the product space $\overline{\Omega}\times\mathbb{S}^{d-1}$. Indeed, introducing the varifold lift $V_k := |\nabla\chi_k| \llcorner \Omega \otimes (\delta_{n_k(x)})_{x\in\Omega}$ one may equivalently express the right hand side of (2.10) in terms of the functional $B \mapsto -\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s\otimes s) : \nabla B \,\mathrm{d}V_k(x,s)$ which is now stable with respect to weak* convergence in the space of finite Radon measures on $\overline{\Omega}\times\mathbb{S}^{d-1}$. Note also that by the choice of working in a varifold setting, one expects $\sigma \int_{\overline{\Omega}} 1 \,\mathrm{d}|V|_{\mathbb{S}^{d-1}}$ instead of $\sigma \int_\Omega 1 \,\mathrm{d}|\nabla\chi|$ as the interfacial energy contribution in (2.8), where the finite Radon measure $|V|_{\mathbb{S}^{d-1}}$ denotes the mass of the varifold V.

Motivated by the previous discussion, we give a full formulation of a varifold solution concept to two-phase fluid flow with surface tension and constant ninety degree contact angle in Definition 11 below. This definition is nothing else but the suitable analogue of the definition by Abels [1], who provides for the full-space setting a global-in-time existence theory for such varifold solutions with respect to rather general initial data. Unfortunately, in the bounded domain case with non-zero interfacial surface tension, to the best of our knowledge a global-in-time existence result for varifold solutions is missing. In particular, such a result is not contained in the work of Abels [1]. For this reason, we include in this work at least a sketch of an existence proof. To this end, one may follow on one side the higher-level structure of the argument given by Abels [1] for the full-space setting. On the other side, additional arguments are of course necessary due to the specified boundary conditions for the geometry and the fluids, respectively. These additional arguments are outlined in Section 2.8.

2.1.4 Weak-strong uniqueness for varifold solutions of two-phase fluid flow

In case the two fluids occupy the full space $\mathbb{R}^d$, $d \in \{2, 3\}$, a weak-strong uniqueness result for Abels' [1] varifold solutions of the system (2.5)–(2.7) was recently established by Fischer and the first author [45]. Given sufficiently regular initial data, it is shown that on the time horizon of existence of the associated unique strong solution, any varifold solution in the sense of Abels [1] starting from the same initial data has to coincide with this strong solution.

This result is achieved by extending a by now several decades old idea in the analysis of classical PDE models from continuum mechanics to a previously not covered class of problems: a relative entropy method for surface tension driven interface evolution. The gist of this method can be described as follows. Based on a dissipated energy functional, one first tries to build an error functional — the relative entropy — which penalizes the difference between two solutions in a sufficiently strong sense. A minimum requirement is to ensure that the error functional vanishes if and only if the two solutions coincide. In a second step, one proceeds by computing the time evolution of this error functional. In a third step, one tries to identify all the terms appearing in this computation as contributions which either are controlled by the error functional itself or otherwise may be absorbed into a residual quadratic term represented essentially by the difference of the dissipation energies. One finally concludes by an application of Gronwall's lemma.

The novelty of the work [45] consists of an implementation of this strategy for the full-space version of the energy functional (2.8). More precisely, the relative entropy as it was originally constructed in the full-space setting in [45] essentially consists of two contributions. The first aims for a penalization of the difference of the underlying geometries of the two solutions. This in fact is performed at the level of the interfaces by introducing a tilt-excess type error functional with respect to the two associated unit normal vector fields. To this end, the construction of a suitable extension of the unit normal vector field of the interface of the strong solution in the vicinity of its space-time trajectory is required. Furthermore, the length of this vector field is required to decrease quadratically fast as one moves away from the interface of the strong solution. The merit of this is that one also obtains a measure of the interface error in terms of the distance between them.

Due to the inclusion of contact point dynamics in form of a constant ninety degree contact angle, some additional ingredients are needed for the present work. We refer to Subsection 2.2.2 below for a detailed and mathematical account on the geometric part of the relative entropy functional. There are however two notable additional difficulties in comparison to [45] which are worth emphasizing already at this point. Both are related to the required extension ξ of the unit normal vector field associated with the evolving interface of the strong solution. The first is concerned with the correct boundary condition for the extension ξ along $\partial\Omega$. Since along the contact set the interface intersects the boundary of the domain orthogonally, it is natural to enforce ξ to be tangential along $\partial\Omega$. This indeed turns out to be the right condition as it allows by an integration by parts to rewrite the interfacial part of the relative entropy as the sum of interfacial energy of the weak solution and a linear functional with respect to the characteristic function χ of the weak solution. This is crucial to even attempt computing the time evolution.

The second difference concerns the actual construction of the extension ξ. In contrast to [45], where only a finite number of sufficiently regular closed curves ($d = 2$) or closed surfaces ($d = 3$) are allowed at the level of the strong solution, this results in a nontrivial and subtle task in the context of the present work due to the necessarily singular geometry in contact angle problems. The main difficulty roughly speaking is to provide a construction which on one side respects the required boundary condition and on the other side is regular enough to support the computations and estimates in the Gronwall-type argument. For a complete list of the required conditions for the extension ξ, we refer to Definition 2 below.

We finally turn to a brief discussion of the second contribution in the total relative entropy functional from [45]. In principle, this term on first sight should be nothing else than the relative entropy analogue to the kinetic part of the energy of the system, thus controlling the squared L^2-distance between the fluid velocities of the two solutions. However, as recognized in [45] a major problem arises for the two-phase fluid problem in the regime of different viscosities $\mu^+ \neq \mu^-$: without performing a very careful (and in its implementation highly technical) perturbation of this naive ansatz for the fluid velocity error, a Gronwall-type argument will not be realizable; cf. for more details the discussion in [45, Subsection 3.4]. Since the main focus of the present work lies on the inclusion of the ninety degree contact angle condition, we do not delve into these issues and simply assume for the rest of this work that the viscosities of the two fluids coincide: $\mu := \mu^+ = \mu^-$. We emphasize, however, that at least for the construction of the extension ξ and the verification of its properties we in fact do not rely on this assumption.

2.2 Main results

2.2.1 Weak-strong uniqueness and stability of evolutions

The main result of this chapter reads as follows.

Theorem 1. *Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_u, u, V) be a varifold solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 11 on a time interval $[0, T_w)$. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0, T_s)$ where $T_s \leq T_w$.*

Then, for every $T \in (0, T_s)$ there exists a constant $C = C(\chi_v, v, T) > 0$ such that the relative entropy functional (2.29) *and the bulk error functional* (2.31) *satisfy stability estimates of the form*

$$E[\chi_u, u, V|\chi_v, v](t) \leq Ce^{Ct}\Big(E[\chi_u, u, V|\chi_v, v](0) + E_{\mathrm{vol}}[\chi_u|\chi_v](0)\Big), \tag{2.11}$$

$$E_{\mathrm{vol}}[\chi_u|\chi_v](t) \leq Ce^{Ct}\Big(E[\chi_u, u, V|\chi_v, v](0) + E_{\mathrm{vol}}[\chi_u|\chi_v](0)\Big) \tag{2.12}$$

for almost every $t \in [0, T]$.

In particular, in case the initial data for the varifold solution and strong solution coincide, it follows that

$$\chi_u(\cdot, t) = \chi_v(\cdot, t), \quad u(\cdot, t) = v(\cdot, t) \qquad \text{a.e. in } \Omega \text{ for a.e. } t \in [0, T_s), \tag{2.13}$$

$$V_t = (|\nabla\chi_u(\cdot,t)|\llcorner\Omega) \otimes \Big(\delta_{\frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}(x)}\Big)_{x\in\Omega} \qquad \text{for a.e. } t\in[0,T_s). \tag{2.14}$$

Before proceeding with a discussion on the proof of Theorem 1, we comment on its validity in the regime of different shear viscosities $\mu^+ \neq \mu^-$ of the two fluids (cf. also the detailed discussion in [45, Subsection 3.4]). In this case, one would have to deal with an additional term in the derivation of the Gronwall inequality (2.11) of the form

$$-\int_0^{T'}\int_\Omega (\mu^+ - \mu^-)(\chi_u - \chi_v)2(\nabla^{\mathrm{sym}}u - \nabla^{\mathrm{sym}}v) : \nabla v\,dx. \tag{2.15}$$

A major problem then results from the observation that, even for strong solutions, the normal derivative of the tangential velocity is discontinuous across the associated interface in case $\mu^+ \neq \mu^-$. As a consequence, the term (2.15) is in fact only of linear order in our error functionals which makes the derivation of a stability estimate as in (2.11) infeasible (cf. the example given in [45, Subsection 3.4]).

The key idea for the weak-strong uniqueness result in the different viscosities regime in the full space setting [45] was to adapt the kinetic energy contribution of the relative entropy: instead of comparing u with v directly, one carefully constructs an auxiliary divergence free vector field w and compares u with $v + w$. The two desired main properties of w are as follows. First, the L^2 norm of w shall be controlled by the interfacial error contribution of the relative entropy, so that the adapted relative entropy does not lose coercivity with respect to the error in the velocity fields. Second, ∇w should be designed such that it essentially compensates the linear order error term (2.15). The main idea for the latter is to adapt ∇v through ∇w to the different location of the interface of the varifold solution.

Of course, also in our bounded domain setting with constant $90°$ degree contact angle and pure slip condition, this additional adaptation of the relative entropy is needed to conclude about the validity of Theorem 1 in case of different viscosities for the two fluids. In principle, we expect this to be possible in the setting of the present work. However, adapting the construction of the compensating vector field w from [45] in the vicinity of the domain boundary (in order to satisfy required boundary conditions) together with then verifying all of its desired properties may certainly require a substantial amount of technical work (e.g., due to the singular nature of the geometry at the contact set). For this reason, we omit the rigorous study of the different viscosities regime in this work and we leave it as a possible further development of our result. Finally, as for the validity of Theorem 1 for non-Newtonian fluids, we mention that this is an open problem in both the full space setting and the setting of the present work.

Returning to the regime of same viscosities $\mu^+ = \mu^-$, we explain throughout the next two subsections the key ideas underlying the proof of Theorem 1.

2.2.2 Quantitative stability by a relative entropy approach

Following the general strategy of [45], our weak-strong uniqueness result essentially relies on two ingredients: *i)* the construction of a suitable extension ξ of the unit normal vector field of the interface of a strong solution, and *ii)* based on this extension, the introduction of a suitably defined error functional penalizing the interface error between a varifold and a strong solution in a sufficiently strong sense. In comparison to [45], the extension of the unit normal has to be carefully constructed in the sense that the vector field ξ is required to be tangent to the domain boundary $\partial\Omega$ (which is the natural boundary condition in case of a $90°$ contact angle). Due to the singular nature of the geometry at the contact set, this is a nontrivial task. The precise conditions on the extension ξ are summarized as follows.

Definition 2 (Boundary adapted extension of the interface unit normal)**.** *Let $d \in \{2,3\}$, and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable and smooth boundary. Let $T \in (0,\infty)$ be a finite time horizon. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on the time interval $[0,T]$.*

In this setting, we call a vector field $\xi\colon \overline{\Omega}\times[0,T]\to\mathbb{R}^d$ a boundary adapted extension of n_{I_v} for two-phase fluid flow (χ_v, v) with $90°$ contact angle *if the following conditions are satisfied:*

- *In terms of regularity, it holds $\xi \in \big(C^0_tC^2_x \cap C^1_tC^0_x\big)\big(\overline{\Omega\times[0,T]} \setminus (I_v \cap (\partial\Omega\times[0,T]))\big)$.*
- *The vector field ξ extends the unit normal vector field n_{I_v} (pointing inside Ω^+_v) of the interface I_v subject to the conditions*

$$|\xi| \leq \max\Big\{0, 1 - C\operatorname{dist}^2(\cdot, I_v)\Big\} \qquad \text{in } \Omega\times[0,T], \tag{2.16a}$$

$$\xi\cdot n_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega\times[0,T], \tag{2.16b}$$

$$\nabla\cdot\xi = -H_{I_v} \qquad \text{on } I_v, \tag{2.16c}$$

  for some $C>0$. Here, H_{I_v} denotes the scalar mean curvature of the interface I_v (oriented with respect to the normal n_{I_v}).
- *The fluid velocity approximately transports the vector field ξ in form of*

$$\partial_t\xi + (v\cdot\nabla)\xi + (\mathrm{Id} - \xi\otimes\xi)(\nabla v)^\mathsf{T}\xi = O(\operatorname{dist}(\cdot, I_v)\wedge 1) \qquad \text{in } \Omega\times[0,T], \tag{2.16d}$$

$$\partial_t|\xi|^2 + (v\cdot\nabla)|\xi|^2 = O(\operatorname{dist}^2(\cdot, I_v)\wedge 1) \qquad \text{in } \Omega\times[0,T]. \tag{2.16e}$$

Let us comment on the motivation behind this definition. Given a vector field ξ with respect to a fixed strong solution (χ_v, v) as in the previous definition, we may introduce for any varifold solution (χ_u, u, V) and for all $t\in[0,T]$ a functional

$$E[\chi_u, V|\chi_v](t) := \sigma\int_{\overline{\Omega}} 1\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} - \sigma\int_{I_u(t)} \frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}\cdot\xi(\cdot,t)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{2.17}$$

where $I_u(t) := \operatorname{supp}|\nabla\chi_u(\cdot,t)|\cap\Omega$ denotes the interface associated to the varifold solution. The functional $E[\chi_u, V|\chi_v]$ is a measure for the interfacial error between the two solutions for the following reasons. First of all, it is a consequence of the definition of a varifold solution, cf. the compatibility condition (2.42), that for almost every $t\in[0,T]$ it holds $|\nabla\chi_u(\cdot,t)|\llcorner\Omega \leq |V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega$ in the sense of measures on Ω. In particular, it follows that the functional $E[\chi_u, V|\chi_v]$ controls its "BV-analogue"

$$0 \leq E[\chi_u|\chi_v](t) := \sigma\int_{I_u(t)} 1 - \frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}\cdot\xi(\cdot,t)\,\mathrm{d}\mathcal{H}^{d-1} \leq E[\chi_u, V|\chi_v](t). \tag{2.18}$$

Introducing the Radon–Nikodým derivative $\theta_t := \frac{\mathrm{d}|\nabla\chi_u(\cdot,t)|\llcorner\Omega}{\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega}$, one can be even more precise in the sense that

$$E[\chi_u, V|\chi_v](t) = \sigma\int_{\partial\Omega} 1\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} + \sigma\int_{\Omega} 1 - \theta_t\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} + E[\chi_u|\chi_v](t). \tag{2.19}$$

This representation of the functional $E[\chi_u, V|\chi_v]$ as well as the length constraint (2.16a) for the vector field ξ lead to the following two observations. First, the functional $E[\chi_u, V|\chi_v]$ controls the mass of hidden boundaries and higher multiplicity interfaces (i.e., where $\theta_t \in [0,1)$) in the sense of

$$\sigma \int_{\partial\Omega} 1 \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} + \sigma \int_{\Omega} 1 - \theta_t \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} \leq E[\chi_u, V|\chi_v](t). \tag{2.20}$$

Second, because of (2.16a) it measures the interface error in the sense that

$$\sigma \int_{I_u(t)} \frac{1}{2}\left|\frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|} - \xi\right|^2 \mathrm{d}\mathcal{H}^{d-1} \leq E[\chi_u|\chi_v](t), \tag{2.21}$$

$$\sigma \int_{I_u(t)} \min\left\{1, C\,\mathrm{dist}^2(\cdot, I_v(t))\right\} \mathrm{d}\mathcal{H}^{d-1} \leq E[\chi_u|\chi_v](t). \tag{2.22}$$

On a different note, the compatibility condition (2.42) satisfied by a varifold solution together with the boundary condition (2.16b) also allows to represent the error functional $E[\chi_u, V|\chi_v]$ in the alternative form

$$E[\chi_u, V|\chi_v](t) = \sigma \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} 1 - s\cdot\xi \,\mathrm{d}V_t, \tag{2.23}$$

which then entails as a consequence of (2.16a)

$$\sigma \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} \frac{1}{2}|s-\xi|^2 \,\mathrm{d}V_t \leq E[\chi_u, V|\chi_v](t), \tag{2.24}$$

$$\sigma \int_{\overline{\Omega}} \min\left\{1, C\,\mathrm{dist}^2(\cdot, I_v(t))\right\} \mathrm{d}|V_t|_{\mathbb{S}^{d-1}} \leq E[\chi_u, V|\chi_v](t). \tag{2.25}$$

Finally, let us quickly discuss what is implied by $E[\chi_u, V|\chi_v](t) = 0$. We claim that (2.14) and $I_u(t) \subset I_v(t)$ up to $\mathcal{H}^{d-1}$-negligible sets have to be satisfied. Indeed, the latter follows directly from (2.18) and (2.22). The former is best seen when representing the varifold $V_t \llcorner (\Omega\times\mathbb{S}^{d-1})$ by its disintegration $(|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega) \otimes (\nu_{x,t})_{x\in\Omega}$. Then, it follows on one side from (2.20) that $|V_t|_{\mathbb{S}^{d-1}}\llcorner\partial\Omega = 0$ and $|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega = |\nabla\chi_u(\cdot,t)|\llcorner\Omega$ as measures on $\partial\Omega$ and Ω, respectively, and then on the other side that $\nu_{x,t} = \delta_{\frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}(x)}$ for $|\nabla\chi_u(\cdot,t)|$-a.e. $x\in\Omega$ due to

$$\begin{aligned}
&\int_{\Omega}\int_{\mathbb{S}^{d-1}} \frac{1}{2}\left|s - \frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}(x)\right|^2 \mathrm{d}\nu_{x,t}(s)\,\mathrm{d}(|\nabla\chi_u(\cdot,t)|\llcorner\Omega)(x) \\
&= \int_{\Omega}\int_{\mathbb{S}^{d-1}} 1 - s\cdot\frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}(x)\,\mathrm{d}\nu_{x,t}(s)\,\mathrm{d}(|\nabla\chi_u(\cdot,t)|\llcorner\Omega)(x) = 0,
\end{aligned}$$

where for the last equality we simply plugged in the compatibility condition (2.42) and again $|V_t|_{\mathbb{S}^{d-1}}\llcorner\partial\Omega = 0$ as well as $|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega = |\nabla\chi_u(\cdot,t)|\llcorner\Omega$.

Apart from these coercivity conditions, it is equally important to be able to estimate the time evolution of the error functional $E[\chi_u, V|\chi_v]$. The main observation in this regard is that the functional can be rewritten as a perturbation of the interface energy $E[\chi_u, V](t) := \sigma\int_{\overline{\Omega}} 1 \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}$ which is linear in the dependence on the indicator function χ_u. Indeed, thanks to the boundary condition (2.16b) for the extension ξ, a simple integration by parts readily reveals

$$E[\chi_u, V|\chi_v](t) = E[\chi_u, V](t) + \sigma\int_{\Omega} \chi_u(\cdot,t)(\nabla\cdot\xi)(\cdot,t)\,\mathrm{d}x. \tag{2.26}$$

This structure is in fact the very reason why we call $E[\chi_u, V|\chi_v]$ a relative entropy. Computing the time evolution of $E[\chi_u|\chi_v]$ then only requires to exploit the dissipation of energy and using $\nabla \cdot \xi$ as a test function in the evolution equation of the phase indicator χ_u of the varifold solution. The latter in turn requires knowledge on the time evolution of ξ itself, which is encoded in terms of the fluid velocity v through the equations (2.16d) and (2.16e). The condition (2.16c) is natural in view of the interpretation of ξ as an extension of the unit normal n_{I_v} away from the interface I_v.

Even though all of this may already be quite promising, there is one small caveat: obviously, one can not deduce from $E[\chi_u, V|\chi_v] = 0$ that $\chi_u = \chi_v$ (e.g., χ_u representing an empty phase is consistent with having vanishing relative entropy). This lack of coercivity in the regime of vanishing interface measure motivates to introduce a second error functional which directly controls the deviation of χ_u from χ_v. The main input to such a functional is captured in the following definition.

Definition 3 (Transported weight). *Let $d \in \{2,3\}$, and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable and smooth boundary. Let $T \in (0,\infty)$ be a finite time horizon, consider a solenoidal vector field $v \in L^2([0,T]; H^1(\Omega;\mathbb{R}^d))$ with $(v \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$, and let $(\Omega_v^+(t))_{t\in[0,T]}$ be a family of sets of finite perimeter in Ω. Denote by $I_v(t)$, $t \in [0,T]$, the reduced boundary of $\Omega_v^+(t)$ in Ω. Writing $\chi_v(\cdot,t)$ for the indicator function associated to $\Omega_v^+(t)$, assume that $\partial_t\chi_v = -\nabla \cdot (\chi_v v)$ in a weak sense.*

In this setting, we call a map $\vartheta\colon \overline{\Omega} \times [0,T] \to [-1,1]$ a transported weight *with respect to (χ_v, v) if the following conditions are satisfied:*

- *(Regularity) It holds $\vartheta \in W^{1,\infty}_{x,t}(\Omega \times [0,T])$.*
- *(Coercivity) Throughout the essential interior of Ω_v^+ (relative to Ω) it holds $\vartheta < 0$, throughout the essential exterior of Ω_v^+ (relative to Ω) it holds $\vartheta > 0$, and along $I_v \cup \partial\Omega$ we have $\vartheta = 0$. There also exists $C > 0$ such that*

$$\operatorname{dist}(\cdot,\partial\Omega) \wedge \operatorname{dist}(\cdot, I_v) \wedge 1 \leq C|\vartheta| \quad \text{in } \Omega \times [0,T]. \tag{2.27}$$

- *(Transport equation) There exists $C > 0$ such that*

$$|\partial_t\vartheta + (v \cdot \nabla)\vartheta| \leq C|\vartheta| \quad \text{in } \Omega \times [0,T]. \tag{2.28}$$

The merit of the previous two definitions is now the following result. It reduces the proof of Theorem 1 to the existence of a boundary adapted extension ξ of the interface unit normal and a transported weight ϑ with respect to a strong solution (χ_v, v), respectively.

Proposition 4 (Conditional weak-strong uniqueness principle). *Let $d \in \{2,3\}$, and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable and smooth boundary. Let (χ_u, u, V) be a varifold solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 11 on a time interval $[0,T]$. Consider in addition a strong solution (χ_v, v) to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$.*

Assume there exists a boundary adapted extension ξ of the unit normal n_{I_v} as well as a transported weight ϑ with respect to (χ_v, v) in the sense of Definition 2 and Definition 3, respectively. Then the stability estimates (2.11) *and* (2.12) *for the relative entropy functional* (2.29) *and the bulk error functional* (2.31) *are satisfied, respectively. Moreover, if the initial data of the varifold solution and the strong solution coincide, we may conclude that*

$$\chi_u(\cdot,t) = \chi_v(\cdot,t), \quad u(\cdot,t) = v(\cdot,t) \qquad \text{a.e. in } \Omega \text{ for a.e. } t \in [0,T],$$

$$V_t = (|\nabla\chi_u(\cdot,t)|\llcorner\Omega) \otimes \Big(\delta_{\frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}(x)}\Big)_{x\in\Omega} \qquad \textit{for a.e. } t\in[0,T].$$

A proof of this conditional weak-strong uniqueness principle is presented in Subsection 2.3.3 below. We emphasize again that it is valid for $d\in\{2,3\}$. The key ingredient to the stability estimate (2.11) is the following relative entropy inequality. We refer to Subsection 2.3.1 for a proof.

Proposition 5 (Relative entropy inequality in case of a 90° contact angle)**.** *Let $d\in\{2,3\}$, and let $\Omega\subset\mathbb{R}^d$ be a smooth and bounded domain. Let (χ_u,u,V) be a varifold solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 11 on a time interval $[0,T]$. In particular, let θ be the density $\theta_t := \frac{\mathrm{d}|\nabla\chi_u(\cdot,t)|\llcorner\Omega}{\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega}$ as defined in* (2.43)*. Furthermore, let (χ_v,v) be a strong solution in the sense of Definition 10 on the same time interval $[0,T]$, and assume there exists a boundary adapted extension ξ of the interface unit normal n_{I_v} with respect to (χ_v,v) as in Definition 2.*

Then, the total relative entropy defined by (recall the definition (2.17) *of the interface contribution $E[\chi_u,V|\chi_v]$)*

$$E[\chi_u,u,V|\chi_v,v](t) := \int_\Omega \frac{1}{2}\rho(\chi_u(\cdot,t))|u(\cdot,t)-v(\cdot,t)|^2\,\mathrm{d}x + E[\chi_u,V|\chi_v](t) \tag{2.29}$$

satisfies the relative entropy inequality

$$\begin{aligned} &E[\chi_u,u,V|\chi_v,v](T') + \int_0^{T'}\int_\Omega \frac{\mu}{2}|\nabla(u-v)+\nabla(u-v)^\mathsf{T}|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\leq E[\chi_u,u,V|\chi_v,v](0) + R_{dt} + R_{adv} + R_{surTen}, \end{aligned}\tag{2.30}$$

for almost every $T'\in[0,T]$, where we made use of the abbreviations (denote by $n_u := \frac{\nabla\chi_u}{|\nabla\chi_u|}$ the measure-theoretic unit normal)

$$R_{dt} = -\int_0^{T'}\int_\Omega (\rho(\chi_v)-\rho(\chi_u))(u-v)\cdot\partial_t v\,\mathrm{d}x\,\mathrm{d}t,$$

$$\begin{aligned} R_{adv} = &-\int_0^{T'}\int_\Omega (\rho(\chi_u)-\rho(\chi_v))(u-v)\cdot(v\cdot\nabla)v\,\mathrm{d}x\,\mathrm{d}t \\ &-\int_0^{T'}\int_\Omega \rho(\chi_u)(u-v)\cdot((u-v)\cdot\nabla)v\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

as well as

$$\begin{aligned} R_{surTen} = &-\sigma\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (s-\xi)\cdot((s-\xi)\cdot\nabla)v\,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\ &+\sigma\int_0^{T'}\int_\Omega (1-\theta_t)\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\ &+\sigma\int_0^{T'}\int_{\partial\Omega} \xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\ &+\sigma\int_0^{T'}\int_\Omega (\chi_u-\chi_v)((u-v)\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&- \sigma \int_0^{T'} \int_\Omega (n_u - \xi) \cdot (\partial_t \xi + (v \cdot \nabla)\xi + (\mathrm{Id} - \xi \otimes \xi)(\nabla v)^\mathsf{T} \xi) \, \mathrm{d}|\nabla \chi_u| \, \mathrm{d}t \\
&- \sigma \int_0^{T'} \int_\Omega ((n_u - \xi) \cdot \xi)(\xi \otimes \xi : \nabla v) \, \mathrm{d}|\nabla \chi_u| \, \mathrm{d}t \\
&- \sigma \int_0^{T'} \int_\Omega \Big(\partial_t \frac{1}{2}|\xi|^2 + (v \cdot \nabla)\frac{1}{2}|\xi|^2 \Big) \, \mathrm{d}|\nabla \chi_u| \, \mathrm{d}t \\
&+ \sigma \int_0^{T'} \int_\Omega (1 - n_u \cdot \xi)(\nabla \cdot v) \, \mathrm{d}|\nabla \chi_u| \, \mathrm{d}t.
\end{aligned}
$$

The stability estimate (2.12) for the bulk error functional is in turn based on the following auxiliary result; see Subsection 2.3.2 for a proof.

Lemma 6 (Time evolution of the bulk error)**.** *Let $d \in \{2,3\}$, and let $\Omega \subset \mathbb{R}^d$ be a smooth and bounded domain. Let $T \in (0,\infty)$ be a finite time horizon, and let (χ_v, v) be as in Definition 3 of a transported weight. Let (χ_u, u, V) be a varifold solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 11 on $[0,T]$. Assume there exists a transported weight ϑ with respect to (χ_v, v) in the sense of Definition 3, and define the bulk error functional*

$$
E_{\mathrm{vol}}[\chi_u|\chi_v](t) := \int_\Omega |\chi_u(\cdot,t) - \chi_v(\cdot,t)||\vartheta(\cdot,t)| \, \mathrm{d}x. \tag{2.31}
$$

Then the following identity holds true for almost every $T' \in [0,T]$

$$
\begin{aligned}
E_{\mathrm{vol}}[\chi_u|\chi_v](T') = E_{\mathrm{vol}}[\chi_u|\chi_v](0) &+ \int_0^{T'} \int_\Omega (\chi_u - \chi_v)(\partial_t \vartheta + (v \cdot \nabla)\vartheta) \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^{T'} \int_\Omega (\chi_u - \chi_v)\big((u - v) \cdot \nabla\big)\vartheta \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{2.32}
$$

2.2.3 Existence of boundary adapted extensions of the interface unit normal and transported weights in planar case

To upgrade the conditional weak-strong uniqueness principle of Proposition 4 to the statement of Theorem 1, it remains to construct a boundary adapted extension ξ of n_{I_v} and a transported weight ϑ associated to a given strong solution (χ_v, v). In the context of the present work, we perform this task for simplicity in the planar regime $d = 2$. However, it is expected that the principles of the construction carry over to the case $d = 3$ involving contact lines.

Proposition 7. *Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Then there exists a boundary adapted extension ξ of n_{I_v} w.r.t. (χ_v, v) in the sense of Definition 2.*

A proof of this result is presented in Subsection 2.6.2 below. One major step in the proof consists of reducing the global construction to certain local constructions being supported in the bulk Ω or in the vicinity of contact points along $\partial\Omega$, respectively. The main ingredients for this reduction argument are provided in Subsection 2.6.1. The construction of suitable local vector fields subject to conditions as in Definition 2 is in turn relegated to Section 2.4 (bulk construction) and Section 2.5 (construction near contact points). We finally provide the construction of a transported weight in Section 2.7.

Lemma 8. *Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0, T]$. Then there exists a transported weight ϑ w.r.t. (χ_v, v) in the sense of Definition 3.*

2.2.4 Definition of varifold and strong solutions

In this subsection, we present definitions of strong and varifold solutions for the free-boundary problem of the evolution of two immiscible, incompressible, viscous fluids separated by a sharp interface with surface tension inside a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, with smooth and orientable boundary. Recall in this context that we restrict ourselves to the case of a $90°$ contact angle between the interface and the boundary of the domain Ω. In order to define a notion of strong solutions, we first introduce the notion of a smoothly evolving domain within Ω.

Definition 9 (Smoothly evolving domains and smoothly evolving interfaces with $90°$ contact angle)**.** *Let $d \in \{2, 3\}$, and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable and smooth boundary. Let $T \in (0, \infty)$ be a finite time horizon. Consider an open subset $\Omega_0^+ \subset \Omega$ subject to the following regularity conditions:*

- *Denoting by I_0 the closure of $\partial\Omega_0^+ \cap \Omega$ in $\overline{\Omega}$, we require I_0 to be a $(d-1)$-dimensional uniform C^3_x submanifold of $\overline{\Omega}$ with or without boundary. Moreover, I_0 is compact and consists of finitely many connected components.*
- *Interior points of I_0 are contained in Ω, whereas boundary points of I_0 are contained in $\partial\Omega$. In particular, $I_0 \cap \partial\Omega$ is a $(d-2)$-dimensional uniform C^3_x submanifold of $\partial\Omega$.*
- *Whenever I_0 intersects with $\partial\Omega$, it does so by forming an angle of $90°$.*

Now, consider a set $\Omega^+ = \bigcup_{t\in[0,T]} \Omega^+(t)\times\{t\}$ represented in terms of open subsets $\Omega^+(t) \subset \Omega$ for all $t \in [0, T]$. Denote by $I(t)$ the closure of $\partial\Omega^+(t) \cap \Omega$ in $\overline{\Omega}$, $t \in [0, T]$. We call Ω^+ a smoothly evolving domain in Ω, *and $I = \bigcup_{t\in[0,T]} I(t)\times\{t\}$ a* smoothly evolving interface with $90°$ contact angle, *if there exists a flow map $\psi\colon \overline{\Omega} \times [0, T] \to \overline{\Omega}$ such that the following requirements are satisfied:*

- *$\psi(\cdot, 0) = \mathrm{Id}$. For any $t \in [0, T]$, the map $\psi_t := \psi(\cdot, t)\colon \overline{\Omega} \to \overline{\Omega}$ is a C^3_x diffeomorphism such that $\psi_t(\Omega) = \Omega$, $\psi_t(\partial\Omega) = \partial\Omega$ and $\sup_{t\in[0,T]} \|\psi_t\|_{W^{3,\infty}_x(\overline{\Omega})} < \infty$.*
- *For all $t \in [0, T]$, it holds $\Omega^+(t) = \psi_t(\Omega_0^+)$ and $I(t) = \psi_t(I_0)$.*
- *$\partial_t\psi \in C([0, T]; C^1(\overline{\Omega}))$ such that $\sup_{t\in[0,T]} \|\partial_t\psi(\cdot, t)\|_{W^{1,\infty}_x(\overline{\Omega})} < \infty$.*
- *Whenever $I(t)$, $t \in [0, T]$, intersects $\partial\Omega$ it does so by forming an angle of $90°$.*

With the geometric setup in place, we can proceed with our notion of strong solutions to two-phase Navier–Stokes flow with $90°$ contact angle.

Definition 10 (Strong solution)**.** *Let $d \in \{2, 3\}$, and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable and smooth boundary. Let a surface tension constant $\sigma > 0$, the densities and shear viscosity of the two fluids $\rho^\pm, \mu > 0$, and a finite time $T_s > 0$ be given. Let*

χ_0 denote the indicator function of an open subset $\Omega_0^+ \subset \Omega$ subject to the conditions of Definition 9. Denoting the associated initial interface by $I_v(0)$, let a solenoidal initial velocity profile $v_0 \in L^2(\Omega;\mathbb{R}^d)$ be given such that it holds $v_0 \in C^2(\overline{\Omega} \setminus I_v(0))$. (Of course, additional compatibility conditions in terms of an initial pressure p_0 have to be satisfied by v_0 to allow for the below required regularity of the solution.)

A pair (χ_v, v) consisting of a velocity field $v\colon \overline{\Omega} \times [0,T_s) \to \mathbb{R}^d$ and an indicator function $\chi_v\colon \overline{\Omega} \times [0,T_s) \to \{0,1\}$ is called a strong solution to the free boundary problem for the Navier–Stokes equation for two fluids with $90°$ contact angle and initial data (χ_0, v_0) *if for all $T \in (0,T_s)$ it is a* strong solution on $[0,T]$ *in the following sense:*

- *It holds*

$$
\begin{aligned}
v &\in W^{1,\infty}([0,T];W^{1,\infty}(\Omega;\mathbb{R}^d)),\\
\nabla v &\in L^1([0,T];\mathrm{BV}(\Omega;\mathbb{R}^{d\times d})),\\
\chi_v &\in L^\infty([0,T];\mathrm{BV}(\Omega;\{0,1\})).
\end{aligned}
$$

- *Define $\Omega_v^+(t) := \{x \in \Omega : \chi_v(x,t) = 1\}$. Then, $\Omega_v^+ = \bigcup_{t\in[0,T]} \Omega_v^+(t)\times\{t\}$ is a smoothly evolving domain in Ω in the sense of Definition 9 with $\Omega_v^+(0) = \Omega_0^+$. Denoting by $I_v(t)$ the closure of $\partial\Omega_v^+(t) \cap \Omega$ in $\overline{\Omega}$ for all $t \in [0,T]$, the set $I_v = \bigcup_{t\in[0,T]} I_v(t)\times\{t\}$ is a smoothly evolving interface with $90°$ contact angle in the sense of Definition 9. In particular, for every $t \in [0,T]$ and every contact point $c(t) \in I_v(t) \cap \partial\Omega$*

$$
n_{\partial\Omega}(c(t)) \cdot n_{I_v}(c(t),t) = 0. \tag{2.33}
$$

Moreover, for every $t \in [0,T]$ and every $c(t) \in I_v(t) \cap \partial\Omega$ the following higher-order compatibility condition is required to hold:

$$
-\Big((n_{\partial\Omega}\cdot\nabla)(n_{I_v}\cdot v)\Big)(c(t),t) = H_{\partial\Omega}(c(t))(n_{I_v}\cdot v)(c(t),t), \tag{2.34}
$$

where $H_{\partial\Omega}$ denotes the scalar mean curvature of $\partial\Omega$ (with respect to the inward pointing unit normal $n_{\partial\Omega}$).

- *The velocity field v has vanishing divergence $\nabla\cdot v = 0$, and it satisfies the boundary conditions*

$$
v(\cdot,t)\cdot n_{\partial\Omega} = 0 \qquad \textit{along } \partial\Omega, \tag{2.35}
$$

$$
\Big(n_{\partial\Omega}\cdot\mu(\nabla v + \nabla v^{\mathsf{T}})(\cdot,t)B\Big) = 0 \qquad \textit{along } \partial\Omega \tag{2.36}
$$

for all $t \in [0,T]$ and all tangential vector fields B along $\partial\Omega$. Moreover, the equation for the momentum balance

$$
\begin{aligned}
&\int_\Omega \rho(\chi_v(\cdot,T'))v(\cdot,T')\cdot\eta(\cdot,T')\,\mathrm{d}x - \int_\Omega \rho(\chi_0))v_0\cdot\eta(\cdot,0)\,\mathrm{d}x\\
&= \int_0^{T'}\int_\Omega \rho(\chi_v)v\cdot\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'}\int_\Omega \rho(\chi_v)v\otimes v : \nabla\eta\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_0^{T'}\int_\Omega \mu(\nabla v + \nabla v^{\mathsf{T}}) : \nabla\eta\,\mathrm{d}x\,\mathrm{d}t + \sigma\int_0^{T'}\int_{I_v(t)} \mathrm{H}_{I_v}\cdot\eta\,\mathrm{d}S\,\mathrm{d}t
\end{aligned} \tag{2.37}
$$

holds true for almost every $T' \in [0,T]$ and every $\eta \in C^\infty(\overline{\Omega}\times[0,T];\mathbb{R}^d)$ such that $\nabla\cdot\eta = 0$ as well as $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega} = 0$. Here, $\mathrm{H}_{I_v}(\cdot,t)$ denotes the mean curvature vector of the interface $I_v(t)$. For the sake of brevity, we have used the abbreviation $\rho(\chi) := \rho^+\chi + \rho^-(1-\chi)$.

- *The indicator function* χ_v *is transported by the fluid velocity* v *in form of*

$$\int_\Omega \chi_v(\cdot,T')\varphi(\cdot,T')\,\mathrm{d}x - \int_\Omega \chi_0\varphi(\cdot,0)\,\mathrm{d}x = \int_0^{T'}\int_\Omega \chi_v(\partial_t\varphi+(v\cdot\nabla)\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{2.38}$$

  for almost every $T'\in[0,T]$ *and all* $\varphi\in C^\infty(\overline{\Omega}\times[0,T])$.

- *It holds* $v\in C^1_tC^0_x(\overline{\Omega\times[0,T]}\setminus I_v)\cap C^0_tC^2_x(\overline{\Omega\times[0,T]}\setminus I_v)$.

Short-time existence of strong solutions in the precise sense of the previous definition may in principle be established based on the results of Wilke [127] resp. Watanabe [125], which in turn are based on a maximal L^p_x-L^p_t resp. L^q_x-L^p_t regularity approach (cf. [113], [104] and [112] for further maximal L^q_x-L^p_t regularity results in the context of two-phase Navier–Stokes flow with surface tension). In these works, the evolving interface is represented in terms of the graph of a time-dependent height function over the initial interface, whereas the evolving phase of one of the fluids is represented in terms of the associated Hanzawa transform.

However, it has to be said that the results of [127] and [125] are not immediately sufficient to guarantee the required higher regularity of the interface and the fluid velocity from Definition 10 (in particular, the regularity up to time $t=0$). One may expect that this higher regularity can be derived along the lines of [45, Remark 7, Remark 36, and Remark 37], where for our purposes next to the higher regularity of the fluid velocity from each side of the evolving interface one also has to provide similar arguments near the domain boundary. Needless to say, one has to be particularly careful in the vicinity of contact points or contact lines, for which our mathematically idealized setting of pure slip and constant ninety degree contact angle may prove beneficial (cf. the discussion in [107] or [50]). In summary, a detailed proof of the required higher regularity is certainly worth a paper on its own and thus out of the scope of this article.

We conclude the discussion on strong solutions with a series of remarks. First, by standard arguments one may deduce from (2.38), the solenoidality of v, and the boundary condition $(v\cdot n_{\partial\Omega})|_{\partial\Omega}=0$ that $V_{I_v}=v\cdot n_{I_v}$ holds true along the interface I_v for the normal speed V_{I_v} of I_v (oriented with respect to n_{I_v}). Second, as a consequence of the contact point condition (2.33) it holds for all $t\in[0,T_s)$

$$\int_{I_v(t)} \mathrm{H}_{I_v}\cdot\eta\,\mathrm{d}S = -\int_{I_v(t)} \Big(\mathrm{Id}-n_{I_v}(\cdot,t)\otimes n_{I_v}(\cdot,t)\Big) : \nabla\eta\,\mathrm{d}S$$

for all test fields $\eta\in C^\infty(\overline{\Omega};\mathbb{R}^d)$ subject to $\nabla\cdot\eta=0$ and $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega}=0$. Third, note that Definition 10 implies that all pairs of two distinct contact points at the initial time remain distinct at all later times within a finite time horizon. This in fact is a consequence of the regularity of the velocity field and the evolving interface. Indeed, denoting by $t\mapsto c(t)\in I_v(t)\cap\partial\Omega$ resp. $t\mapsto\hat{c}(t)\in I_v(t)\cap\partial\Omega$ the trajectories of two distinct contact points, we may estimate the time evolution of their squared distance $\alpha(t):=\frac{1}{2}|c(t)-\hat{c}(t)|^2$ by means of

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) = \Big(c(t)-\hat{c}(t)\Big)\cdot\Big(v(c(t),t)-v(\hat{c}(t),t)\Big) \geq -2\|\nabla v\|_{L^\infty_{x,t}}\alpha(t).$$

Using Gronwall's Lemma, we can conclude that $\alpha(t)\geq\alpha(0)\exp(-2\|\nabla v\|_{L^\infty_{x,t}}t)$.

Fourth, we remark that it actually suffices to require the compatibility conditions (2.33) and (2.34) at the initial time $t=0$ only. For later times $t\in(0,T]$, they are in fact consequences of the regularity of a strong solution, which can be seen as follows. For the sake fo simplicity, consider the case $d = 2$. By means of the chain rule, the fact that $v \cdot n_{\partial\Omega} = 0$ along $\partial\Omega$, and the formulas for $\nabla n_{\partial\Omega}$ and $\nabla \tau_{\partial\Omega}$ from Lemma 19, we may rewrite the boundary condition $(\mu(\nabla v + \nabla v^{\mathsf{T}}) : n_{\partial\Omega} \otimes \tau_{\partial\Omega}) = 0$ along $\partial\Omega$ as

$$H_{\partial\Omega}(v \cdot \tau_{\partial\Omega}) + (n_{\partial\Omega} \cdot \nabla)(v \cdot \tau_{\partial\Omega}) = 0 \quad \text{along } \partial\Omega,$$

which holds in particular at a contact point $c(t)$ for any $t \in [0, T]$. Then, since the quantities $|\tau_{\partial\Omega} \cdot \tau_{I_v}| = |n_{I_v} \cdot n_{\partial\Omega}|$, $|\tau_{\partial\Omega} - n_{I_v}|$, $|n_{\partial\Omega} + \tau_{I_v}|$ evaluated at a contact point can all be bounded from above by $\sqrt{1 - n_{I_v} \cdot \tau_{\partial\Omega}}$, we may compute by adding zeros (see also the formulas for $\nabla n_{\partial\Omega}$ and $\nabla \tau_{\partial\Omega}$ as well as the expressions for $\frac{\mathrm{d}}{\mathrm{d}t}\tau_{\partial\Omega}(c(t))$ and $\frac{\mathrm{d}}{\mathrm{d}t}n_{I_v}(c(t), t)$ from Lemma 19 and Lemma 20, respectively)

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[1 - n_{I_v}(c(t), t) \cdot \tau_{\partial\Omega}(c(t))] \\
&\quad = -\Big((n_{I_v} \cdot n_{\partial\Omega})((n_{\partial\Omega} \cdot \nabla)(v \cdot \tau_{\partial\Omega}) + (\tau_{I_v} \cdot \nabla)(v \cdot n_{I_v}))\Big)\Big|_{(c(t),t)} \\
&\quad = -\Big((n_{I_v} \cdot n_{\partial\Omega})(\nabla v : (\tau_{\partial\Omega} - n_{I_v}) \otimes n_{\partial\Omega} + \nabla v : n_{I_v} \otimes (n_{\partial\Omega} + \tau_{I_v}) \\
&\qquad - H_{I_v}(v \cdot \tau_{I_v})(\tau_{\partial\Omega} \cdot \tau_{I_v}))\Big)\Big|_{(c(t),t)} \\
&\quad \leq C\|\nabla v\|_{L^\infty_{x,t}}[1 - n_{I_v}(c(t), t) \cdot \tau_{\partial\Omega}(c(t))]
\end{aligned}$$

for some $C > 0$ and any $t \in [0, T]$. From an application of a Gronwall-type argument and the validity of the contact angle condition (2.33) at the initial time $t = 0$, we may conclude that (2.33) is indeed satisfied for any $t \in [0, T]$. The compatibility condition (2.34) in turn follows from differentiating in time the angle condition (2.33) along a smooth trajectory $t \mapsto c(t) \in I_v(t) \cap \partial\Omega$ of a contact point, see for details the proof of Lemma 20.

We proceed with the notion of a varifold solution.

Definition 11 (Varifold solution in case of 90° contact angle condition)**.** *Let a surface tension constant $\sigma > 0$, the densities and shear viscosity of the two fluids $\rho^\pm, \mu > 0$, a finite time $T_w > 0$, a solenoidal initial velocity profile $u_0 \in L^2(\Omega; \mathbb{R}^d)$, and an indicator function $\chi_0 \in \mathrm{BV}(\Omega)$ be given.*

A triple (χ_u, u, V) consisting of a velocity field u, an indicator function χ_u, and an oriented varifold V with

$$\begin{aligned}
&u \in L^2([0, T_w]; H^1(\Omega; \mathbb{R}^d)) \cap L^\infty([0, T_w]; L^2(\Omega; \mathbb{R}^d)), \\
&\chi_u \in L^\infty([0, T_w]; \mathrm{BV}(\Omega; \{0, 1\})), \\
&V \in L^\infty_w([0, T_w]; \mathcal{M}(\overline{\Omega} \times \mathbb{S}^{d-1})),
\end{aligned}$$

is called a varifold solution to the free boundary problem for the Navier-Stokes equation for two fluids with 90° contact angle and initial data (χ_0, u_0) *if the following conditions are satisfied:*

- *The velocity field u has vanishing divergence $\nabla \cdot u = 0$, its trace a vanishing normal component on the boundary of the domain $(u \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$, and the equation for the momentum balance*

$$\int_\Omega \rho(\chi_u(\cdot, T))u(\cdot, T) \cdot \eta(\cdot, T)\,\mathrm{d}x - \int_\Omega \rho(\chi_0))u_0 \cdot \eta(\cdot, 0)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^T \int_\Omega \rho(\chi_u) u \cdot \partial_t \eta \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \rho(\chi_u) u \otimes u : \nabla \eta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_\Omega \mu(\nabla u + \nabla u^\mathsf{T}) : \nabla \eta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \sigma \int_0^T \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s \otimes s) : \nabla \eta \,\mathrm{d}V_t(x,s)\,\mathrm{d}t
\end{aligned} \tag{2.39}
$$

is satisfied for almost every $T \in [0, T_w)$ and for every test vector field η subject to $\eta \in C^\infty([0,T_w); C^1(\overline{\Omega};\mathbb{R}^d) \cap \bigcap_{p\geq 2} W^{2,p}(\Omega;\mathbb{R}^d))$, $\nabla \cdot \eta = 0$ as well as $(\eta \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$. We again made use of the abbreviation $\rho(\chi) := \rho^+\chi + \rho^-(1-\chi)$.

- *The indicator χ_u satisfies the weak formulation of the transport equation*

$$
\int_\Omega \chi_u(\cdot,T)\varphi(\cdot,T)\,\mathrm{d}x - \int_\Omega \chi_0 \varphi(\cdot,0)\,\mathrm{d}x = \int_0^T \int_\Omega \chi_u(\partial_t\varphi + (u\cdot\nabla)\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{2.40}
$$

  for almost every $T \in [0,T_w)$ and all $\varphi \in C^\infty(\overline{\Omega}\times[0,T_w))$.

- *The energy dissipation inequality*

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\rho(\chi_u(\cdot,T))|u(\cdot,T)|^2\,\mathrm{d}x + \sigma|V_T|_{\mathbb{S}^{d-1}}(\overline{\Omega}) + \int_0^T\int_\Omega \frac{\mu}{2}|\nabla u + \nabla u^\mathsf{T}|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_\Omega \frac{1}{2}\rho(\chi_0(\cdot))|u_0(\cdot)|^2\,\mathrm{d}x + \sigma|\nabla\chi_0|(\Omega)
\end{aligned} \tag{2.41}
$$

  is satisfied for almost every $T \in [0,T_w)$.

- *The phase boundary $\partial^*\{\chi_u(\cdot,t)=0\}\cap\Omega$ and the varifold V_t satisfy the compatibility condition*

$$
\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} \psi(x)\cdot s\,\mathrm{d}V_t(x,s) = \int_\Omega \psi(x)\cdot\,\mathrm{d}\nabla\chi_u(x,t) \tag{2.42}
$$

  for almost every $t \in [0,T_w)$ and every smooth function $\psi \in C^\infty(\overline{\Omega};\mathbb{R}^d)$ such that $(\psi\cdot n_{\partial\Omega})|_{\partial\Omega} = 0$.

Finally, if (χ_u, V) satisfy (2.14) *we call the pair (χ_u, u) a* BV solution to the free boundary problem for the Navier-Stokes equation for two fluids with $90°$ contact angle and initial data (χ_0, u_0).

We conclude with a remark concerning the notion of varifold solutions. Denote by $V_t \in \mathcal{M}(\overline{\Omega}\times\mathbb{S}^{d-1})$ the non-negative measure representing at time $t \in [0,T_w)$ the varifold associated to a varifold solution (χ_u, u, V). The compatibility condition (2.42) entails that $|\nabla\chi_u(\cdot,t)|\llcorner\Omega$ is absolutely continuous with respect to $|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega$; in fact, $|\nabla\chi_u(\cdot,t)|\llcorner\Omega \leq |V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega$ in the sense of measures on Ω. Hence, we may define the Radon–Nikodym derivative

$$
\theta_t := \frac{\mathrm{d}|\nabla\chi_u(\cdot,t)|\llcorner\Omega}{\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega}, \tag{2.43}
$$

which is a $(|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega)$-measurable function with $|\theta_t| \leq 1$ valid $(|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega)$-almost everywhere in Ω. In other words, the quantity $\frac{1}{\theta_t}$ represents the multiplicity of the varifold (in the interior). With this notation in place, it then holds

$$
\int_\Omega f(x)\,\mathrm{d}|\nabla\chi_u(\cdot,t)|(x) = \int_\Omega \theta_t(x) f(x)\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}(x) \tag{2.44}
$$

for every $f \in L^1(\Omega, |\nabla\chi_u(\cdot,t)|)$ and almost every $t \in [0,T_w)$.

2.2.5 Summary of strategy

To summarize, the proof of our weak-strong uniqueness result (Theorem 1) is divided into two parts. The first part is concerned with the derivation of Gronwall stability estimates (cf. Proposition 4, Proposition 5 and Lemma 6) of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}E[\chi_u, u, V|\chi_v, v] \leq C(E[\chi_u, u, V|\chi_v, v] + E_{\mathrm{vol}}[\chi_u|\chi_v]),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{vol}}[\chi_u|\chi_v] \leq C(E[\chi_u, u, V|\chi_v, v] + E_{\mathrm{vol}}[\chi_u|\chi_v]),$$

where $E[\chi_u, u, V|\chi_v, v]$ and $E_{\mathrm{vol}}[\chi_u|\chi_v]$ are two suitably constructed error functionals between a varifold (cf. Definition 11) and a strong solution (cf. Definition 10). The functional $E[\chi_u, u, V|\chi_v, v]$ has the form of a relative entropy and penalizes, amongst other things, the error in the two velocity fields (cf. (2.29)) and the error in the locations of the two interfaces (cf. (2.18)–(2.25)). The other functional $E_{\mathrm{vol}}[\chi_u|\chi_v]$ in turn directly controls the difference between the phase indicator functions of the respective first fluids of the two solutions (cf. (2.31)). These coercivity properties are not only sufficient to establish the above Gronwall estimates, but also that the two solutions have to coincide if both error functionals are zero.

The argument in the first part of the proof is conditional in the sense that both error functionals rely on suitable inputs which have to be constructed from the strong solution. More precisely, the interfacial part of the relative entropy $E[\chi_u, u, V|\chi_v, v]$ is defined in terms of a suitable extension ξ of the interface unit normal n_{I_v} of the strong solution (cf. Definition 2), whereas $E_{\mathrm{vol}}[\chi_u|\chi_v]$ is defined based on a suitable weight ϑ essentially representing a truncated signed distance function with respect to the phase of the first fluid of the strong solution (cf. Definition 3). Once these two inputs are rigorously constructed, one may infer our main result Theorem 1 from the corresponding conditional one of Proposition 4.

The second part of the proof therefore takes care of establishing the existence of such ξ and ϑ for strong solutions (cf. Proposition 7 and Lemma 8). In the following, we provide some comments on the construction of the former (which is the more challenging task). Away from the domain boundary, and therefore in particular away from contact points, one may simply follow the ansatz from [45] which is

$$\xi(x,t) := \eta_{I_v}(\mathrm{sdist}(x, I_v(t)))n_{I_v}(P_{I_v}(x,t),t), \tag{2.45}$$

where η_{I_v} is a quadratic cutoff localizing to the width of a regular tubular neighborhood of the interface $I_v(t)$, $\mathrm{sdist}(\cdot, I_v(t))$ denotes the signed distance to $I_v(t)$, and $P_{I_v}(\cdot,t)$ represents the nearest point projection onto $I_v(t)$.

Near contact points $\partial I_v(t)$, the above ansatz (2.45) requires a careful adaptation because one of the main requirements for ξ is to be tangential along the domain boundary: $(\xi \cdot n_{\partial\Omega})|_{\partial\Omega} \equiv 0$. To achieve this, it is first easiest to think about fixing the values of ξ along either the interface I_v or the domain boundary $\partial\Omega$:

$$\xi(x,t) := \eta_{\partial I_v(t)}(x,t)\widetilde{\xi}^{I_v}(x,t), \quad \widetilde{\xi}^{I_v}(x,t) = n_{I_v}(x,t) \qquad \text{along } I_v(t),$$
$$\xi(x,t) := \eta_{\partial I_v(t)}(x,t)\widetilde{\xi}^{\partial\Omega}(x,t), \quad \widetilde{\xi}^{\partial\Omega}(x,t) = \tau_{\partial\Omega}(x,t) \qquad \text{along } \partial\Omega,$$

where $\eta_{\partial I_v(t)}$ is a quadratic cutoff localizing to a neighborhood of the contact points $\partial I_v(t)$, and where $\tau_{\partial\Omega}(\cdot,t)$ is a tangent vector field along $\partial\Omega$ extending locally for each contact point $c \in \partial I_v(t)$ the normal $n_{I_v}(c,t)$. Due to the 90° degree contact angle condition, this is indeed meaningful and guarantees continuity of ξ along $I_v(t) \cup \partial\Omega$.

Now, in order to define ξ in a full neighborhood of the contact points $\partial I_v(t)$, the basic idea is to interpolate between the two auxiliary fields $\widetilde{\xi}^{I_v}$ and $\widetilde{\xi}^{\partial\Omega}$. However, some care has to be taken here due to the required regularity of ξ. This is the reason why we employ an expansion ansatz for both $\widetilde{\xi}^{I_v}$ and $\widetilde{\xi}^{\partial\Omega}$ of the structure

$$\widetilde{\xi}^{I_v} := n_{I_v} + \alpha_{I_v}\,\mathrm{sdist}(\cdot, I_v)\tau_{I_v} - \frac{1}{2}\alpha_{I_v}^2\,\mathrm{sdist}^2(\cdot, I_v)n_{I_v},$$
$$\widetilde{\xi}^{\partial\Omega} := \tau_{\partial\Omega} + \alpha_{\partial\Omega}\,\mathrm{sdist}(\cdot, \partial\Omega)n_{\partial\Omega} - \frac{1}{2}\alpha_{\partial\Omega}^2\,\mathrm{sdist}^2(\cdot, \partial\Omega)\tau_{\partial\Omega},$$

where the normal-tangent frames (n_{I_v}, τ_{I_v}) and $(n_{\partial\Omega}, \tau_{\partial\Omega})$ as well as the coefficients α_{I_v} and $\alpha_{\partial\Omega}$ are extended constantly in the respective normal directions. The point then is to choose the coefficients in a suitable way such that $\nabla\widetilde{\xi}^{I_v}$ and $\nabla\widetilde{\xi}^{\partial\Omega}$ agree at contact points. With this in place, one may then interpolate between the two constructions so that the resulting vector field ξ satisfies the required regularity. The second-order terms in the above expansions are only needed for a length correction of the first-order perturbations. We finally remark that controlling the time evolution of the interpolation construction requires the higher-order compatibility condition at contact points following from differentiating in time the $90°$ contact angle condition.

With the constructions of suitable candidates for ξ in place, one technical problem remains. Namely, the domains of definition for the above two outlined constructions away and near contact points overlap. The solution for this technicality consists of carefully designing the quadratic cutoff functions η_{I_v} and $\eta_{\partial I_v}$ so that they form on one hand a partition of unity along the interface of the strong solution, and that they on the other hand get transported along the fluid flow. Once this is established, the construction of ξ is finished.

In terms of organization, the remaining parts of the chapter are structured as follows. The first part of the proof as outlined above is conducted in Section 2.3. The construction of the vector field ξ, which is the main step of the second part of the proof, is distributed across Section 2.4 (construction away from contact points), Section 2.5 (construction near contact points) and Section 2.6 (global construction by partition of unity). We conclude the paper with the construction of the weight ϑ in Section 2.7.

2.2.6 Notation

Throughout the present work, we employ the notational conventions of [45]. A notable addition is the following convention. If $D \subset \mathbb{R}^d$ is an open subset and $\Gamma \subset D$ a closed subset of Hausdorff-dimension $k \in \{0, \ldots, d-1\}$, we write $C^k(\overline{D} \setminus \Gamma)$ for all maps $f\colon D \to \mathbb{R}$ which are k-times continuously differentiable throughout $D \setminus \Gamma$ such that the function together with all its derivatives stays bounded throughout $D \setminus \Gamma$. Analogously, one defines the space $C_t^k C_x^m(\overline{D} \setminus \Gamma)$ for $D = \bigcup_{t\in[0,T]} D(t) \times \{t\}$ and $\Gamma = \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}$, where $(D(t))_{t\in[0,T]}$ is a family of open subsets of $\mathbb{R}^d$ and $(\Gamma(t))_{t\in[0,T]}$ is a family of closed subsets $\Gamma(t) \subset D(t)$ of constant Hausdorff-dimension $k \in \{0, \ldots, d-1\}$.

2.3 Proof of main results

2.3.1 Relative entropy inequality: Proof of Proposition 5

The general structure of the proof is in parts similar to the proof of [45, Proposition 10]. In what follows, we thus mainly focus on how to exploit the boundary conditions for the velocity fields (u,v) and a boundary adapted extension ξ of the strong interface unit normal in these computations.

Step 1: Since $\rho(\chi_v)$ is an affine function of χ_v, it consequently satisfies

$$\int_\Omega \rho(\chi_v(\cdot,T'))\varphi(\cdot,T')\,\mathrm{d}x - \int_\Omega \rho(\chi_v^0)\varphi(\cdot,0)\,\mathrm{d}x = \int_0^T \int_\Omega \rho(\chi_v)(\partial_t\varphi + (v\cdot\nabla)\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{2.46}$$

for almost every $T' \in [0,T]$ and all $\varphi \in C^\infty(\overline{\Omega}\times[0,T])$. By the regularity of v and an approximation argument, we may test this equation with $v\cdot\eta$ for any $\eta \in C^\infty(\overline{\Omega}\times[0,T];\mathbb{R}^d)$, yielding

$$\begin{aligned} &\int_\Omega \rho(\chi_v(\cdot,T'))v(\cdot,T')\cdot\eta(\cdot,T')\,\mathrm{d}x - \int_\Omega \rho(\chi_v^0)v(\cdot,0)\cdot\eta(\cdot,0)\,\mathrm{d}x \\ &= \int_0^{T'}\int_\Omega \rho(\chi_v)(v\cdot\partial_t\eta + \eta\cdot\partial_t v)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^{T'}\int_\Omega \rho(\chi_v)(\eta\cdot(v\cdot\nabla)v + v\cdot(v\cdot\nabla)\eta)\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{2.47}$$

for almost every $T' \in [0,T]$. Next, we subtract from (2.47) the equation for the momentum balance (2.37) of the strong solution. It follows that the velocity field v of the strong solution satisfies

$$\begin{aligned} 0 &= \int_0^{T'}\int_\Omega \rho(\chi_v)\eta\cdot\partial_t v\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'}\int_\Omega \rho(\chi_v)\eta\cdot(v\cdot\nabla)v\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^{T'}\int_\Omega \mu(\nabla v + \nabla v^T):\nabla\eta\,\mathrm{d}x\,\mathrm{d}t - \sigma\int_0^{T'}\int_{I_v(t)} \mathrm{H}_{I_v}\cdot\eta\,\mathrm{d}S\,\mathrm{d}t \end{aligned} \tag{2.48}$$

for almost every $T' \in [0,T]$ and every test vector field $\eta \in C^\infty(\overline{\Omega}\times[0,T];\mathbb{R}^d)$ such that $\nabla\cdot\eta = 0$ and $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega} = 0$. For any such test vector field η, note that by means of (2.16c), the incompressibility of η as well as $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega} = 0$, we may rewrite

$$\begin{aligned} -\sigma\int_0^{T'}\int_{I_v(t)} \mathrm{H}_{I_v}\cdot\eta\,\mathrm{d}S\,\mathrm{d}t &= \sigma\int_0^{T'}\int_{I_v(t)} (\nabla\cdot\xi)\eta\cdot n_{I_v}\,\mathrm{d}S\,\mathrm{d}t \\ &= -\sigma\int_0^{T'}\int_\Omega \chi_v(\eta\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.49}$$

Hence, we deduce from inserting (2.49) back into (2.48) that

$$\begin{aligned} 0 &= \int_0^{T'}\int_\Omega \rho(\chi_v)\eta\cdot\partial_t v\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'}\int_\Omega \rho(\chi_v)\eta\cdot(v\cdot\nabla)v\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^{T'}\int_\Omega \mu(\nabla v + \nabla v^T):\nabla\eta\,\mathrm{d}x\,\mathrm{d}t - \sigma\int_0^{T'}\int_\Omega \chi_v(\eta\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{2.50}$$

for almost every $T' \in [0,T]$ and every test vector field $\eta \in C^\infty(\overline{\Omega}\times[0,T];\mathbb{R}^d)$ such that $\nabla\cdot\eta = 0$ and $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega} = 0$. The merit of rewriting (2.48) into the form (2.50) consists of the following observation. Consider a test vector field $\eta \in C^\infty([0,T];H^1(\Omega;\mathbb{R}^d))$ such that $\nabla\cdot\eta = 0$ and $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega} = 0$. Denoting by ψ a standard mollifier, for every $k \in \mathbb{N}$ by $\psi_k := k^d\psi(k\cdot)$ its usual rescaling, and by P_Ω the Helmholtz projection associated with the smooth domain Ω, it follows from standard theory (e.g., by a combination of [116] and standard $W^{m,2}(\Omega)$-elliptic regularity theory – see also Section 2.8) that $\eta_k := P_\Omega(\psi_k * \eta)$ is an admissible test vector field for (2.50). Moreover, taking the limit $k \to \infty$ in (2.50) with η_k as test vector fields is admissible and results in

$$\begin{aligned} 0 = &\int_0^{T'} \int_\Omega \rho(\chi_v)\eta \cdot \partial_t v \,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'} \int_\Omega \rho(\chi_v)\eta \cdot (v \cdot \nabla)v \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^{T'} \int_\Omega \mu(\nabla v + \nabla v^T) : \nabla\eta \,\mathrm{d}x\,\mathrm{d}t - \sigma \int_0^{T'} \int_\Omega \chi_v(\eta \cdot \nabla)(\nabla \cdot \xi) \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{2.51}$$

for almost every $T' \in [0, T]$ and every test vector field $\eta \in C^\infty([0, T]; H^1(\Omega; \mathbb{R}^d))$ such that $\nabla \cdot \eta = 0$ and $(\eta \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$. As an important consequence, because of the boundary condition for the velocity fields (u, v) and their solenoidality, we may choose (after performing a mollification argument in the time variable) $\eta = u - v$ as a test function in (2.51) which entails for almost every $T' \in [0, T]$

$$\begin{aligned} 0 = &\int_0^{T'} \int_\Omega \rho(\chi_v)(u - v) \cdot \partial_t v \,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'} \int_\Omega \rho(\chi_v)(u - v) \cdot (v \cdot \nabla)v \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^{T'} \int_\Omega \mu(\nabla v + \nabla v^\mathsf{T}) : \nabla(u - v) \,\mathrm{d}x\,\mathrm{d}t - \sigma \int_0^{T'} \int_\Omega \chi_v((u - v) \cdot \nabla)(\nabla \cdot \xi) \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.52}$$

We proceed by testing the analogue of (2.46) for the phase-dependent density $\rho(\chi_u)$ with the test function $\frac{1}{2}|v|^2$, obtaining for almost every $T' \in [0, T]$

$$\begin{aligned} &\int_\Omega \frac{1}{2}\rho(\chi_u(\cdot, T'))|v(\cdot, T')|^2 \,\mathrm{d}x - \int_\Omega \frac{1}{2}\rho(\chi_u^0)|v_0(\cdot)|^2 \,\mathrm{d}x \\ &= \int_0^{T'} \int_\Omega \rho(\chi_u)v \cdot \partial_t v \,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'} \int_\Omega \rho(\chi_u)v \cdot (u \cdot \nabla)v \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.53}$$

We next want to test (2.39) with the fluid velocity v. Modulo a mollification argument in the time variable, we have to argue that ∇v does not jump across the interface so that v is an admissible test function. Indeed, since the tangential derivative $(\tau_{I_v} \cdot \nabla)v$ is continuous across the interface it follows from $\nabla \cdot v = 0$ that also $n_{I_v} \cdot (n_{I_v} \cdot \nabla)v$ does not jump across I_v. The only component which may jump is thus $\tau_{I_v} \cdot (n_{I_v} \cdot \nabla)v$. However, this is ruled out by the equilibrium condition for the stresses along I_v together with having $\mu_+ = \mu_-$. In summary, using v in (2.39) implies

$$\begin{aligned} &- \int_\Omega \rho(\chi_u(\cdot, T'))u(\cdot, T') \cdot v(\cdot, T') \,\mathrm{d}x + \int_\Omega \rho(\chi_u^0))u_0 \cdot v_0(\cdot) \,\mathrm{d}x \\ &- \int_0^{T'} \int_\Omega \mu(\nabla u + \nabla u^\mathsf{T}) : \nabla v \,\mathrm{d}x\,\mathrm{d}t \\ &= - \int_0^{T'} \int_\Omega \rho(\chi_u)u \cdot \partial_t v \,\mathrm{d}x\,\mathrm{d}t - \int_0^{T'} \int_\Omega \rho(\chi_u)u \cdot (u \cdot \nabla)v \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \sigma \int_0^{T'} \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s \otimes s) : \nabla v \,\mathrm{d}V_t(x, s) \,\mathrm{d}t \end{aligned} \tag{2.54}$$

for almost every $T' \in [0, T]$. We finally use $\sigma(\nabla \cdot \xi)$ as a test function in the transport equation (2.40) for the indicator function χ_u of the varifold solution. Hence, we obtain

$$\sigma \int_\Omega \chi_u(\cdot, T')(\nabla \cdot \xi)(\cdot, T') \,\mathrm{d}x - \int_\Omega \chi_u^0(\nabla \cdot \xi)(\cdot, 0) \,\mathrm{d}x$$

$$= \sigma \int_0^{T'} \int_\Omega \chi_u(\nabla \cdot \partial_t \xi + (u \cdot \nabla)(\nabla \cdot \xi)) \,\mathrm{d}x \,\mathrm{d}t.$$

for almost every $T' \in [0,T]$. Based on the boundary condition (2.16b), which in turn in particular implies $(\partial_t \xi \cdot n_{\partial\Omega})|_{\partial\Omega} = \partial_t(\xi \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$, we may integrate by parts to upgrade the previous display to

$$\begin{aligned}
&- \sigma \int_\Omega n_u(\cdot,T') \cdot \xi(\cdot,T') \,\mathrm{d}|\nabla\chi_u(\cdot,T)| + \int_\Omega n_u^0 \cdot \xi(\cdot,0) \,\mathrm{d}|\nabla\chi_u(\cdot,0)| \\
&= -\sigma \int_0^{T'} \int_\Omega n_u \cdot \partial_t \xi \,\mathrm{d}|\nabla\chi_u| \,\mathrm{d}t + \sigma \int_0^{T'} \int_\Omega \chi_u (u \cdot \nabla)(\nabla \cdot \xi) \,\mathrm{d}x \,\mathrm{d}t
\end{aligned} \tag{2.55}$$

for almost every $T' \in [0,T]$.

Step 2: Summing (2.52), (2.53), (2.41) as well as (2.54), we obtain

$$\begin{aligned}
&LHS_{kin}(T') + LHS_{visc} + LHS_{surEn}(T') \\
&\leq RHS_{kin}(0) + RHS_{surEn}(0) + RHS_{dt} + RHS_{adv} + RHS_{surTen},
\end{aligned} \tag{2.56}$$

where the individual terms are given by (cf. the proof of [45, Proposition 10])

$$LHS_{kin}(T') := \int_\Omega \frac{1}{2}\rho(\chi_u(\cdot,T'))|u-v|^2(\cdot,T') \,\mathrm{d}x, \tag{2.57}$$

$$RHS_{kin}(0) := \int_\Omega \frac{1}{2}\rho(\chi_u^0)|u_0 - v_0|^2 \,\mathrm{d}x, \tag{2.58}$$

$$LHS_{surEn}(T') := \sigma|\nabla\chi_u(\cdot,T')|(\Omega) + \sigma \int_\Omega (1-\theta_{T'}) \,\mathrm{d}|V_{T'}|_{\mathbb{S}^{d-1}}(x), \tag{2.59}$$

$$RHS_{surEn}(0) := \sigma|\nabla\chi_u^0(\cdot)|(\Omega), \tag{2.60}$$

$$LHS_{visc} := \int_0^{T'} \int_\Omega \frac{\mu}{2}|\nabla(u-v) + \nabla(u-v)^\mathsf{T}|^2 \,\mathrm{d}x \,\mathrm{d}t, \tag{2.61}$$

$$RHS_{dt} := -\int_0^{T'} \int_\Omega (\rho(\chi_v) - \rho(\chi_u))(u-v) \cdot \partial_t v \,\mathrm{d}x \,\mathrm{d}t, \tag{2.62}$$

$$\begin{aligned}
RHS_{adv} := &-\int_0^{T'} \int_\Omega (\rho(\chi_u) - \rho(\chi_v))(u-v) \cdot (v \cdot \nabla)v \,\mathrm{d}x \,\mathrm{d}t \\
&-\int_0^{T'} \int_\Omega \rho(\chi_u)(u-v) \cdot ((u-v) \cdot \nabla)v \,\mathrm{d}x \,\mathrm{d}t,
\end{aligned} \tag{2.63}$$

$$\begin{aligned}
RHS_{surTen} := &-\sigma \int_0^{T'} \int_\Omega \chi_v((u-v) \cdot \nabla)(\nabla \cdot \xi) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \sigma \int_0^{T'} \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s \otimes s) : \nabla v \,\mathrm{d}V_t(x,s) \,\mathrm{d}t.
\end{aligned} \tag{2.64}$$

Adding zeros, $\nabla \cdot v = 0$, the boundary condition $n_{\partial\Omega} \cdot (\nabla v + (\nabla v)^\mathsf{T})\xi = n_{\partial\Omega} \cdot (\nabla v + (\nabla v)^\mathsf{T})(\mathrm{Id} - n_{\partial\Omega} \otimes n_{\partial\Omega})\xi = 0$ due to (2.36) and (2.16b), and the compatibility condition (2.42) allow to rewrite the second term of (2.64) as follows

$$\begin{aligned}
&\sigma \int_0^{T'} \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s \otimes s) : \nabla v \,\mathrm{d}V_t(x,s) \,\mathrm{d}t \\
&= - \sigma \int_0^{T'} \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (s-\xi) \cdot ((s-\xi) \cdot \nabla)v \,\mathrm{d}V_t(x,s) \,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&-\sigma\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} s\cdot(\nabla v+(\nabla v)^{\mathsf{T}})\xi\,\mathrm{d}V_t(x,s)\,\mathrm{d}t\\
&+\sigma\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} \xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}V_t(x,s)\,\mathrm{d}t\\
=&-\sigma\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (s-\xi)\cdot((s-\xi)\cdot\nabla)v\,\mathrm{d}V_t(x,s)\,\mathrm{d}t \qquad (2.65)\\
&-\sigma\int_0^{T'}\int_{\Omega} \xi\cdot(n_u\cdot\nabla)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t-\sigma\int_0^{T'}\int_{\Omega} n_u\cdot(\xi\cdot\nabla)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&+\sigma\int_0^{T'}\int_{\overline{\Omega}} \xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t.
\end{aligned}
$$

Furthermore, because of (2.44) we obtain

$$
\begin{aligned}
&\sigma\int_0^{T'}\int_{\overline{\Omega}} \xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \qquad (2.66)\\
&=\sigma\int_0^{T'}\int_{\Omega}(1-\theta_t)\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t+\sigma\int_0^{T'}\int_{\Omega}\theta_t\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t\\
&\quad+\sigma\int_0^{T'}\int_{\partial\Omega}\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t\\
&=\sigma\int_0^{T'}\int_{\Omega}(1-\theta_t)\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t+\sigma\int_0^{T'}\int_{\Omega}\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&\quad+\sigma\int_0^{T'}\int_{\partial\Omega}\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t.
\end{aligned}
$$

The combination of (2.64), (2.65) and (2.66) together with $\nabla\cdot v=0$ then implies

$$
\begin{aligned}
RHS_{surTen}=&-\sigma\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}}(s-\xi)\cdot((s-\xi)\cdot\nabla)v\,\mathrm{d}V_t(x,s)\,\mathrm{d}t \qquad (2.67)\\
&+\sigma\int_0^{T'}\int_{\Omega}(1-\theta_t)\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t\\
&+\sigma\int_0^{T'}\int_{\partial\Omega}\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t\\
&-\sigma\int_0^{T'}\int_{\Omega}\chi_v((u-v)\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t\\
&-\sigma\int_0^{T'}\int_{\Omega}\xi\cdot((n_u-\xi)\cdot\nabla)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&-\sigma\int_0^{T'}\int_{\Omega}(n_u-\xi)\cdot(\xi\cdot\nabla)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&+\sigma\int_0^{T'}\int_{\Omega}(\mathrm{Id}-\xi\otimes\xi):\nabla v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t.
\end{aligned}
$$

In summary, plugging back (2.57)–(2.63) and (2.67) into (2.56), and then summing (2.55) to the resulting inequality yields in view of the definition (2.29) of the relative entropy

$$
E[\chi_u,u,V|\chi_v,v](T')+\int_0^{T'}\int_{\Omega}\frac{\mu}{2}|\nabla(u-v)+\nabla(u-v)^{\mathsf{T}}|^2\,\mathrm{d}x\,\mathrm{d}t
$$

$$\leq E[\chi_u, u, V|\chi_v, v](0) + R_{dt} + R_{adv} + R^{(1)}_{surTen} + R^{(2)}_{surTen} \tag{2.68}$$

for almost every $T' \in [0,T]$, where in addition to the notation of Proposition 5 we also defined the two auxiliary quantities

$$\begin{aligned} R^{(1)}_{surTen} := &-\sigma \int_0^{T'} \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (s-\xi)\cdot((s-\xi)\cdot\nabla)v \,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\ &+\sigma \int_0^{T'} \int_{\Omega} (1-\theta_t)\xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\ &+\sigma \int_0^{T'} \int_{\partial\Omega} \xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t, \end{aligned} \tag{2.69}$$

$$\begin{aligned} R^{(2)}_{surTen} := &\,\sigma \int_0^{T} \int_{\Omega} \chi_u(u\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t \\ &-\sigma \int_0^{T'} \int_{\Omega} \chi_v((u-v)\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t \\ &-\sigma \int_0^{T'} \int_{\Omega} \xi\cdot((n_u-\xi)\cdot\nabla)v \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &-\sigma \int_0^{T'} \int_{\Omega} (n_u-\xi)\cdot(\xi\cdot\nabla)v \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &+\sigma \int_0^{T'} \int_{\Omega} (\mathrm{Id}-\xi\otimes\xi):\nabla v \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &-\sigma \int_0^{T'} \int_{\Omega} n_u\cdot\partial_t\xi \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t. \end{aligned} \tag{2.70}$$

The remainder of the proof is concerned with the post-processing of the term $R^{(2)}_{surTen}$.

Step 3: By adding zeros, we can rewrite the last right hand side term of (2.70) as

$$\begin{aligned} &-\sigma \int_0^{T'} \int_{\Omega} n_u\cdot\partial_t\xi \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &= -\sigma \int_0^{T'} \int_{\Omega} (n_u-\xi)\cdot(\partial_t\xi+(v\cdot\nabla)\xi+(\mathrm{Id}-\xi\otimes\xi)(\nabla v)^{\mathsf{T}}\xi)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &\quad -\sigma \int_0^{T'} \int_{\Omega} ((n_u-\xi)\cdot\xi)(\xi\otimes\xi:\nabla v)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &\quad -\sigma \int_0^{T'} \int_{\Omega} \Big(\partial_t\frac{1}{2}|\xi|^2+(v\cdot\nabla)\frac{1}{2}|\xi|^2\Big)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &\quad +\sigma \int_0^{T'} \int_{\Omega} \xi\otimes(n_u-\xi):\nabla v \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &\quad +\sigma \int_0^{T'} \int_{\Omega} n_u\cdot((v\cdot\nabla)\xi)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t. \end{aligned} \tag{2.71}$$

We proceed by manipulating the last term in the latter identity. To this end, we compute applying the product rule in the first step and then adding zero

$$\sigma \int_0^{T'} \int_{\Omega} n_u\cdot((v\cdot\nabla)\xi)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t$$

$$= \sigma \int_0^{T'} \int_\Omega n_u \cdot (\nabla \cdot (\xi \otimes v))\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \qquad (2.72)$$
$$+ \sigma \int_0^{T'} \int_\Omega (1 - n_u \cdot \xi)(\nabla \cdot v)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t - \sigma \int_0^{T'} \int_\Omega \mathrm{Id} : \nabla v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t.$$

Noting that for symmetry reasons $\nabla \cdot (\nabla \cdot (\xi \otimes v)) = \nabla \cdot (\nabla \cdot (v \otimes \xi))$, an integration by parts based on the boundary conditions (2.16b) and $(v \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$ entails

$$\begin{aligned}
&\sigma \int_0^{T'} \int_\Omega n_u \cdot (\nabla \cdot (\xi \otimes v))\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\
&= -\sigma \int_0^{T'} \int_\Omega \chi_u \nabla \cdot (\nabla \cdot (v \otimes \xi))\,\mathrm{d}x\,\mathrm{d}t - \sigma \int_0^{T'} \int_{\partial\Omega} \chi_u (n_{\partial\Omega} \otimes v : \nabla\xi)\,\mathrm{d}S\,\mathrm{d}t \\
&= \sigma \int_0^{T'} \int_\Omega n_u \cdot (\nabla \cdot (v \otimes \xi))\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\
&\quad + \sigma \int_0^{T'} \int_{\partial\Omega} \chi_u (n_{\partial\Omega} \cdot ((\xi \cdot \nabla)v - (v \cdot \nabla)\xi))\,\mathrm{d}S\,\mathrm{d}t.
\end{aligned}$$

We next observe that the last right hand side term of the previous display is zero. Indeed, note first that thanks to the boundary conditions (2.16b) and $(v \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$ the involved gradients are in fact tangential gradients along $\partial\Omega$. Since the tangential gradient of a function only depends on its definition along the manifold, we are free to substitute $(\xi \cdot \tau_{\partial\Omega})\tau_{\partial\Omega}$ for ξ resp. $(v \cdot \tau_{\partial\Omega})\tau_{\partial\Omega}$ for v, obtaining in the process

$$\begin{aligned}
&\int_0^{T'} \int_{\partial\Omega} \chi_u (n_{\partial\Omega} \cdot ((\xi \cdot \nabla)v - (v \cdot \nabla)\xi))\,\mathrm{d}S\,\mathrm{d}t \\
&= \int_0^{T'} \int_{\partial\Omega} \chi_u [(\xi \cdot \nabla)(v \cdot \tau_{\partial\Omega}) - (v \cdot \nabla)(\xi \cdot \tau_{\partial\Omega})](\tau_{\partial\Omega} \cdot n_{\partial\Omega})\,\mathrm{d}S\,\mathrm{d}t \\
&\quad + \int_0^{T'} \int_{\partial\Omega} \chi_u [((v \cdot \tau_{\partial\Omega})\xi - (\xi \cdot \tau_{\partial\Omega})v) \cdot \nabla)\tau_{\partial\Omega}] \cdot n_{\partial\Omega}\,\mathrm{d}S\,\mathrm{d}t = 0.
\end{aligned}$$

The combination of the previous two displays together with an integration by parts and an application of the product rule thus yields

$$\begin{aligned}
&\sigma \int_0^{T'} \int_\Omega n_u \cdot (\nabla \cdot (\xi \otimes v))\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\
&= \sigma \int_0^{T'} \int_\Omega (n_u \cdot v)(\nabla \cdot \xi)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t + \sigma \int_0^{T'} \int_\Omega n_u \otimes \xi : \nabla v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t.
\end{aligned}$$

By another integration by parts, relying in the process also on $\nabla \cdot v = 0$ and $(v \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$, we may proceed computing

$$\begin{aligned}
&\sigma \int_0^{T'} \int_\Omega n_u \cdot (\nabla \cdot (\xi \otimes v))\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\
&= -\sigma \int_0^{T'} \int_\Omega \chi_u \nabla \cdot (v(\nabla \cdot \xi))\,\mathrm{d}x\,\mathrm{d}t + \sigma \int_0^{T'} \int_\Omega n_u \otimes \xi : \nabla v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\
&= -\sigma \int_0^{T'} \int_\Omega \chi_u (v \cdot \nabla)(\nabla \cdot \xi)\,\mathrm{d}x\,\mathrm{d}t + \sigma \int_0^{T'} \int_\Omega n_u \otimes \xi : \nabla v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t. \qquad (2.73)
\end{aligned}$$

In summary, taking together (2.71)–(2.73) and adding for a last time zero yields

$$
\begin{aligned}
&-\sigma \int_0^{T'} \int_\Omega n_u \cdot \partial_t \xi \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&= -\sigma \int_0^{T'} \int_\Omega \chi_u (v \cdot \nabla)(\nabla \cdot \xi) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad - \sigma \int_0^{T'} \int_\Omega (n_u - \xi) \cdot (\partial_t \xi + (v \cdot \nabla)\xi + (\mathrm{Id} - \xi \otimes \xi)(\nabla v)^\mathsf{T} \xi) \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad - \sigma \int_0^{T'} \int_\Omega ((n_u - \xi) \cdot \xi)(\xi \otimes \xi : \nabla v) \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad - \sigma \int_0^{T'} \int_\Omega \Big(\partial_t \frac{1}{2}|\xi|^2 + (v \cdot \nabla)\frac{1}{2}|\xi|^2 \Big) \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad + \sigma \int_0^{T'} \int_\Omega (1 - n_u \cdot \xi)(\nabla \cdot v) \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad + \sigma \int_0^{T'} \int_\Omega (n_u - \xi) \otimes \xi : \nabla v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t + \sigma \int_0^{T'} \int_\Omega \xi \otimes (n_u - \xi) : \nabla v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad - \sigma \int_0^{T'} \int_\Omega (\mathrm{Id} - \xi \otimes \xi) : \nabla v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t.
\end{aligned} \tag{2.74}
$$

Inserting (2.74) into (2.70) then implies that $R^{(1)}_{surTen} + R^{(2)}_{surTen}$ combines to the desired term R_{surTen}. In particular, the estimate (2.68) upgrades to (2.30) as asserted. □

2.3.2 Time evolution of the bulk error: Proof of Lemma 6

Note that the sign conditions for the transported weight ϑ, see Definition 3, ensure that

$$
E_{\mathrm{vol}}[\chi_u|\chi_v](t) = \int_\Omega \Big(\chi_u(\cdot, t) - \chi_v(\cdot, t) \Big) \vartheta(\cdot, t) \,\mathrm{d}x
$$

for all $t \in [0, T]$. Hence, as a consequence of the transport equations for χ_v and χ_u (see Definition 10 and Definition 11, respectively) one obtains

$$
\begin{aligned}
E_{\mathrm{vol}}[\chi_u|\chi_v](T') = E_{\mathrm{vol}}[\chi_u|\chi_v](0)& \\
+ \int_0^{T'} \int_\Omega (\chi_u - \chi_v) \partial_t \vartheta \,\mathrm{d}x \,\mathrm{d}t &+ \int_0^{T'} \int_\Omega (\chi_u u - \chi_v v) \cdot \nabla \vartheta \,\mathrm{d}x \,\mathrm{d}t
\end{aligned} \tag{2.75}
$$

for almost every $T' \in [0, T]$. Note that for any sufficiently regular solenoidal vector field F with $(F \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$, since $\vartheta = 0$ along I_v (see Definition 3), an integration by parts yields

$$
\int_\Omega \chi_v (F \cdot \nabla) \vartheta \,\mathrm{d}x = 0. \tag{2.76}
$$

Adding zero in (2.75) and making use of (2.76) with respect to the choices $F = u$ and $F = v$ in form of $\int_\Omega \chi_v \big((u - v) \cdot \nabla \big) \vartheta \,\mathrm{d}x = 0$ then updates (2.75) to (2.32). This concludes the proof of Lemma 6. □

2.3.3 Conditional weak-strong uniqueness: Proof of Proposition 4

Starting point for a proof of the conditional weak-strong uniqueness principle is the following important coercivity estimate (cf. [45, Lemma 20]).

Lemma 12. *Let the assumptions and notation of Proposition 4 be in place. Then there exists a constant $C = C(\chi_v, v, T) > 0$ such that for all $\delta \in (0,1]$ it holds*

$$\begin{aligned}\int_0^{T'}\int_\Omega |\chi_v - \chi_u||u-v| \,\mathrm{d}x\,\mathrm{d}t \leq \frac{C}{\delta}\int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) + E_{\mathrm{vol}}[\chi_u|\chi_v](t)\,\mathrm{d}t \\ + \delta \int_0^{T'}\int_\Omega |\nabla u - \nabla v|^2 \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{2.77}$$

for all $T' \in [0,T]$.

Proof. It turns out to be convenient to introduce a decomposition of the interface I_v into its topological features: the connected components of $I_v \cap \Omega$ and the connected components of $I_v \cap \partial\Omega$. Let $N \in \mathbb{N}$ denote the total number of such topological features of I_v, and split $\{1, \ldots, N\} =: \mathcal{I} \cup \mathcal{C}$ as follows. The subset $\mathcal{I}$ enumerates the space-time connected components of $I_v \cap \Omega$ (being time-evolving connected *interfaces*), whereas the subset $\mathcal{C}$ enumerates the space-time connected components of $I_v \cap \partial\Omega$ (being time-evolving *contact points* if $d = 2$, or time-evolving connected *contact lines* if $d = 3$). If $i \in \mathcal{I}$, we let $\mathcal{T}_i$ denote the space-time trajectory in Ω of the corresponding connected interface. Furthermore, for every $c \in \mathcal{C}$ we write $\mathcal{T}_c$ representing the space-time trajectory in $\partial\Omega$ of the corresponding contact point (if $d = 2$) or line (if $d = 3$). Finally, let us write $i \sim c$ for $i \in \mathcal{I}$ and $c \in \mathcal{C}$ if and only if $\mathcal{T}_i$ ends at $\mathcal{T}_c$. With this language and notation in place, the proof is now split into five steps.

Step 1: (Choice of a suitable localization scale) Denote by $n_{\partial\Omega}$ the unit normal vector field of $\partial\Omega$ pointing into Ω, and by $n_{I_v}(\cdot, t)$ the unit normal vector field of $I_v(t)$ pointing into $\Omega_v(t)$. Because of the uniform C^2_x regularity of the boundary $\partial\Omega$ and the uniform $C_tC^2_x$ regularity of the interface $I_v(t)$, $t \in [0,T]$, we may choose a scale $r \in (0, \frac{1}{2}]$ such that for all $t \in [0,T]$ and all $i \in \mathcal{I}$ the maps

$$\Psi_{\partial\Omega} \colon \partial\Omega \times (-3r, 3r) \to \mathbb{R}^d, \quad (x,y) \mapsto x + y n_{\partial\Omega}(x), \tag{2.78}$$

$$\Psi_{\mathcal{T}_i(t)} \colon \mathcal{T}_i(t) \times (-3r, 3r) \to \mathbb{R}^d, \quad (x,y) \mapsto x + y n_{I_v}(x,t) \tag{2.79}$$

are C^1 diffeomorphisms onto their image. By uniform regularity of $\partial\Omega$ and I_v (the latter in space-time), we have bounds

$$\sup_{\partial\Omega\times[-r,r]} |\nabla\Psi_{\partial\Omega}| \leq C, \qquad \sup_{\Psi_{\partial\Omega}(\partial\Omega\times[-r,r])} |\nabla\Psi^{-1}_{\partial\Omega}| \leq C, \tag{2.80}$$

$$\sup_{t\in[0,T]}\sup_{\mathcal{T}_i(t)\times[-r,r]} |\nabla\Psi_{\mathcal{T}_i(t)}| \leq C, \quad \sup_{t\in[0,T]}\sup_{\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r])} |\nabla\Psi^{-1}_{\mathcal{T}_i(t)}| \leq C \tag{2.81}$$

for all $i \in \mathcal{I}$. By possibly choosing $r \in (0, \frac{1}{2}]$ even smaller, we may also guarantee that for all $t \in [0,T]$ and all $i \in \mathcal{I}$ it holds

$$\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r]) \cap \Psi_{\mathcal{T}_{i'}(t)}(\mathcal{T}_{i'}(t)\times[-r,r]) = \emptyset \text{ for all } i' \in \mathcal{I},\, i' \neq i, \tag{2.82}$$

$$\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r]) \cap \Psi_{\partial\Omega}(\partial\Omega\times[-r,r]) \neq \emptyset \Leftrightarrow \exists c \in \mathcal{C} \colon i \sim c, \tag{2.83}$$

$$\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r]) \cap \Psi_{\partial\Omega}(\partial\Omega\times[-r,r]) \subset B_{2r}(\mathcal{T}_c(t)) \text{ if } \exists c \in \mathcal{C} \colon i \sim c \tag{2.84}$$

$$B_{2r}(\mathcal{T}_c(t)) \cap B_{2r}(\mathcal{T}_{c'}(t)) = \emptyset \text{ for all } c, c' \in \mathcal{C},\ c' \neq c. \tag{2.85}$$

Note finally that because of the $90°$ contact angle condition and by possibly choosing $r \in (0, \frac{1}{2}]$ even smaller, we can furthermore ensure that

$$\begin{aligned} &\Omega \setminus \Big(\Psi_{\partial\Omega}(\partial\Omega \times [-r,r]) \cup \bigcup_{i\in\mathcal{I}} \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t) \times [-r,r])\Big) \\ &\subset \Omega \cap \{x \in \mathbb{R}^d \colon \operatorname{dist}(x,\partial\Omega) \wedge \operatorname{dist}(x, I_v(t)) > r\} \end{aligned} \tag{2.86}$$

for all $t \in [0,T]$. Indeed, for $x \in \Omega \setminus \big(\Psi_{\partial\Omega}(\partial\Omega \times [-r,r]) \cup \bigcup_{i\in\mathcal{I}} \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t) \times [-r,r])\big)$ it follows that $\operatorname{dist}(x,\partial\Omega) > r$. In case the interface $I_v(t)$ intersects $\partial\Omega$ it may not be immediately clear that also $\operatorname{dist}(x, I_v(t)) > r$ holds true. Assume there exists a point $x \in \Omega \setminus \big(\Psi_{\partial\Omega}(\partial\Omega \times [-r,r]) \cup \bigcup_{i\in\mathcal{I}} \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t) \times [-r,r])\big)$ such that $\operatorname{dist}(x, I_v(t)) \leq r$. Then necessarily $x \in (\Omega \cap B_r(c(t))) \setminus \bigcup_{i\in\mathcal{I}} \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t) \times [-r,r])$ for some boundary point $c(t) \in \partial\Omega \cap I_v(t)$. Hence, because of the uniform C^2_x regularity of $\partial\Omega$ and $I_v(t)$ intersecting $\partial\Omega$ at an angle of $90°$, one may choose $r \in (0, \frac{1}{2}]$ small enough such that $x \in (\Omega \cap B_r(c(t)))$ implies $\operatorname{dist}(x,\partial\Omega) \leq r$. As we have already seen, this contradicts $x \in \Omega \setminus \Psi_{\partial\Omega}(\partial\Omega \times [-r,r])$.

Step 2: (A reduction argument) We may estimate by a union bound and (2.86)

$$\begin{aligned} &\int_0^{T'} \int_\Omega |\chi_v - \chi_u||u-v| \,\mathrm{d}x\,\mathrm{d}t \\ &\leq \int_0^{T'} \int_{\Omega \cap \Psi_{\partial\Omega}(\partial\Omega \times [-r,r]) \setminus \bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))} |\chi_v - \chi_u||u-v| \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \sum_{i\in\mathcal{I}} \int_0^{T'} \int_{\Omega \cap \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t) \times [-r,r]) \setminus \bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))} |\chi_v - \chi_u||u-v| \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + C \sum_{c\in\mathcal{C}} \int_0^{T'} \int_{\Omega \cap B_{2r}(\mathcal{T}_c(t))} |\chi_v - \chi_u||u-v| \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^{T'} \int_{\Omega \cap \{\operatorname{dist}(\cdot,\partial\Omega) \wedge \operatorname{dist}(\cdot, I_v(t)) > r\}} |\chi_v - \chi_u||u-v| \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.87}$$

An application of Hölder's inequality and Young's inequality, the definition (2.29) of the relative entropy functional, the coercivity estimate (2.27) for the transported weight, and the definition (2.31) of the bulk error functional further imply

$$\begin{aligned} &\int_0^{T'} \int_{\Omega \cap \{\operatorname{dist}(\cdot,\partial\Omega) \wedge \operatorname{dist}(\cdot, I_v(t)) > r\}} |\chi_v - \chi_u||u-v| \,\mathrm{d}x\,\mathrm{d}t \\ &\leq C \int_0^{T'} \int_{\Omega \cap \{\operatorname{dist}(\cdot,\partial\Omega) \wedge \operatorname{dist}(\cdot, I_v(t)) > r\}} |\chi_v - \chi_u| \,\mathrm{d}x\,\mathrm{d}t + C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t \\ &\leq C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) + E_{\mathrm{vol}}[\chi_u, \chi_v](t) \,\mathrm{d}t. \end{aligned}$$

Hence, it remains to estimate the first three terms on the right hand side of (2.87).

Step 3: (Estimate near the interface but away from contact points) First of all, because of the localization properties (2.82)–(2.84) it holds for all $i \in \mathcal{I}$

$$\operatorname{dist}(\cdot, \mathcal{T}_i) = \operatorname{dist}(\cdot, \partial\Omega) \wedge \operatorname{dist}(\cdot, I_v(t)) \tag{2.88}$$

in $\Omega \cap \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r]) \setminus \bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))$. Hence, the local interface error height as measured in the direction of n_{I_v} on $\mathcal{T}_i$

$$h_{\mathcal{T}_i}(x,t) := \int_{-r}^{r} |\chi_u - \chi_v|(\Psi_{\mathcal{T}_i(t)}(x,y),t)\,\mathrm{d}y, \quad x \in \mathcal{T}_i(t),\, t\in[0,T],$$

is, because of (2.88) and the coercivity estimate (2.27) of the transported weight ϑ, subject to the estimate

$$\begin{aligned} h_{\mathcal{T}_i}^2(x,t) &\leq C\int_{-r}^{r} |\chi_u - \chi_v|(\Psi_{\mathcal{T}_i(t)}(x,y),t)y\,\mathrm{d}y \\ &\leq C\int_{-r}^{r} |\chi_u - \chi_v|(\Psi_{\mathcal{T}_i(t)}(x,y),t)|\vartheta|(\Psi_{\mathcal{T}_i(t)}(x,y),t)\,\mathrm{d}y \end{aligned} \tag{2.89}$$

for all $x \in \mathcal{T}_i(t) \setminus \bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))$, all $t\in[0,T]$ and all $i\in\mathcal{I}$. Carrying out the slicing argument of the proof of [45, Lemma 20] in $\Omega \cap \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r]) \setminus \bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))$ by means of $\Psi_{\mathcal{T}_i(t)}$, which is indeed admissible thanks to (2.79), (2.81) and (2.89), shows that one obtains an estimate of required form

$$\begin{aligned} &\sum_{i\in\mathcal{I}}\int_0^{T'}\int_{\Omega\cap\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r])\setminus\bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))} |\chi_v - \chi_u||u-v|\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{C}{\delta}\int_0^{T'} E[\chi_u,u,V|\chi_v,v](t) + E_{\mathrm{vol}}[\chi_u|\chi_v](t)\,\mathrm{d}t + \delta\int_0^{T'}\int_\Omega |\nabla u - \nabla v|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Step 4: (Estimate near the boundary of the domain but away from contact points) The argument is similar to the one of the previous step, with the only major difference being that the slicing argument of the proof of [45, Lemma 20] is now carried out in $\Omega\cap\Psi_{\partial\Omega}(\partial\Omega\times[-r,r])\setminus \bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))$ by means of $\Psi_{\partial\Omega}$. This in turn is facilitated by the following facts. First, the localization properties (2.82)–(2.84) ensure

$$\mathrm{dist}(\cdot,\partial\Omega) = \mathrm{dist}(\cdot,\partial\Omega)\wedge\mathrm{dist}(\cdot,I_v(t)) \tag{2.90}$$

in $\Omega \cap \Psi_{\partial\Omega}(\partial\Omega\times[-r,r]) \setminus \bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))$. Second, as a consequence of (2.90) and the coercivity estimate (2.27) of the transported weight ϑ, the local interface error height as measured in the direction of $n_{\partial\Omega}$

$$h_{\partial\Omega}(x,t) := \int_{-r}^{r} |\chi_u - \chi_v|(\Psi_{\partial\Omega}(x,y),t)\,\mathrm{d}y, \quad x\in\partial\Omega,\, t\in[0,T],$$

satisfies the estimate

$$\begin{aligned} h_{\partial\Omega}^2(x,t) &\leq C\int_{-r}^{r} |\chi_u - \chi_v|(\Psi_{\partial\Omega}(x,y),t)y\,\mathrm{d}y \\ &\leq C\int_{-r}^{r} |\chi_u - \chi_v|(\Psi_{\partial\Omega}(x,y),t)|\vartheta|(\Psi_{\partial\Omega}(x,y),t)\,\mathrm{d}y. \end{aligned} \tag{2.91}$$

Hence, we obtain

$$\int_0^{T'}\int_{\Omega\cap\Psi_{\partial\Omega}(\partial\Omega\times[-r,r])\setminus\bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))} |\chi_v - \chi_u||u-v|\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \frac{C}{\delta} \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) + E_{\mathrm{vol}}[\chi_u|\chi_v](t)\,\mathrm{d}t + \delta \int_0^{T'} \int_\Omega |\nabla u - \nabla v|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Step 5: (Estimate near contact points) Fix $c \in \mathcal{C}$, and let $i \in \mathcal{I}$ denote the unique connected interface $\mathcal{T}_i$ such that $i \sim c$. Because of the regularity of $\partial\Omega$, the regularity of $\mathcal{T}_i$, and the 90° contact angle condition we may decompose the neighborhood $\Omega \cap B_{2r}(\mathcal{T}_c(t))$—by possibly reducing the localization scale $r \in (0, \frac{1}{2}]$ even further—into three pairwise disjoint open sets $W_{\partial\Omega}(t)$, $W_{\mathcal{T}_i}(t)$ and $W_{\partial\Omega\sim\mathcal{T}_i}(t)$ such that $\Omega \cap B_{2r}(\mathcal{T}_c(t)) \setminus \big(W_{\partial\Omega}(t) \cup W_{\mathcal{T}_i}(t) \cup W_{\partial\Omega\sim\mathcal{T}_i}(t)\big)$ is an $\mathcal{H}^d$ null set and

$$\mathrm{dist}(\cdot, \partial\Omega) = \mathrm{dist}(\cdot, \partial\Omega) \wedge \mathrm{dist}(\cdot, I_v(t)) \quad \text{in } W_{\partial\Omega}(t), \tag{2.92}$$
$$\mathrm{dist}(\cdot, \mathcal{T}_i(t)) = \mathrm{dist}(\cdot, \partial\Omega) \wedge \mathrm{dist}(\cdot, I_v(t)) \quad \text{in } W_{\mathcal{T}_i}(t), \tag{2.93}$$
$$\mathrm{dist}(\cdot, \partial\Omega) \sim \mathrm{dist}(\cdot, \mathcal{T}_i(t)) \sim \mathrm{dist}(\cdot, I_v(t)) \quad \text{in } W_{\partial\Omega\sim\mathcal{T}_i}(t), \tag{2.94}$$

as well as

$$W_{\partial\Omega}(t) \subset \Psi_{\partial\Omega}(\partial\Omega \times (-3r, 3r)), \tag{2.95}$$
$$W_{\mathcal{T}_i}(t) \subset \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t) \times (-3r, 3r)), \tag{2.96}$$
$$W_{\partial\Omega\sim\mathcal{T}_i}(t) \subset \Psi_{\partial\Omega}(\partial\Omega \times (-3r, 3r)) \cap \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t) \times (-3r, 3r)). \tag{2.97}$$

(Up to a rigid motion, these sets can in fact be defined independent of $t \in [0, T]$.) Hence, applying the argument of *Step 3* based on (2.93) and (2.96) with respect to $\Omega \cap B_{2r}(\mathcal{T}_c(t)) \cap W_{\mathcal{T}_i}(t)$, the argument of *Step 4* based on (2.92) and (2.95) with respect to $\Omega \cap B_{2r}(\mathcal{T}_c(t)) \cap W_{\partial\Omega}(t)$, and either the argument of *Step 3* or *Step 4* based on (2.94) and (2.97) with respect to $\Omega \cap B_{2r}(\mathcal{T}_c(t)) \cap W_{\partial\Omega\sim\mathcal{T}_i}(t)$ entails

$$\begin{aligned}&\sum_{c\in\mathcal{C}} \int_0^{T'} \int_{\Omega\cap B_{2r}(\mathcal{T}_c(t))} |\chi_v - \chi_u||u - v| \,\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{C}{\delta} \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) + E_{\mathrm{vol}}[\chi_u|\chi_v](t)\,\mathrm{d}t + \delta \int_0^{T'} \int_\Omega |\nabla u - \nabla v|^2 \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

This in turn concludes the proof of Lemma 12. □

Proof of Proposition 4. The proof proceeds in three steps.

Step 1: (Post-processing the relative entropy inequality (2.30)*)* It follows immediately from the $L^\infty_{x,t}$-bound for $\partial_t v$ and $\rho(\chi_v) - \rho(\chi_u) = (\rho^+ - \rho^-)(\chi_v - \chi_u)$ that

$$|R_{dt}| \leq C \int_0^{T'} \int_\Omega |\chi_v - \chi_u||u - v| \,\mathrm{d}x\,\mathrm{d}t \tag{2.98}$$

for almost every $T' \in [0, T]$. Furthermore, the $L^\infty_t W^{1,\infty}_x$-bound for v, the definition (2.29) of the relative entropy functional, and again the identity $\rho(\chi_v) - \rho(\chi_u) = (\rho^+ - \rho^-)(\chi_v - \chi_u)$ imply that

$$|R_{adv}| \leq C \int_0^{T'} \int_\Omega |\chi_v - \chi_u||u - v| \,\mathrm{d}x\,\mathrm{d}t + C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t)\,\mathrm{d}t \tag{2.99}$$

for almost every $T' \in [0, T]$. For a bound on the interface contribution R_{surTen}, we rely on the $L^\infty_t W^{1,\infty}_x$-bound for v, the $L^\infty_t W^{2,\infty}_x$-bound for ξ, the $L^\infty_t W^{1,\infty}_x$-bound for B, the definition (2.29) of the relative entropy functional, as well as the estimates (2.16d) and (2.16e) of a boundary adapted extension ξ of n_{I_v} to the effect that

$$
\begin{aligned}
|R_{surTen}| \leq C \int_0^{T'} \int_\Omega |\chi_v - \chi_u||u - v| \,\mathrm{d}x\,\mathrm{d}t \\
&+ C \int_0^{T'} \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} |s-\xi|^2 \,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\
&+ C \int_0^{T'} \int_\Omega 1-\theta_t \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\
&+ C \int_0^{T'} \int_{\partial\Omega} 1 \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\
&+ C \int_0^{T'} \int_\Omega |n_u - \xi|^2 \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\
&+ C \int_0^{T'} \int_\Omega \mathrm{dist}^2(\cdot, I_v) \wedge 1 \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\
&+ C \int_0^{T'} \int_\Omega |\xi\cdot(\xi - n_u)| \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\
&+ C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t)\,\mathrm{d}t
\end{aligned}
\tag{2.100}
$$

for almost every $T' \in [0,T]$. It follows from property (2.16a) of a boundary adapted extension ξ and the trivial estimates $|\xi\cdot(\xi-n_u)| \leq (1-|\xi|^2) + (1-n_u\cdot\xi) \leq 2(1-|\xi|) + (1-n_u\cdot\xi)$ and $1-|\xi| \leq 1-n_u\cdot\xi$ that

$$
\begin{aligned}
&\int_0^{T'} \int_\Omega \mathrm{dist}^2(\cdot, I_v) \wedge 1 \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t + \int_0^{T'} \int_\Omega |\xi\cdot(\xi - n_u)| \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\
&\leq C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t)\,\mathrm{d}t.
\end{aligned}
\tag{2.101}
$$

Moreover, the trivial estimate $|n_u - \xi|^2 \leq 2(1-n_u\cdot\xi)$ implies

$$
\int_0^{T'} \int_\Omega |n_u - \xi|^2 \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \leq C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t)\,\mathrm{d}t. \tag{2.102}
$$

Recall finally from (2.24) and (2.20) that

$$
\begin{aligned}
\int_0^{T'} \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} |s-\xi|^2 \,\mathrm{d}V_t(x,s)\,\mathrm{d}t &\leq C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t)\,\mathrm{d}t, \\
\int_0^{T'} \int_\Omega 1-\theta_t \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t + \int_0^{T'} \int_{\partial\Omega} 1 \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t &\leq C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t)\,\mathrm{d}t.
\end{aligned}
\tag{2.103}
$$

By inserting back the estimates (2.98)–(2.103) into the relative entropy inequality (2.30), then making use of the coercivity estimate (2.77) and Korn's inequality, and finally carrying out an absorption argument, it follows that there exist two constants $c = c(\chi_v, v, T) > 0$ and $C = C(\chi_v, v, T) > 0$ such that for almost every $T' \in [0,T]$

$$
E[\chi_u, u, V|\chi_v, v](T') + c \int_0^{T'} \int_\Omega |\nabla(u-v) + \nabla(u-v)^{\mathsf{T}}|^2 \,\mathrm{d}x\,\mathrm{d}t
$$

$$\leq E[\chi_u, u, V|\chi_v, v](0) + C\int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) + E_{\mathrm{vol}}[\chi_u|\chi_v](t)\,\mathrm{d}t. \tag{2.104}$$

Step 2: (Post-processing the identity (2.32)*)* By the $L_t^\infty W_x^{1,\infty}$-bound for the transported weight ϑ, the estimate (2.28) on the advective derivative of the transported weight ϑ, and the definition (2.31) of the bulk error functional we infer that

$$\begin{aligned} E_{\mathrm{vol}}[\chi_u|\chi_v](T') \leq E_{\mathrm{vol}}[\chi_u|\chi_v](0) + C\int_0^{T'} E_{\mathrm{vol}}[\chi_u|\chi_v](t)\,\mathrm{d}t \\ + C\int_0^{T'}\int_\Omega |\chi_v - \chi_u||u - v|\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

for almost every $T' \in [0,T]$. Adding (2.104) to the previous display, and making use of the coercivity estimate (2.77) in combination with Korn's inequality and an absorption argument thus implies that for almost every $T' \in [0,T]$

$$\begin{aligned} &E[\chi_u, u, V|\chi_v, v](T') + E_{\mathrm{vol}}[\chi_u|\chi_v](T') + c\int_0^{T'}\int_\Omega |\nabla(u-v) + \nabla(u-v)^\mathsf{T}|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\leq E[\chi_u, u, V|\chi_v, v](0) + E_{\mathrm{vol}}[\chi_u|\chi_v](0) \\ &\quad + C\int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) + E_{\mathrm{vol}}[\chi_u|\chi_v](t)\,\mathrm{d}t. \end{aligned} \tag{2.105}$$

Step 3: (Conclusion) The stability estimates (2.11) and (2.12) are an immediate consequence of the estimate (2.105) by an application of Gronwall's lemma. In case of coinciding initial conditions, it follows that $E_{\mathrm{vol}}[\chi_u|\chi_v](t) = 0$ for almost every $t \in [0,T]$. This in turn implies that $\chi_u(\cdot,t) = \chi_v(\cdot,t)$ almost everywhere in Ω for almost every $t \in [0,T]$. The asserted representation of the varifold follows from the fact that $E[\chi_u, u, V|\chi_v, v](t) = 0$ for almost every $t \in [0,T]$. This concludes the proof of the conditional weak-strong uniqueness principle. □

2.3.4 Proof of Theorem 1

This is now an immediate consequence of Proposition 4 and the existence results of Proposition 7 and Lemma 8, respectively. □

2.4 Bulk extension of the interface unit normal

The aim of this short section is the construction of an extension of the interface unit normal in the vicinity of a space-time trajectory in Ω of a connected component of the interface I_v corresponding to a strong solution in the sense of Definition 10 on a time interval $[0,T]$.

Mainly for reference purposes in later sections, it turns out to be beneficial to introduce already at this stage some notation in relation to a decomposition of the interface I_v into its topological features: the connected components of $I_v \cap \Omega$ and the connected components of $I_v \cap \partial\Omega$. Denoting by $N \in \mathbb{N}$ the total number of such topological features present in the interface I_v we split $\{1,\ldots,N\} =: \mathcal{I} \cup \mathcal{C}$ by means of two disjoint subsets. In particular, the subset $\mathcal{I}$ enumerates the space-time connected components of $I_v \cap \Omega$, i.e., time-evolving connected *interfaces*, whereas the subset $\mathcal{C}$ enumerates the space-time connected components of $I_v \cap \partial\Omega$, i.e., time-evolving *contact points.* If $i \in \mathcal{I}$, we denote by $\mathcal{T}_i := \bigcup_{t\in[0,T]} \mathcal{T}_i(t)\times\{t\} \subset I_v \cap (\Omega\times[0,T])$ the space-time trajectory of the corresponding connected interfaces $\mathcal{T}_i(t) \subset I_v(t) \cap \Omega$, $t \in [0,T]$.

For each $i \in \mathcal{I}$, we want to define a vector field ξ^i subject to conditions as in Definition 2; at least in a suitable neighborhood of $\mathcal{T}_i$. We first formalize what we mean by the latter in form of the following definition.

Definition 13. *Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Fix a two-phase interface $i \in \mathcal{I}$. We call $r_i \in (0,1]$ an* admissible localization radius *for the interface $\mathcal{T}_i \subset I_v \cap (\Omega\times[0,T])$ if the map*

$$\Psi_{\mathcal{T}_i} \colon \mathcal{T}_i \times (-2r_i, 2r_i) \to \mathbb{R}^2 \times [0,T], \quad (x,t,s) \mapsto \Big(x + s n_{I_v}(x,t), t\Big) \tag{2.106}$$

is bijective onto its image $\mathrm{im}(\Psi_{\mathcal{T}_i}) := \Psi_{\mathcal{T}_i}\Big(\mathcal{T}_i\times(-2r_i,2r_i)\Big)$, and its inverse is a diffeomorphism of class $C^0_tC^2_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})}) \cap C^1_tC^0_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})})$.

In case such a scale $r_i \in (0,1]$ exists, we may express the inverse by means of $\Psi^{-1}_{\mathcal{T}_i} =: (P_{\mathcal{T}_i}, \mathrm{Id}, s_{\mathcal{T}_i}) \colon \mathrm{im}(\Psi_{\mathcal{T}_i}) \to \mathcal{T}_i\times(-2r_i,2r_i)$. Hence, the map $P_{\mathcal{T}_i}$ represents in each time slice the nearest-point projection *onto the interface $\mathcal{T}_i(t) \subset I_v(t) \cap \Omega$, $t \in [0,T]$, whereas $s_{\mathcal{T}_i}$ bears the interpretation of a* signed distance function *with orientation fixed by $\nabla s_{\mathcal{T}_i} = n_{I_v}$. In particular, $s_{\mathcal{T}_i} \in C^0_tC^3_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})}) \cap C^1_tC^1_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})})$ as well as $P_{\mathcal{T}_i} \in C^0_tC^2_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})}) \cap C^1_tC^0_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})})$.*

By a slight abuse of notation, we extend to $\mathrm{im}(\Psi_{\mathcal{T}_i})$ the definition of the normal vector field resp. the scalar mean curvature of $\mathcal{T}_i$ by means of

$$n_{I_v} \colon \mathrm{im}(\Psi_{\mathcal{T}_i}) \to \mathbb{S}^1, \quad (x,t) \mapsto n_{I_v}(P_{\mathcal{T}_i}(x,t),t) = \nabla s_{\mathcal{T}_i}(x,t), \tag{2.107}$$

$$H_{I_v} \colon \mathrm{im}(\Psi_{\mathcal{T}_i}) \to \mathbb{R}, \quad (x,t) \mapsto -(\Delta s_{\mathcal{T}_i})(P_{\mathcal{T}_i}(x,t),t). \tag{2.108}$$

Hence, we may register that $n_{I_v} \in C^0_tC^2_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})}) \cap C^1_tC^0_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})})$ as well as $H_{I_v} \in C^0_tC^1_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})})$.

It is clear from Definition 10 of a strong solution to the incompressible Navier–Stokes equation for two fluids, in particular Definition 9 of smoothly evolving domains and interfaces, that all interfaces admit an admissible localization radius in the sense of Definition 13 as a consequence of the tubular neighborhood theorem.

Construction 14. *Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Fix a two-phase interface $i \in \mathcal{I}$ and let $r_i \in (0,1]$ be an admissible localization radius for the interface $\mathcal{T}_i \subset I_v$ in the sense of Definition 13. Then* a bulk extension of the unit normal n_{I_v} along a smooth interface $\mathcal{T}_i$ *is the vector field ξ^i defined by*

$$\xi^i(x,t) := n_{I_v}(x,t), \qquad (x,t) \in \mathrm{im}(\Psi_{\mathcal{T}_i}) \cap (\Omega\times[0,T]). \tag{2.109}$$

We record the required properties of the vector field ξ^i.

Proposition 15. *Let the assumptions and notation of Construction 14 be in place. Then, in terms of regularity it holds that* $\xi^i \in C_t^0C_x^2 \cap C_t^1C_x^0(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i}) \cap (\Omega{\times}[0,T])})$. *Moreover, we have*

$$\nabla\cdot\xi^i + H_{I_v} = O(\mathrm{dist}(\cdot,\mathcal{T}_i)), \tag{2.110}$$

$$\partial_t\xi^i + (v\cdot\nabla)\xi^i + (\mathrm{Id} - \xi^i\otimes\xi^i)(\nabla v)^\mathsf{T}\xi^i = O(\mathrm{dist}(\cdot,\mathcal{T}_i)), \tag{2.111}$$

$$\partial_t|\xi^i|^2 + (v\cdot\nabla)|\xi^i|^2 = 0 \tag{2.112}$$

throughout the space-time domain $\mathrm{im}(\Psi_{\mathcal{T}_i}) \cap (\Omega{\times}[0,T])$.

Proof. The asserted regularity of ξ^i is a direct consequence of its definition (2.109) and the regularity of n_{I_v} from Definition 13. In view of the definitions (2.109), (2.107) and (2.108), the estimate (2.110) is directly implied by a Lipschitz estimate based on the regularity of H_{I_v} from Definition 13. The equation (2.112) is trivially fulfilled because ξ^i is a unit vector, cf. the definition (2.109).

For a proof of (2.111), we first note that $\partial_t s_{\mathcal{T}_i}(x,t) = -\big(v(P_{\mathcal{T}_i}(x,t),t)\cdot\nabla\big)s_{\mathcal{T}_i}(x,t)$ for all $(x,t) \in \mathrm{im}(\Psi_{\mathcal{T}_i}) \cap (\Omega{\times}[0,T])$. Indeed, $\partial_t s_{\mathcal{T}_i}$ equals the normal speed (oriented with respect to $-n_{I_v}$) of the nearest point on the connected interface $\mathcal{T}_i$, which in turn by $n_{I_v} = \nabla s_{\mathcal{T}_i}$ is precisely given by the asserted right hand side term. Differentiating the equation for the time evolution of $s_{\mathcal{T}_i}$ then yields (2.111) by means of $\nabla P_{\mathcal{T}_i} = \mathrm{Id} - n_{I_v}\otimes n_{I_v} - s_{\mathcal{T}_i}\nabla n_{I_v}$, the chain rule, and the regularity of v. Note carefully that this argument is actually valid regardless of the assumption $\mu_- = \mu_+$ since $(\tau_{I_v}\cdot\nabla)v$ does not jump across the interface $\mathcal{T}_i$. □

2.5 Extension of the interface unit normal at a 90° contact point

This section constitutes the core of the present work. We establish the existence of a boundary adapted extension of the interface unit normal in the vicinity of a space-time trajectory of a 90° contact point on the boundary $\partial\Omega$.

The vector field from the previous section serves as the main building block for an extension of n_{I_v} away from the domain boundary $\partial\Omega$. However, it is immediately clear that the bulk construction in general does not respect the necessary boundary condition $n_{\partial\Omega}\cdot\xi = 0$ along $\partial\Omega$. (Even more drastically, on non-convex parts of $\partial\Omega$ the domain of definition for the bulk construction from the previous section may not even include $\partial\Omega$!) Hence, in the vicinity of contact points a careful perturbation of the rather trivial construction from the previous section is required to enforce the boundary condition. That this can indeed be achieved is summarized in the following Proposition 16, representing the main result of this section.

For its formulation, it is convenient for the purposes of Section 2.6 to recall the notation in relation to the decomposition of the interface I_v in terms of its topological features. More precisely, denoting by $N \in \mathbb{N}$ the total number of such topological features present in the interface I_v, we split $\{1,\ldots,N\} =: \mathcal{I} \cup \mathcal{C}$, where $\mathcal{I}$ enumerates the time-evolving connected *interfaces* of $I_v \cap \Omega$, whereas $\mathcal{C}$ enumerates the time-evolving *contact points* of $I_v \cap \partial\Omega$. If $i \in \mathcal{I}$, $\mathcal{T}_i := \bigcup_{t\in[0,T]}\mathcal{T}_i(t){\times}\{t\} \subset I_v \cap (\Omega{\times}[0,T])$ denotes the space-time trajectory of the corresponding connected interface, whereas if $c \in \mathcal{C}$, we denote by $\mathcal{T}_c := \bigcup_{t\in[0,T]}\mathcal{T}_c(t){\times}\{t\} \subset I_v \cap (\partial\Omega{\times}[0,T])$ the space-time trajectory of the corresponding contact point. Finally, we write $i \sim c$ for $i \in \mathcal{I}$ and $c \in \mathcal{C}$ if and only if $\mathcal{T}_i$ ends at $\mathcal{T}_c$; otherwise $i \not\sim c$.

Proposition 16. *Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary $\partial\Omega$. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0, T]$. Fix a contact point $c \in \mathcal{C}$ and let $i \in \mathcal{I}$ be such that $i \sim c$. Let $r_c \in (0, 1]$ be an associated admissible localization radius in the sense of Definition 17 below.*

There exists a potentially smaller radius $\widehat{r}_c \in (0, r_c]$, and a vector field

$$\xi^c \colon \mathcal{N}_{\widehat{r}_c,c}(\Omega) \to \mathbb{S}^1$$

defined on the space-time domain $\mathcal{N}_{\widehat{r}_c,c}(\Omega) := \bigcup_{t\in[0,T]} \Big(B_{\widehat{r}_c}(\mathcal{T}_c(t)) \cap \Omega\Big) \times \{t\}$, such that the following conditions are satisfied:

i) It holds $\xi^c \in \Big(C^0_t C^2_x \cap C^1_t C^0_x\Big)\Big(\overline{\mathcal{N}_{\widehat{r}_c,c}(\Omega)} \setminus \mathcal{T}_c\Big)$.

ii) We have $\xi^c(\cdot, t) = n_{I_v}(\cdot, t)$ and $\nabla \cdot \xi^c(\cdot, t) = -H_{I_v}(\cdot, t)$ along $\mathcal{T}_i(t) \cap B_{\widehat{r}_c}(\mathcal{T}_c(t))$ for all $t \in [0, T]$.

iii) The required boundary condition is satisfied even away from the contact point, namely $\xi^c \cdot n_{\partial\Omega} = 0$ along $\mathcal{N}_{\widehat{r}_c,c}(\Omega) \cap (\partial\Omega \times [0, T])$.

iv) The following estimates on the time evolution of ξ^c hold true in $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$

$$\partial_t \xi^c + (v \cdot \nabla)\xi^c + (\mathrm{Id} - \xi^c \otimes \xi^c)(\nabla v)^{\mathsf{T}} \xi^c = O\Big(\operatorname{dist}(\cdot, \mathcal{T}_i)\Big), \tag{2.113}$$

$$\partial_t |\xi^c|^2 + (v \cdot \nabla)|\xi^c|^2 = 0. \tag{2.114}$$

v) Let $r_i \in (0, 1]$ be an admissible localization radius for the interface $\mathcal{T}_i$, and let ξ^i be the bulk extension of the interface unit normal on scale r_i as provided by Proposition 15. The vector field ξ^c is a perturbation of the bulk extension ξ^i in the sense that the following compatibility bounds hold true

$$|\xi^i(\cdot, t) - \xi^c(\cdot, t)| + |\nabla \cdot \xi^i(\cdot, t) - \nabla \cdot \xi^c(\cdot, t)| \leq C \operatorname{dist}(\cdot, \mathcal{T}_i(t)), \tag{2.115}$$

$$|\xi^i(\cdot, t) \cdot (\xi^i - \xi^c)(\cdot, t)| \leq C \operatorname{dist}^2(\cdot, \mathcal{T}_i(t)) \tag{2.116}$$

within $B_{\widehat{r}_c \wedge r_i}(\mathcal{T}_c(t)) \cap \Big(W^c_{\mathcal{T}_i}(t) \cup W^c_{\Omega^\pm_v}(t)\Big)$ for all $t \in [0, T]$, cf. Definition 17.

A vector field ξ^c subject to these requirements will be referred to as a contact point extension of the interface unit normal on scale *$\widehat{r}_c$.*

A proof of Proposition 16 is provided in Subsection 2.5.4. The preceding three subsections collect all the ingredients required for the construction.

2.5.1 Description of the geometry close to a moving contact point, choice of orthonormal frames, and a higher-order compatibility condition

We provide a suitable decomposition for a space-time neighborhood of a moving contact point $\mathcal{T}_c$, $c \in \mathcal{C}$. The main ingredient is given by the following notion of an admissible localization radius. Though rather technical and lengthy in appearance, all requirements in the definition are essentially a direct consequence of the regularity of a strong solution. The main purpose of the notion of an admissible localization radius is to collect in a unified way notation and properties which will be referred to numerous times in the sequel.

Definition 17. *Let $d=2$, and let $\Omega\subset\mathbb{R}^2$ be a bounded domain with orientable and smooth boundary $\partial\Omega$. Let (χ_v,v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Fix a contact point $c\in\mathcal{C}$ and let $i\in\mathcal{I}$ be such that $i\sim c$. Let $r_i\in(0,1]$ be an admissible localization radius for the connected interface $\mathcal{T}_i$ in the sense of Definition 13. We call $r_c\in(0,r_i]$ an* admissible localization radius for the moving 90° contact point $\mathcal{T}_c$ *if the following list of properties is satisfied:*

i) Let the map $\Psi_{\partial\Omega}\colon \partial\Omega\times(-2r_c,2r_c)\to\mathbb{R}^2$ be given by $(x,s)\mapsto x+sn_{\partial\Omega}(x)$. We require $\Psi_{\partial\Omega}$ to be bijective onto its image $\mathrm{im}(\Psi_{\partial\Omega}):=\Psi_{\partial\Omega}\big(\partial\Omega\times(-2r_c,2r_c)\big)$, and its inverse $\Psi_{\partial\Omega}^{-1}$ is a diffeomorphism of class $C^2_x(\overline{\mathrm{im}(\Psi_{\partial\Omega})})$. We may express the inverse by means of $\Psi_{\partial\Omega}^{-1}=:(P_{\partial\Omega},s_{\partial\Omega})\colon \mathrm{im}(\Psi_{\partial\Omega})\to\partial\Omega\times(-2r_c,2r_c)$. Hence, $P_{\partial\Omega}$ represents the nearest-point projection *onto $\partial\Omega$, whereas $s_{\partial\Omega}$ is the* signed distance function *with orientation fixed by $\nabla s_{\partial\Omega}=n_{\partial\Omega}$. In particular, $s_{\partial\Omega}\in C^3_x(\overline{\mathrm{im}(\Psi_{\partial\Omega})})$ and $P_{\partial\Omega}\in C^2_x(\overline{\mathrm{im}(\Psi_{\partial\Omega})})$. By a slight abuse of notation, we extend to $\mathrm{im}(\Psi_{\partial\Omega})$ the definition of the normal vector field resp. the scalar mean curvature of $\partial\Omega$ by means of*

$$n_{\partial\Omega}\colon \mathrm{im}(\Psi_{\partial\Omega})\to\mathbb{S}^1,\quad (x,t)\mapsto n_{\partial\Omega}(P_{\partial\Omega}(x))=\nabla s_{\partial\Omega}(x), \tag{2.117}$$

$$H_{\partial\Omega}\colon \mathrm{im}(\Psi_{\partial\Omega})\to\mathbb{R},\quad (x,t)\mapsto -(\Delta s_{\partial\Omega})(P_{\partial\Omega}(x)). \tag{2.118}$$

Hence, we note that $n_{\partial\Omega}\in C^2_x(\overline{\mathrm{im}(\Psi_{\partial\Omega})})$ and $H_{\partial\Omega}\in C^1_x(\overline{\mathrm{im}(\Psi_{\partial\Omega})})$.

ii) There exist sets $W^c_{\mathcal{T}_i}=\bigcup_{t\in[0,T]}W^c_{\mathcal{T}_i}(t)\times\{t\}$, $W^c_{\Omega^\pm_v}=\bigcup_{t\in[0,T]}W^c_{\Omega^\pm_v}(t)\times\{t\}$ and $W^{\pm,c}_{\partial\Omega}=\bigcup_{t\in[0,T]}W^{\pm,c}_{\partial\Omega}(t)\times\{t\}$ with the following properties:

First, for every $t\in[0,T]$, the sets $W^c_{\mathcal{T}_i}(t)$, $W^c_{\Omega^\pm_v}(t)$ and $W^{\pm,c}_{\partial\Omega}(t)$ are non-empty subsets of $\overline{B_{r_c}(\mathcal{T}_c(t))}$ with pairwise disjoint interior. For all $t\in[0,T]$, each of these sets is represented by a cone with apex at the contact point $\mathcal{T}_c(t)$ intersected with $\overline{B_{r_c}(\mathcal{T}_c(t))}$. More precisely, there exist six time-dependent pairwise distinct unit-length vectors $X^\pm_{\mathcal{T}_i}$, $X_{\Omega^\pm_v}$ and $X^\pm_{\partial\Omega}$ of class $C^1_t([0,T])$ such that for all $t\in[0,T]$ it holds

$$W^c_{\mathcal{T}_i}(t)=\Big(\mathcal{T}_c(t)+\{\alpha X^+_{\mathcal{T}_i}(t)+\beta X^-_{\mathcal{T}_i}(t)\colon \alpha,\beta\in[0,\infty)\}\Big)\cap\overline{B_{r_c}(\mathcal{T}_c(t))}, \tag{2.119}$$

$$W^c_{\Omega^\pm_v}(t)=\Big(\mathcal{T}_c(t)+\{\alpha X_{\Omega^\pm_v}(t)+\beta X^\pm_{\mathcal{T}_i}(t)\colon \alpha,\beta\in[0,\infty)\}\Big)\cap\overline{B_{r_c}(\mathcal{T}_c(t))}, \tag{2.120}$$

$$W^{\pm,c}_{\partial\Omega}(t)=\Big(\mathcal{T}_c(t)+\{\alpha X^\pm_{\partial\Omega}(t)+\beta X_{\Omega^\pm_v}(t)\colon \alpha,\beta\in[0,\infty)\}\Big)\cap\overline{B_{r_c}(\mathcal{T}_c(t))}. \tag{2.121}$$

The opening angles of these cones are constant, and numerically fixed by

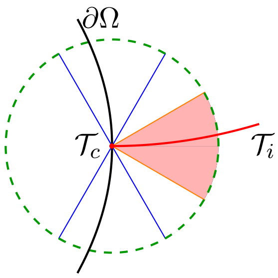
$$X^\pm_{\partial\Omega}\cdot X_{\Omega^\pm_v}=X^+_{\mathcal{T}_i}\cdot X^-_{\mathcal{T}_i}=\cos(\pi/3),\quad X_{\Omega^\pm_v}\cdot X^\pm_{\mathcal{T}_i}=\cos(\pi/6). \tag{2.122}$$

Second, for every $t\in[0,T]$, the sets $W^c_{\mathcal{T}_i}(t)$, $W^c_{\Omega^\pm_v}(t)$ and $W^{\pm,c}_{\partial\Omega}(t)$ provide a decomposition of $B_{r_c}(\mathcal{T}_c(t))$ in form of

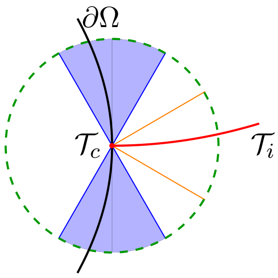
$$\begin{aligned}&\overline{B_{r_c}(\mathcal{T}_c(t))}\cap\overline{\Omega}\\&=\Big(W^c_{\mathcal{T}_i}(t)\cup W^c_{\Omega^+_v}(t)\cup W^c_{\Omega^-_v}(t)\cup W^{+,c}_{\partial\Omega}(t)\cup W^{-,c}_{\partial\Omega}(t)\Big)\cap\overline{\Omega}.\end{aligned} \tag{2.123}$$

Third, for each $t\in[0,T]$, the following inclusions hold true (recall from Definition 13 the notation for the diffeomorphism $\Psi_{\mathcal{T}_i}$):

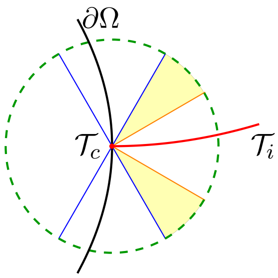
$$\overline{B_{r_c}(\mathcal{T}_c(t))}\cap\mathcal{T}_i(t)\subset\Big(W^c_{\mathcal{T}_i}(t)\setminus\mathcal{T}_c(t)\Big)\subset\{x\in\Omega\colon (x,t)\in\mathrm{im}(\Psi_{\mathcal{T}_i})\}, \tag{2.124}$$



(a) Interface wedge $W^c_{\mathcal{T}_i}$.



(b) Boundary wedges $W^{\pm}_{\partial\Omega}$.



(c) Interpolation wedges $W^c_{\Omega^{\pm}_v}$.

Figure 2.1: Decomposition for a space-time neighborhood of $\mathcal{T}_c$.

$$\overline{B_{r_c}(\mathcal{T}_c(t))} \cap \partial\Omega \subset W^{+,c}_{\partial\Omega}(t) \cup W^{-,c}_{\partial\Omega}(t), \tag{2.125}$$
$$W^{\pm,c}_{\partial\Omega}(t) \subset \{x \in \mathbb{R}^2 \colon x \in \mathrm{im}(\Psi_{\partial\Omega})\}, \tag{2.126}$$
$$W^c_{\Omega^{\pm}_v}(t) \setminus \mathcal{T}_c(t) \subset \Omega^{\pm}_v(t) \cap \{x \in \Omega \colon (x,t) \in \mathrm{im}(\Psi_{\mathcal{T}_i}),\, x \in \mathrm{im}(\Psi_{\partial\Omega})\}. \tag{2.127}$$

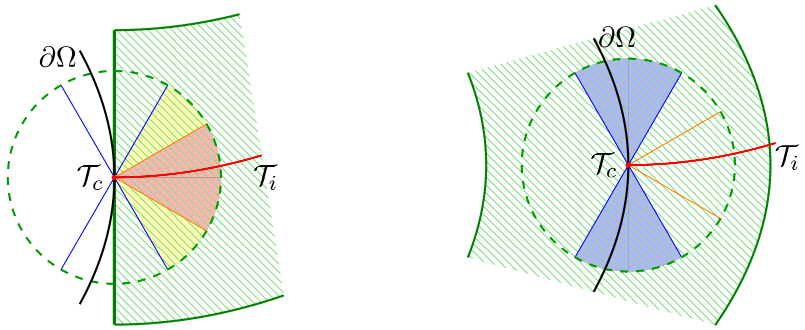
iii) Finally, there exists a constant $C > 0$ such that

$$\mathrm{dist}(\cdot, \mathcal{T}_c) \vee \mathrm{dist}(\cdot, \partial\Omega) \leq C\, \mathrm{dist}(\cdot, \mathcal{T}_i) \qquad \text{on } W^c_{\Omega^{\pm}_v} \cup W^{\pm,c}_{\partial\Omega}, \tag{2.128}$$

We refer from here onwards to $W^c_{\mathcal{T}_i}$ as the interface wedge, *$W^{\pm,c}_{\partial\Omega}$ as* boundary wedges, *and $W^c_{\Omega^{\pm}_v}$ as* interpolation wedges.

Figures 2.1–2.2 contain several illustrations of the previous definition. Before moving on, we briefly discuss the existence of an admissible localization radius.

Lemma 18. *Let the assumptions and notation of Definition 17 be in place. There exists a constant $C = C(\partial\Omega, \chi_v, v, T) \geq 1$ such that each $r_c \in (0, \frac{1}{C}]$ is an admissible localization radius for the contact point $\mathcal{T}_c$ in the sense of Definition 17.*



(a) Inclusion in the image of $\Psi_{\mathcal{T}_i}$. (b) Inclusion in the image of $\Psi_{\partial\Omega}$.

Figure 2.2: Inclusion properties of diffeomorphisms.

Proof. The first item in the definition of an admissible localization radius is an immediate consequence of the tubular neighborhood theorem, which in turn is facilitated by the regularity of the domain boundary $\partial\Omega$.

For a construction of the wedges, we only have to provide a definition for the vectors $X^{\pm}_{\mathcal{T}_i}$, $X_{\Omega^{\pm}_v}$ and $X^{\pm}_{\partial\Omega}$ A possible choice is the following. Fix $t \in [0,T]$ and let $\{c(t)\} = \mathcal{T}_c(t)$. The desired unit vectors are obtained through rotation of the inward-pointing unit normal $n_{\partial\Omega}(c(t))$. Note that $\big(n_{\partial\Omega}(c(t)),\, n_{I_v}(c(t),t)\big)$ form an orthonormal basis of $\mathbb{R}^2$ thanks to the contact angle condition (2.33). We then let $X^{\pm}_{\mathcal{T}_i}(t)$ be the unique unit vector with $X^{\pm}_{\mathcal{T}_i}(t) \cdot n_{\partial\Omega}(c(t)) = \frac{\sqrt{3}}{2}$ as well as $\operatorname{sign}\big(X^{\pm}_{\mathcal{T}_i}(t) \cdot n_{I_v}(c(t),t)\big) = \pm 1$. Similarly, $X_{\Omega^{\pm}_v}(t)$ represents the unique unit vector with $X_{\Omega^{\pm}_v}(t) \cdot n_{\partial\Omega}(c(t)) = \frac{1}{2}$ and $\operatorname{sign}\big(X_{\Omega^{\pm}_v}(t) \cdot n_{I_v}(c(t),t)\big) = \pm 1$. Finally, $X^{\pm}_{\partial\Omega}(t)$ denotes the unique unit vector with $X^{\pm}_{\Omega}(t) \cdot n_{\partial\Omega}(c(t)) = -\frac{1}{2}$ and $\operatorname{sign}\big(X^{\pm}_{\Omega}(t) \cdot n_{I_v}(c(t),t)\big) = \pm 1$. For an illustration, we refer again to Figure 2.1.

The wedges $W^c_{\mathcal{T}_i}(t)$, $W^c_{\Omega^{\pm}_v}(t)$ and $W^{\pm,c}_{\partial\Omega}(t)$ may now be defined through the right hand sides of (2.119), (2.120) and (2.121), respectively. The properties (2.123)–(2.128) are then obviously valid for sufficiently small radii as a consequence of the regularity of the domain boundary $\partial\Omega$, the regularity of the interface I_v due to Definition 10 of a strong solution, as well as the 90° contact angle condition (2.33). □

A main step in the construction of a contact point extension of the interface unit normal consists of perturbing the bulk construction of Section 2.4 by introducing suitable tangential terms, cf. Subsection 2.5.2 below. (This in turn becomes necessary due to the boundary constraint $n_{\partial\Omega} \cdot \xi^c = 0$ along $\partial\Omega$.) To this end, the following constructions and formulas will be of frequent use.

Lemma 19. *Let the assumptions and notation of Definition 13 and Definition 17 be in place. Let r_c be an admissible localization radius of a contact point $\mathcal{T}_c$ and let $i \in \mathcal{I}$ such that $i \sim c$. Define $\mathcal{N}_{r_c,c}(\Omega) := \bigcup_{t\in[0,T]} \big(B_{r_c}(\mathcal{T}_c(t)) \cap \overline{\Omega}\big) \times \{t\}$. We fix unit-length tangential vector fields $\widetilde{\tau}_{I_v}$ resp. $\widetilde{\tau}_{\partial\Omega}$ along $\mathcal{N}_{r_c,c}(\Omega) \cap \mathcal{T}_i$ resp. $\partial\Omega$ with orientation chosen such that $\widetilde{\tau}_{I_v} = -n_{\partial\Omega}$ resp. $\widetilde{\tau}_{\partial\Omega} = n_{I_v}$ hold true at the contact point $\mathcal{T}_c$. We then define extensions*

$$\tau_{I_v} \colon \mathcal{N}_{r_c,c}(\Omega) \cap \operatorname{im}(\Psi_{\mathcal{T}_i}) \to \mathbb{S}^1, \quad (x,t) \mapsto \widetilde{\tau}_{I_v}(P_{\mathcal{T}_i}(x,t),t),$$
$$\tau_{\partial\Omega} \colon \operatorname{im}(\Psi_{\partial\Omega}) \to \mathbb{S}^1, \quad x \mapsto \widetilde{\tau}_{\partial\Omega}(P_{\partial\Omega}(x)),$$

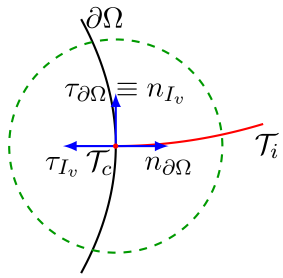


Figure 2.3: Orientation of normal and tangential vectors at $\mathcal{T}_c$.

Then, it holds $\tau_{I_v} \in C_t^0 C_x^2(\overline{\mathcal{N}_{r_c,c}(\Omega) \cap \mathrm{im}(\Psi_{\mathcal{T}_i})}) \cap C_t^1 C_x^0(\overline{\mathcal{N}_{r_c,c}(\Omega) \cap \mathrm{im}(\Psi_{\mathcal{T}_i})})$ *as well as* $\tau_{\partial\Omega} \in C_x^2(\mathrm{im}(\Psi_{\partial\Omega}))$. *Moreover,*

$$\nabla n_{I_v} = -H_{I_v} \tau_{I_v} \otimes \tau_{I_v} + O(\mathrm{dist}(\cdot, \mathcal{T}_i)) \qquad \text{in } \mathcal{N}_{r_c,c}(\Omega) \cap \mathrm{im}(\Psi_{\mathcal{T}_i}), \tag{2.129}$$

$$\nabla \tau_{I_v} = H_{I_v} n_{I_v} \otimes \tau_{I_v} + O(\mathrm{dist}(\cdot, \mathcal{T}_i)) \qquad \text{in } \mathcal{N}_{r_c,c}(\Omega) \cap \mathrm{im}(\Psi_{\mathcal{T}_i}). \tag{2.130}$$

Analogous formulas hold on $\mathrm{im}(\Psi_{\partial\Omega})$ *for the orthonormal frame* $(n_{\partial\Omega}, \tau_{\partial\Omega})$.

Proof. By the choice of the orientations, there exists a constant matrix R representing rotation by 90° so that $n_{I_v} = R\tau_{I_v}$ and $n_{\partial\Omega} = R\tau_{\partial\Omega}$. The regularity of the tangential fields τ_{I_v} and $\tau_{\partial\Omega}$ thus follows from Definition 13 and Definition 17, respectively. Moreover, the formula (2.130) simply follows from (2.129) and the product rule. For a proof of (2.129), note first that $(n_{I_v} \cdot \nabla) n_{I_v} = \nabla \frac{1}{2} |n_{I_v}|^2 = 0$ and, as a consequence of $\nabla n_{I_v} = \nabla^2 s_{\mathcal{T}_i}$ being symmetric, that $(\nabla n_{I_v})^\mathsf{T} n_{I_v} = (n_{I_v} \cdot \nabla) n_{I_v} = 0$. The only surviving component of ∇n_{I_v} is thus the one in the direction of $\tau_{I_v} \otimes \tau_{I_v}$, which on the interface in turn evaluates to $-H_{I_v}$, see (2.108). The regularity of the map H_{I_v} from Definition 13 then entails (2.129). Of course, the exact same argument works in terms of the orthonormal frame $(n_{\partial\Omega}, \tau_{\partial\Omega})$. $\square$

The values of a contact point extension in the sense of Proposition 16 are highly constrained along the domain boundary $\partial\Omega$ (i.e., $n_{\partial\Omega} \cdot \xi^c = 0$) or along the interface $\mathcal{T}_i$ (i.e., $\xi^c = n_{I_v}$), respectively. This will be reflected in the construction by stitching together certain local building blocks (i.e., $\xi^c_{\partial\Omega}$ and $\xi^c_{\mathcal{T}_i}$, see Subsection 2.5.2 below) which in turn take care of these restrictions on an individual basis (i.e., $n_{\partial\Omega} \cdot \xi^c_{\partial\Omega} = 0$ along $\partial\Omega$, or $\xi^c_{\mathcal{T}_i} = n_{I_v}$ along $\mathcal{T}_i$, in the vicinity of the contact point). These local building blocks will be unified into a single vector field by interpolation (see Subsection 2.5.3 below). With this in mind, it is of no surprise that compatibility conditions (including a higher-order one) at the contact point are needed to implement this procedure. Indeed, recall from Proposition 16 that a contact point extension requires a certain amount of regularity in combination with a control on its time evolution. We therefore collect for reference purposes the necessary compatibility conditions in the following result.

Lemma 20. *Let the assumptions and notation of Definition 13, Definition 17 and Lemma 19 be in place. Then it holds*

$$n_{I_v}(\cdot, t) = \tau_{\partial\Omega}(\cdot), \quad \tau_{I_v}(\cdot, t) = -n_{\partial\Omega}(\cdot) \qquad \text{at } \mathcal{T}_c(t),\ t \in [0, T], \tag{2.131}$$

$$\Big(\tau_{I_v}(\cdot, t) \cdot \nabla\Big)(n_{I_v} \cdot v)(\cdot, t) = H_{\partial\Omega}(\cdot)(n_{I_v} \cdot v)(\cdot, t) \qquad \text{at } \mathcal{T}_c(t),\ t \in [0, T]. \tag{2.132}$$

Proof. The relations (2.131) are immediate from the choices made in the statement of Lemma 19. Let $\{c(t)\} = \mathcal{T}_c(t)$ for all $t \in [0,T]$. The compatibility condition (2.132) follows from differentiating in time the condition $n_{I_v}(c(t),t) = \tau_{\partial\Omega}(c(t))$. Indeed, one one side we may compute by means of the chain rule, the analogue of (2.130) for $\tau_{\partial\Omega}$, (2.131), and $\frac{\mathrm{d}}{\mathrm{d}t}c(t) = \big(n_{I_v}(c(t),t)\cdot v(c(t),t)\big)n_{I_v}(c(t),t)$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_{\partial\Omega}(c(t)) = H_{\partial\Omega}(c(t))\Big(n_{I_v}(c(t),t)\cdot v(c(t),t)\Big)n_{\partial\Omega}(c(t)).$$

On the other side, it follows from an application of the chain rule, the formula (2.129), the previous expression of $\frac{\mathrm{d}}{\mathrm{d}t}c(t)$, $\partial_t s_{\mathcal{T}_i}(\cdot,t) = -n_{I_v}(\cdot,t)\cdot v(P_{\mathcal{T}_i}(\cdot,t),t)$, as well as $n_{I_v} = \nabla s_{\mathcal{T}_i}$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}n_{I_v}(c(t),t) = -\Big(\tau_{I_v}(c(t),t)\cdot\nabla\Big)\Big(n_{I_v}\cdot v\Big)(c(t),t)\tau_{I_v}(c(t),t).$$

The second condition of (2.131) together with the previous two displays thus imply the compatibility condition (2.132) as asserted. □

2.5.2 Construction and properties of local building blocks

We have everything in place to proceed on with the first major step in the construction of a contact point extension in the sense of Proposition 16. We define auxiliary extensions $\xi^c_{\mathcal{T}_i}$ resp. $\xi^c_{\partial\Omega}$ of the unit normal vector field in the space-time domains $\mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i})$ resp. $\mathcal{N}_{r_c,c}(\Omega)\cap(\mathrm{im}(\Psi_{\partial\Omega})\times[0,T])$. In other words, we construct the extensions separately in the regions close to the interface or close to the boundary (but always near to the contact point).

Definition and regularity properties of local building blocks for the extension of the unit normal

A suitable ansatz for the two vector fields $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$ may be provided as follows.

Construction 21. *Let the assumptions and notation of Definition 13, Definition 17 and Lemma 19 be in place. Expressing $\{c(t)\} = \mathcal{T}_c(t)$ for all $t\in[0,T]$, we define coefficients*

$$\alpha_{\mathcal{T}_i}\colon \mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i})\to\mathbb{R},\quad (x,t)\mapsto -H_{\partial\Omega}(c(t),t), \tag{2.133}$$
$$\alpha_{\partial\Omega}\colon \mathcal{N}_{r_c,c}(\Omega)\cap(\mathrm{im}(\Psi_{\partial\Omega})\times[0,T])\to\mathbb{R},\quad (x,t)\mapsto -H_{I_v}(c(t),t). \tag{2.134}$$

Based on these coefficient functions, we then define extensions

$$\xi^c_{\mathcal{T}_i}\colon \mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i})\to\mathbb{R}^2,\quad \xi^c_{\partial\Omega}\colon \mathcal{N}_{r_c,c}(\Omega)\cap\Big(\mathrm{im}(\Psi_{\partial\Omega})\times[0,T]\Big)\to\mathbb{R}^2$$

of the normal vector field n_{I_v} by means of an expansion ansatz

$$\xi^c_{\mathcal{T}_i} := n_{I_v} + \alpha_{\mathcal{T}_i}s_{\mathcal{T}_i}\tau_{I_v} - \frac{1}{2}\alpha^2_{\mathcal{T}_i}s^2_{\mathcal{T}_i}n_{I_v}, \tag{2.135}$$
$$\xi^c_{\partial\Omega} := \tau_{\partial\Omega} + \alpha_{\partial\Omega}s_{\partial\Omega}n_{\partial\Omega} - \frac{1}{2}\alpha^2_{\partial\Omega}s^2_{\partial\Omega}\tau_{\partial\Omega}. \tag{2.136}$$

Regularity properties of $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$, in particular compatibility up to first order at the contact point, are the content of the following result.

Lemma 22. *Let the assumptions and notation of Construction 21 be in place. Then the auxiliary vector fields satisfy* $\xi^c_{\mathcal{T}_i} \in (C^0_t C^2_x \cap C^1_t C^0_x)(\overline{\mathcal{N}_{r_c,c}(\Omega) \cap \mathrm{im}(\Psi_{\mathcal{T}_i})})$ *and* $\xi^c_{\partial\Omega} \in (C^0_t C^2_x \cap C^1_t C^0_x)(\overline{\mathcal{N}_{r_c,c}(\Omega) \cap (\mathrm{im}(\Psi_{\partial\Omega}) \times [0,T])})$*, with corresponding estimates for* $k \in \{0,1,2\}$

$$|\nabla^k \xi^c_{\mathcal{T}_i}| + |\partial_t \xi^c_{\mathcal{T}_i}| \leq C, \quad \textit{on } \overline{\mathcal{N}_{r_c,c}(\Omega) \cap \mathrm{im}(\Psi_{\mathcal{T}_i})}, \tag{2.137}$$

$$|\nabla^k \xi^c_{\partial\Omega}| + |\partial_t \xi^c_{\partial\Omega}| \leq C, \quad \textit{on } \overline{\mathcal{N}_{r_c,c}(\Omega) \cap (\mathrm{im}(\Psi_{\partial\Omega}) \times [0,T])}. \tag{2.138}$$

Moreover, the constructions are compatible to first order at the contact point in the sense that

$$\xi^c_{\mathcal{T}_i}(\cdot,t) = \xi^c_{\partial\Omega}(\cdot,t), \quad \nabla \xi^c_{\mathcal{T}_i}(\cdot,t) = \nabla \xi^c_{\partial\Omega}(\cdot,t) \quad \textit{at } \mathcal{T}_c(t),\ t \in [0,T]. \tag{2.139}$$

Proof. Step 1 (Regularity estimates): Note first that $\alpha_{\mathcal{T}_i}, \alpha_{\partial\Omega} \in C^1_t([0,T])$ due to the regularity of the maps H_{I_v} resp. $H_{\partial\Omega}$ from (2.108) resp. (2.118). The asserted bounds (2.137) and (2.138) for the derivatives of the vector fields $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$ can thus be inferred from the definitions (2.135) and (2.136) in combination with the regularity of $s_{\mathcal{T}_i}, n_{I_v}$ from Definition 13, the regularity of $s_{\partial\Omega}, n_{\partial\Omega}$ from Definition 17, as well as the regularity of $\tau_{I_v}, \tau_{\partial\Omega}$ from Lemma 19.

Step 2 (First order compatibility at the contact point): The zeroth order condition of (2.139) is a direct consequence of the definitions (2.135) and (2.136) in combination with the compatibility condition (2.131). In order to prove the first order condition, it directly follows from (2.129)–(2.130) and their analogues for the frame $(n_{\partial\Omega}, \tau_{\partial\Omega})$, as well as the definitions (2.135) and (2.136) that

$$\nabla \xi^c_{\mathcal{T}_i} = -H_{I_v} \tau_{I_v} \otimes \tau_{I_v} + \alpha_{\mathcal{T}_i} \tau_{I_v} \otimes n_{I_v} + O(\mathrm{dist}(\cdot, \mathcal{T}_i)), \tag{2.140}$$

$$\nabla \xi^c_{\partial\Omega} = H_{\partial\Omega} n_{\partial\Omega} \otimes \tau_{\partial\Omega} + \alpha_{\partial\Omega} n_{\partial\Omega} \otimes n_{\partial\Omega} + O(\mathrm{dist}(\cdot, \partial\Omega)). \tag{2.141}$$

Finally, since we have (2.131) due to the conventions adopted, using (2.133) and (2.134) we can deduce the first order compatibility condition of (2.139). □

Evolution equations for local building blocks

The following lemma provides the approximate evolution equations for our local constructions $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$, which will eventually lead us to (2.113)–(2.114).

Lemma 23. *Let the assumptions and notation of Construction 21 be in place. Then it holds*

$$\partial_t \xi^c_{\mathcal{T}_i} + (v \cdot \nabla)\xi^c_{\mathcal{T}_i} + (\mathrm{Id} - \xi^c_{\mathcal{T}_i} \otimes \xi^c_{\mathcal{T}_i})(\nabla v)^{\mathsf{T}} \xi^c_{\mathcal{T}_i} = O(\mathrm{dist}(\cdot, \mathcal{T}_i)), \tag{2.142}$$

$$\partial_t |\xi^c_{\mathcal{T}_i}|^2 + (v \cdot \nabla)|\xi^c_{\mathcal{T}_i}|^2 = O(\mathrm{dist}^3(\cdot, \mathcal{T}_i)), \tag{2.143}$$

$$|1 - |\xi^c_{\mathcal{T}_i}|^2| = O(\mathrm{dist}^4(\cdot, \mathcal{T}_i)) \tag{2.144}$$

throughout the space-time domain $\mathcal{N}_{r_c,c}(\Omega) \cap \mathrm{im}(\Psi_{\mathcal{T}_i})$*. Moreover, we have*

$$\partial_t \xi^c_{\partial\Omega} + (v \cdot \nabla)\xi^c_{\partial\Omega} + (\mathrm{Id} - \xi^c_{\partial\Omega} \otimes \xi^c_{\partial\Omega})(\nabla v)^{\mathsf{T}} \xi^c_{\partial\Omega} = O(\mathrm{dist}(\cdot, \partial\Omega) \vee \mathrm{dist}(\cdot, \mathcal{T}_c)), \tag{2.145}$$

$$\partial_t |\xi^c_{\partial\Omega}|^2 + (v \cdot \nabla)|\xi^c_{\partial\Omega}|^2 = O(\mathrm{dist}^3(\cdot, \partial\Omega)), \tag{2.146}$$

$$|1 - |\xi^c_{\partial\Omega}|^2| = O(\mathrm{dist}^4(\cdot, \partial\Omega)) \tag{2.147}$$

throughout the space-time domain $\mathcal{N}_{r_c,c}(\Omega) \cap \Big(\mathrm{im}(\Psi_{\partial\Omega}) \times [0,T]\Big)$.

Proof. Step 1 (Proof of (2.142)*):* Note that because of the definitions (2.109) and (2.135), it holds $\xi^c_{\mathcal{T}_i} = \xi^i + \alpha_{\mathcal{T}_i} s_{\mathcal{T}_i} \tau_{I_v} - \frac{1}{2}\alpha^2_{\mathcal{T}_i} s^2_{\mathcal{T}_i} n_{I_v}$. Since we already proved (2.111), we only need to show that

$$\alpha_{I_v}(\partial_t s_{\mathcal{T}_i})\tau_{I_v} + \alpha_{I_v}(v\cdot\nabla s_{\mathcal{T}_i})\tau_{I_v} = O(\mathrm{dist}(\cdot,\mathcal{T}_i)).$$

However, the above relation is an immediate consequence of the identity $\partial_t s_{\mathcal{T}_i}(x,t) = -\big(v(P_{\mathcal{T}_i}(x,t),t)\cdot\nabla\big)s_{\mathcal{T}_i}(x,t)$ and the regularity of v, see Definition 10 of a strong solution, through a Lipschitz estimate. This proves (2.142).

Step 2 (Proof of (2.145)*):* From the definition (2.136) and $\alpha_{\partial\Omega}\in C^1_t([0,T])$ it directly follows

$$\partial_t\xi^c_{\partial\Omega} = (\partial_t\alpha_{\partial\Omega})s_{\partial\Omega}n_{\partial\Omega} = O(\mathrm{dist}(\cdot,\partial\Omega)).$$

Having $\xi^c_{\partial\Omega} = \tau_{\partial\Omega} + \alpha_{\partial\Omega}s_{\partial\Omega}n_{\partial\Omega} - \frac{1}{2}\alpha^2_{\partial\Omega}s^2_{\partial\Omega}\tau_{\partial\Omega}$, cf. the definition (2.136), it follows from $\nabla s_{\partial\Omega} = n_{\partial\Omega}$, the analogues of (2.129)–(2.130) for the frame $(n_{\partial\Omega},\tau_{\partial\Omega})$, as well as the boundary condition $v\cdot n_{\partial\Omega}=0$ along $\partial\Omega$ that

$$\begin{aligned}(v\cdot\nabla)\xi^c_{\partial\Omega} &= (v\cdot\nabla)(\tau_{\partial\Omega} + \alpha_{\partial\Omega}s_{\partial\Omega}n_{\partial\Omega}) + O(\mathrm{dist}(\cdot,\partial\Omega))\\ &= (v\cdot\tau_{\partial\Omega})\tau_{\partial\Omega}\cdot(H_{\partial\Omega}\tau_{\partial\Omega}\otimes n_{\partial\Omega} + \alpha_{\partial\Omega}n_{\partial\Omega}\otimes n_{\partial\Omega}) + O(\mathrm{dist}(\cdot,\partial\Omega))\\ &= (v\cdot\tau_{\partial\Omega})H_{\partial\Omega}n_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega)).\end{aligned}$$

Moreover, based on $\xi^c_{\partial\Omega} = \tau_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega))$ due to (2.136), $v(c(t),t) = \big(v(c(t),t)\cdot n_{I_v}(c(t),t)\big)n_{I_v}(c(t),t)$ along the moving contact point $\{c(t)\} = \mathcal{T}_c(t)$, the formula (2.129), and the compatibility conditions (2.131)–(2.132) we infer that

$$\begin{aligned}&(\mathrm{Id} - \xi^c_{\partial\Omega}\otimes\xi^c_{\partial\Omega})(\nabla v)^{\mathsf{T}}\xi^c_{\partial\Omega}\\ &= (\mathrm{Id} - \tau_{\partial\Omega}\otimes\tau_{\partial\Omega})(\nabla v)^{\mathsf{T}}\tau_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega))\\ &= (\tau_{\partial\Omega}\cdot(n_{\partial\Omega}\cdot\nabla)v)n_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega))\\ &= -\Big(n_{I_v}(c(t),t)\cdot\Big(\tau_{I_v}(c(t),t)\cdot\nabla\Big)v(c(t),t)\Big)n_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega)\vee\mathrm{dist}(\cdot,\mathcal{T}_c))\\ &= -\Big(\Big(\tau_{I_v}(c(t),t)\cdot\nabla\Big)(v\cdot n_{I_v})(c(t),t)\Big)n_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega)\vee\mathrm{dist}(\cdot,\mathcal{T}_c))\\ &= -(v\cdot\tau_{\partial\Omega})H_{\partial\Omega}n_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega)\vee\mathrm{dist}(\cdot,\mathcal{T}_c)).\end{aligned}$$

Hence, the estimate (2.145) follows as a consequence of the previous three displays.

Step 3 (Proof of (2.143)–(2.144) *and* (2.146)–(2.147)*):* Simply note that (2.143)–(2.144) as well as (2.146)–(2.147) directly follow from the definitions (2.135) resp. (2.136) of the vector field $\xi^c_{\mathcal{T}_i}$ resp. the vector field $\xi^c_{\partial\Omega}$ in form of

$$|\xi^c_{\mathcal{T}_i}|^2 = \Big(1-\frac{1}{2}\alpha^2_{\mathcal{T}_i}s^2_{\mathcal{T}_i}\Big)^2 + \alpha^2_{\mathcal{T}_i}s^2_{\mathcal{T}_i} = 1 + \frac{1}{4}\alpha^4_{\mathcal{T}_i}s^4_{\mathcal{T}_i}, \tag{2.148}$$

$$|\xi^c_{\partial\Omega}|^2 = \Big(1-\frac{1}{2}\alpha^2_{\partial\Omega}s^2_{\partial\Omega}\Big)^2 + \alpha^2_{\partial\Omega}s^2_{\partial\Omega} = 1 + \frac{1}{4}\alpha^4_{\partial\Omega}s^4_{\partial\Omega}. \tag{2.149}$$

This concludes the proof of Lemma 23. □

2.5.3 From building blocks to contact point extensions by interpolation

As we discussed in the previous subsections, the auxiliary vector fields $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$ provide main building block for a contact point extension of the interface unit normal near the connected interface $\mathcal{T}_i$ or near the domain boundary $\partial\Omega$, respectively. More precisely, we will make use of the auxiliary vector field $\xi^c_{\mathcal{T}_i}$ on the wedges $W^c_{\mathcal{T}_i} \cup W^c_{\Omega^+_v} \cup W^c_{\Omega^-_v}$, and of the auxiliary vector field $\xi^c_{\partial\Omega}$ on the wedges $W^{+,c}_{\partial\Omega} \cup W^{-,c}_{\partial\Omega} \cup W^c_{\Omega^+_v} \cup W^c_{\Omega^-_v}$. Note that this is indeed admissible thanks to the inclusions (2.124), (2.126) and (2.127). As the domains of definition for the auxiliary vector fields overlap, we adopt an interpolation procedure on the interpolation wedges $W^c_{\Omega^\pm_v}$. To this end, we first define suitable interpolation functions.

Lemma 24. *Let the assumptions and notation of Definition 17 be in place. Then there exists a pair of interpolation functions*

$$\lambda^\pm_c \colon \bigcup_{t\in[0,T]} \Big(W^c_{\Omega^\pm_v}(t) \setminus \mathcal{T}_c(t)\Big)\times\{t\} \to [0,1]$$

which satisfies the following list of properties:

i) On the boundary of the interpolation wedges $W^c_{\Omega^\pm_v}$ intersected with $B_{r_c}(\mathcal{T}_c)$, the values of $\lambda^\pm_c$ and its derivatives up to second order are given by

$$\lambda^\pm_c(\cdot,t) = 0 \qquad \text{on } \Big(\partial W^c_{\Omega^\pm_v}(t) \cap \partial W^{\pm,c}_{\partial\Omega}(t)\Big) \setminus \mathcal{T}_c(t), \tag{2.150}$$

$$\lambda^\pm_c(\cdot,t) = 1 \qquad \text{on } \Big(\partial W^c_{\Omega^\pm_v}(t) \cap \partial W^c_{\mathcal{T}_i}(t)\Big) \setminus \mathcal{T}_c(t), \tag{2.151}$$

$$\nabla\lambda^\pm_c(\cdot,t) = 0, \qquad \text{on } \Big(\partial W^c_{\Omega^\pm_v}(t) \cap B_{r_c}(\mathcal{T}_c(t))\Big) \setminus \mathcal{T}_c(t), \tag{2.152}$$

$$\nabla^2\lambda^\pm_c(\cdot,t) = 0,\ \partial_t\lambda^\pm_c(\cdot,t) = 0 \qquad \text{on } \Big(\partial W^c_{\Omega^\pm_v}(t) \cap B_{r_c}(\mathcal{T}_c(t))\Big) \setminus \mathcal{T}_c(t) \tag{2.153}$$

for all $t \in [0,T]$.

ii) There exists a constant C such that the estimates

$$|\partial_t\lambda^\pm_c(\cdot,t)| + |\nabla\lambda^\pm_c(\cdot,t)| \leq C|\operatorname{dist}(\cdot,\mathcal{T}_c(t))|^{-1}, \tag{2.154}$$

$$|\nabla\partial_t\lambda^\pm_c(\cdot,t)| + |\nabla^2\lambda^\pm_c(\cdot,t)| \leq C|\operatorname{dist}(\cdot,\mathcal{T}_c(t))|^{-2} \tag{2.155}$$

hold true on $W^c_{\Omega^\pm_v}(t) \setminus \mathcal{T}_c(t)$ for all $t \in [0,T]$.

iii) We have an improved estimate on the advective derivative in form of

$$\Big|\partial_t\lambda^\pm_c(\cdot,t) + \Big(v\cdot\nabla\Big)\lambda^\pm_c(\cdot,t)\Big| \leq C \tag{2.156}$$

on $W^c_{\Omega^\pm_v}(t) \setminus \mathcal{T}_c(t)$ for all $t \in [0,T]$.

Proof. We fix a smooth function $\widetilde{\lambda} \colon \mathbb{R} \to [0,1]$ such that $\widetilde{\lambda} \equiv 0$ on $[\frac{2}{3},\infty)$ and $\widetilde{\lambda} \equiv 1$ on $(-\infty,\frac{1}{3}]$. Recall the representation (2.120) of the interpolation wedges $W_{\Omega^\pm_v}$, and that their opening angle is determined via $X^\pm_{\mathcal{T}_i} \cdot X_{\Omega^\pm_v} = \cos(\pi/6)$ along $\mathcal{T}_c$, see (2.122). We then define a function $\lambda \colon [-1,1] \to [0,1]$ by $\lambda(u) := \widetilde{\lambda}(\frac{1-u}{1-\cos(\pi/6)})$, and set

$$\lambda^\pm_c(x,t) := \lambda\bigg(X^\pm_{\mathcal{T}_i}(t)\cdot\frac{x-c(t)}{|x-c(t)|}\bigg), \quad t\in[0,T],\ x \in W_{\Omega^\pm_v}(t)\setminus\mathcal{T}_c(t).$$

The assertions of the first two items of Lemma 24 are now immediate consequences of the definitions due to $\frac{\mathrm{d}}{\mathrm{d}t}X^{\pm}_{\mathcal{T}_i} \in C^0([0,T])$, cf. Definition 17.

It remains to prove the estimate (2.156) on the advective derivative. To this end, abbreviating $u^{\pm} := X^{\pm}_{\mathcal{T}_i}(t) \cdot \frac{x-c(t)}{|x-c(t)|}$ we compute

$$
\begin{aligned}
\partial_t \lambda^{\pm}_c(x,t) &= \lambda'(u^{\pm}) X^{\pm}_{\mathcal{T}_i}(t) \cdot \partial_t \frac{x-c(t)}{|x-c(t)|} + \lambda'(u^{\pm}) \frac{x-c(t)}{|x-c(t)|} \cdot \frac{\mathrm{d}}{\mathrm{d}t} X^{\pm}_{\mathcal{T}_i}(t) \\
&= \lambda'(u^{\pm}) X^{\pm}_{\mathcal{T}_i}(t) \cdot \frac{1}{|x-c(t)|}\Big(\mathrm{Id} - \frac{x-c(t)}{|x-c(t)|} \otimes \frac{x-c(t)}{|x-c(t)|}\Big)\frac{\mathrm{d}}{\mathrm{d}t}c(t) \\
&\quad + \lambda'(u^{\pm}) \frac{x-c(t)}{|x-c(t)|} \cdot \frac{\mathrm{d}}{\mathrm{d}t} X^{\pm}_{\mathcal{T}_i}(t) \\
&= -\Big(\frac{\mathrm{d}}{\mathrm{d}t}c(t) \cdot \nabla\Big)\lambda^{\pm}_c(x,t) + \lambda'(u^{\pm}) \frac{x-c(t)}{|x-c(t)|} \cdot \frac{\mathrm{d}}{\mathrm{d}t} X^{\pm}_{\mathcal{T}_i}(t).
\end{aligned}
$$

This in turn yields the asserted estimate (2.156) due to $\frac{\mathrm{d}}{\mathrm{d}t}X^{\pm}_{\mathcal{T}_i} \in C^0([0,T])$, cf. Definition 17, $\frac{\mathrm{d}}{\mathrm{d}t}c(t) = v(c(t),t)$, and a Lipschitz estimate based on the regularity of the fluid velocity v from Definition 10 (which counteracts the blow-up (2.154) of $\nabla\lambda^{\pm}_c$). This concludes the proof. □

We have by now everything in place to state the definition of a vector field which in the end will give rise to a contact point extension of the interface unit normal in the precise sense of Proposition 16.

Construction 25. *Let the assumptions and notation of Definition 17, Construction 21 and Lemma 24 be in place. In particular, let $r_c \in (0,1]$ be an admissible localization radius for the contact point $\mathcal{T}_c$. We define a vector field*

$$
\widehat{\xi}^c \colon \mathcal{N}_{r_c,c}(\Omega) \to \mathbb{R}^2
$$

on the space-time domain $\mathcal{N}_{r_c,c}(\Omega) := \bigcup_{t\in[0,T]} \Big(B_{r_c}(\mathcal{T}_c(t)) \cap \Omega\Big) \times \{t\}$ as follows (recall the decomposition (2.123) *of the neighborhood $B_r(\mathcal{T}_c(t)) \cap \overline{\Omega}$):*

$$
\widehat{\xi}^c(\cdot,t) := \begin{cases} \xi^c_{\mathcal{T}_i}(\cdot,t) & \text{on } W^c_{\mathcal{T}_i}(t) \cap \overline{\Omega}, \\ \xi^c_{\partial\Omega}(\cdot,t) & \text{on } W^{\pm,c}_{\partial\Omega}(t) \cap \overline{\Omega}, \\ \lambda^{\pm}_c(\cdot,t)\xi^c_{\mathcal{T}_i}(\cdot,t) + \Big(1-\lambda^{\pm}_c(\cdot,t)\Big)\xi^c_{\partial\Omega}(\cdot,t) & \text{on } W_{\Omega^{\pm}_v}(t) \setminus \mathcal{T}_c(t) \cap \overline{\Omega}, \end{cases} \tag{2.157}
$$

for all $t \in [0,T]$. Note that the vector field $\widehat{\xi}^c$ is not yet normalized to unit length, which is the reason for denoting it by $\widehat{\xi}^c$ instead of ξ^c. Observe also that (2.157) *is well-defined in view of the inclusions* (2.124), (2.126) *and* (2.127).

2.5.4 Proof of Proposition 16

The proof proceeds in several steps. We first establish the required properties in terms of the vector field $\widehat{\xi}^c$. The penultimate step is devoted to fixing $\widehat{r}_c \in (0, r_c]$ such that $\big|\widehat{\xi}^c\big| \geq \frac{1}{2}$ on $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$, so that one may define $\xi := \big|\widehat{\xi}^c\big|^{-1}\widehat{\xi}^c \in \mathbb{S}^1$ throughout $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$ and transfer the properties of $\widehat{\xi}^c$ to ξ^c. Finally, in the last step we verify the asserted compatibility conditions between a contact point extension and a bulk extension of the interface unit normal.

Step 1: Regularity of $\widehat{\xi}^c$ and properties i)–iii). Because of the inclusion (2.124) as well as the definitions (2.135) and (2.157), it follows that $\widehat{\xi}^c(\cdot,t) = n_{I_v}(\cdot,t)$ along $\mathcal{T}_i(t) \cap B_{r_c}(\mathcal{T}_c(t))$ for all $t \in [0,T]$. By the same reasons, relying also on $\xi^c_{\mathcal{T}_i} = \xi^i + \alpha_{\mathcal{T}_i} s_{\mathcal{T}_i} \tau_{I_v} - \frac{1}{2}\alpha^2_{\mathcal{T}_i} s^2_{\mathcal{T}_i} n_{I_v}$, cf. the definitions (2.109) and (2.135), $\nabla s_{\mathcal{T}_i} = n_{I_v}$ and (2.110), we deduce that $\nabla \cdot \widehat{\xi}^c(\cdot,t) = -H_{I_v}(\cdot,t)$ along $\mathcal{T}_i(t) \cap B_{r_c}(\mathcal{T}_c(t))$ for all $t \in [0,T]$. Moreover, in view of the inclusion (2.125) as well as the definitions (2.136) and (2.157), we obtain $\widetilde{\xi}^c(\cdot,t) \cdot n_{\partial\Omega} = \tau_{\partial\Omega} \cdot n_{\partial\Omega} = 0$ along $B_{r_c}(\mathcal{T}_c(t)) \cap \partial\Omega$. This yields the asserted properties i)–iii) of a contact point extension in terms of $\widehat{\xi}^c$ on scale r_c.

The vector fields $\widehat{\xi}^c$, $\partial_t\widehat{\xi}^c$, $\nabla\widehat{\xi}^c$ and $\nabla^2\widehat{\xi}^c$ exist in a pointwise sense and are continuous throughout $\mathcal{N}_{r_c,c}(\Omega) \setminus \mathcal{T}_c$ due to the definition (2.157) of $\widehat{\xi}^c$, the regularity of the local building blocks $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$ as provided by Lemma 22, as well as the regularity of the interpolation parameter $\lambda^\pm_c$ from Lemma 24. Note in this context that no jumps occur across the boundaries of the interpolation wedges as a consequence of the conditions (2.150)–(2.153). It remains to prove the bounds

$$|\partial_t\widehat{\xi}^c(\cdot,t)| + |\nabla^k\widehat{\xi}^c(\cdot,t)| \leq C \quad \text{on } \Big(\overline{B_{r_c}(\mathcal{T}_c(t))} \setminus \mathcal{T}_c\Big) \cap \overline{\Omega} \tag{2.158}$$

for $k \in \{0,1,2\}$, for all $t \in [0,T]$ and some constant $C > 0$.

In the wedges $W^c_{\mathcal{T}_i}$ and $W^{\pm,c}_{\partial\Omega}$ containing the interface or the boundary of the domain, respectively, the estimate follows directly from the estimates (2.137)–(2.138) and the definition (2.157). On interpolation wedges $W^c_{\Omega^\pm_v}$, we compute recalling (2.157)

$$\begin{aligned}
\partial_t\widehat{\xi}^c &= \lambda^\pm_c \partial_t\xi^c_{\mathcal{T}_i} + (1-\lambda^\pm_c)\partial_t\xi^c_{\partial\Omega} + (\xi^c_{\mathcal{T}_i}-\xi^c_{\partial\Omega})\partial_t\lambda^\pm_c \\
\nabla\widehat{\xi}^c &= \lambda^\pm_c \nabla\xi^c_{\mathcal{T}_i} + (1-\lambda^\pm_c)\nabla\xi^c_{\partial\Omega} + (\xi^c_{\mathcal{T}_i}-\xi^c_{\partial\Omega}) \otimes \nabla\lambda^\pm_c, \\
\nabla^2\widehat{\xi}^c &= \lambda^\pm_c \nabla^2\xi^c_{\mathcal{T}_i} + (1-\lambda^\pm_c)\nabla^2\xi^c_{\partial\Omega} + (\nabla\lambda^\pm_c \otimes \nabla^{\mathrm{sym}})(\xi^c_{\mathcal{T}_i}-\xi^c_{\partial\Omega}) + (\xi^c_{\mathcal{T}_i}-\xi^c_{\partial\Omega}) \otimes \nabla^2\lambda^\pm_c.
\end{aligned}$$

Then we recall the bounds (2.154) and (2.155) for the derivatives of the interpolation functions, the estimates (2.137) and (2.138) as well as the compatibility conditions (2.139) for the auxiliary vector fields $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$. Feeding these into the previous display establishes (2.158) on the interpolation wedges.

Step 2: Evolution equation in terms of $\widehat{\xi}^c$. We claim that

$$\partial_t\widehat{\xi}^c + (v \cdot \nabla)\widehat{\xi}^c + (\nabla v)^{\mathsf{T}}\widehat{\xi}^c = O(\mathrm{dist}(\cdot,\mathcal{T}_i)) \quad \text{in } \mathcal{N}_{r_c,c}(\Omega). \tag{2.159}$$

The validity of (2.159) on the wedges $W^c_{\mathcal{T}_i}$ and $W^{\pm,c}_{\partial\Omega}$ follows directly from the estimates (2.142) resp. (2.145), the definition (2.157) and the bound (2.128). Hence, we only need to prove the bound (2.159) on the interpolation wedges $W^c_{\Omega^\pm_v}$.

To this end, recall first that on the interpolation wedges $W^c_{\Omega^\pm_v}$ the distance with respect to the contact point $\mathcal{T}_c$ or the distance with respect to the domain boundary $\partial\Omega$ is dominated by the distance to the connected interface $\mathcal{T}_i$, see (2.128). Writing $\widehat{\xi}^c = \xi^c_{\mathcal{T}_i} + (1-\lambda^\pm_c)(\xi^c_{\partial\Omega}-\xi^c_{\mathcal{T}_i})$, and resp. $\widehat{\xi}^c = \xi^c_{\partial\Omega} + \lambda^\pm_c(\xi^c_{I_v}-\xi^c_{\partial\Omega})$, we then immediately see that

$$\widehat{\xi}^c \otimes \widehat{\xi}^c = \xi^c_{\mathcal{T}_i} \otimes \xi^c_{\mathcal{T}_i} + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)), \tag{2.160}$$

$$\widehat{\xi}^c \otimes \widehat{\xi}^c = \xi^c_{\partial\Omega} \otimes \xi^c_{\partial\Omega} + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)), \tag{2.161}$$

due to compatibility (2.139) up to first order at the contact point $\mathcal{T}_c$, and the regularity estimates (2.137)–(2.138). Using the product rule and the definition (2.157) of $\widehat{\xi}^c$ on $W^c_{\Omega_v^\pm}$, we thus obtain

$$\begin{aligned}
&\partial_t\widehat{\xi}^c + (v\cdot\nabla)\widehat{\xi}^c + (\mathrm{Id}-\widehat{\xi}^c\otimes\widehat{\xi}^c)(\nabla v)^{\mathsf{T}}\widehat{\xi}^c \\
&= \lambda_c^\pm\Big(\partial_t + (v\cdot\nabla) + (\mathrm{Id}-\xi^c_{\mathcal{T}_i}\otimes\xi^c_{\mathcal{T}_i})(\nabla v)^{\mathsf{T}}\Big)\xi^c_{\mathcal{T}_i} \\
&\quad + (1-\lambda_c^\pm)\Big(\partial_t + (v\cdot\nabla) + (\mathrm{Id}-\xi^c_{\partial\Omega}\otimes\xi^c_{\partial\Omega})(\nabla v)^{\mathsf{T}}\Big)\xi^c_{\partial\Omega} \\
&\quad + (\partial_t\lambda_c^\pm + (v\cdot\nabla)\lambda_c^\pm)(\xi^c_{\mathcal{T}_i}-\xi^c_{\partial\Omega}) + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)).
\end{aligned} \tag{2.162}$$

Hence, we obtain (2.159) on interpolation wedges as a consequence of the estimates (2.142) resp. (2.145), the bound (2.156) on the advective derivative of the interpolation parameter, as well as the compatibility condition (2.139).

Step 3: We next claim that

$$\partial_t\Big|\widehat{\xi}^c\Big|^2 + (v\cdot\nabla)\Big|\widehat{\xi}^c\Big|^2 = O(\mathrm{dist}(\cdot,\mathcal{T}_i)) \quad \text{in } \mathcal{N}_{r_c,c}(\Omega), \tag{2.163}$$

$$\Big|\nabla|\widehat{\xi}^c\Big|^2\Big| = O(\mathrm{dist}(\cdot,\mathcal{T}_i)) \quad \text{in } \mathcal{N}_{r_c,c}(\Omega). \tag{2.164}$$

Outside of interpolation wedges, both claims are already established in view of the estimates (2.143)–(2.144) resp. (2.146)–(2.147), the estimate (2.128) as well as the definition (2.157). Using the latter, we may compute on interpolation wedges $W^c_{\Omega_v^\pm}$

$$\begin{aligned}
|\widehat{\xi}^c|^2 - 1 = {\lambda_c^\pm}^2(|\xi^c_{\mathcal{T}_i}|^2-1) + (1-\lambda_c^\pm)^2(|\xi^c_{\partial\Omega}|^2-1) \\
+ 2\lambda_c^\pm(1-\lambda_c^\pm)(\xi^c_{\mathcal{T}_i}\cdot\xi^c_{\partial\Omega}-1),
\end{aligned} \tag{2.165}$$

and thus

$$\begin{aligned}
\Big(\partial_t+(v\cdot\nabla)\Big)\Big|\widehat{\xi}^c\Big|^2 &= \Big(\partial_t+(v\cdot\nabla)\Big)\Big((\lambda_c^\pm)^2|\xi^c_{\mathcal{T}_i}|^2+(1-\lambda_c^\pm)^2|\xi^c_{\partial\Omega}|^2 + 2\lambda_c^\pm(1-\lambda_c^\pm)\Big) \\
&\quad + (\xi^c_{\mathcal{T}_i}\cdot\xi^c_{\partial\Omega}-1)\Big(\partial_t+(v\cdot\nabla)\Big)\Big(2\lambda_c^\pm(1-\lambda_c^\pm)\Big) \\
&\quad + 2\lambda_c^\pm(1-\lambda_c^\pm)\Big(\partial_t+(v\cdot\nabla)\Big)(\xi^c_{\mathcal{T}_i}\cdot\xi^c_{\partial\Omega}-1).
\end{aligned} \tag{2.166}$$

Because of (2.143)–(2.144) and (2.146)–(2.147), the first right hand side term of (2.166) is of required order. For an estimate of the second and third right hand side term of (2.166), observe that it suffices to prove $\xi^c_{\mathcal{T}_i}\cdot\xi^c_{\partial\Omega}-1 = O(\mathrm{dist}^2(\cdot,\mathcal{T}_i))$ on interpolation wedges as the advective derivative of the interpolation parameter is bounded, see (2.156). However, it follows immediately from the definitions (2.135) and (2.136), the formulas (2.140) and (2.141), as well as the compatibility condition (2.139), that at the contact point $\mathcal{T}_c$ it holds $\xi^c_{\mathcal{T}_i}\cdot\xi^c_{\partial\Omega} = 1$, $(\nabla\xi^c_{\mathcal{T}_i})^{\mathsf{T}}\xi^c_{\partial\Omega} = 0$ and $(\nabla\xi^c_{\partial\Omega})^{\mathsf{T}}\xi^c_{\mathcal{T}_i} = 0$. Hence, $\xi^c_{\mathcal{T}_i}\cdot\xi^c_{\partial\Omega}-1 = O(\mathrm{dist}^2(\cdot,\mathcal{T}_i))$ is a consequence of a Lipschitz estimate making use of the estimates (2.137)–(2.138) and the bound (2.128).

In summary, the above arguments upgrade (2.166) to (2.163), and analogous considerations based on (2.165) also entail (2.164) on interpolation wedges.

Step 4: Choice of $\widehat{r}_c$ and definition of the normalized vector field ξ^c. By the definition (2.157) of the vector field $\widehat{\xi}^c$ we have $|\widehat{\xi}^c(\cdot,t)| = 1$ on $B_{r_c}(\mathcal{T}_c(t))\cap(\partial\Omega\cup\mathcal{T}_i(t))$ for all $t\in[0,T]$. Due to its Lipschitz continuity, see *Step 1* of the proof, we may choose a radius $\widehat{r}_c \leq r_c$ such that $|\widehat{\xi}^c| \geq \frac{1}{2}$ holds true in the space-time domain $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$. We then define $\xi^c := \left|\widehat{\xi}^c\right|^{-1}\widehat{\xi}^c \in \mathbb{S}^1$ throughout $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$, so that it remains to argue that the properties of $\widehat{\xi}^c$ are inherited by ξ^c.

Since $\xi^c(\cdot,t) = \widehat{\xi}^c(\cdot,t)$ on $B_{r_c}(\mathcal{T}_c(t))\cap(\partial\Omega\cup\mathcal{T}_i(t))$ for all $t\in[0,T]$, it immediately follows that $\xi^c(\cdot,t) = n_{I_v}(\cdot,t)$ along $\mathcal{T}_i(t)\cap B_{\widehat{r}_c}(\mathcal{T}_c(t))$ as well as $\xi^c(\cdot,t)\cdot n_{\partial\Omega}(\cdot) = 0$ along $\partial\Omega\cap B_{\widehat{r}_c}(\mathcal{T}_c(t))$ for all $t\in[0,T]$. Moreover, $\nabla\cdot\xi^c = |\widehat{\xi}^c|^{-1}\nabla\cdot\widehat{\xi}^c - \frac{(\widehat{\xi}^c\cdot\nabla)|\widehat{\xi}^c|^2}{2|\widehat{\xi}^c|^3}$ so that $\nabla\cdot\xi^c = -H_{I_v}(\cdot,t)$ holds true on $\mathcal{T}_i(t)\cap B_{\widehat{r}_c}(\mathcal{T}_c(t))$ for all $t\in[0,T]$ because of (2.164), the validity of this equation in terms of $\widehat{\xi}^c$, and the fact that $|\widehat{\xi}^c(\cdot,t)| = 1$ on $\mathcal{T}_i(t)\cap B_{\widehat{r}_c}(\mathcal{T}_c(t))$ for all $t\in[0,T]$. In summary, properties *ii)–iii)* are satisfied.

The required regularity is obtained by the choice of the radius $\widehat{r}_c$, the definition $\xi^c := \left|\widehat{\xi}^c\right|^{-1}\widehat{\xi}^c$, and the fact that the vector field $\widehat{\xi}^c$ already satisfies it as argued in *Step 1* of this proof. Since $\xi^c\in\mathbb{S}^1$ throughout $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$, (2.114) holds true for trivial reasons. For a proof of (2.113), one may argue as follows. Recalling that $|\widehat{\xi}^c| \geq \frac{1}{2}$ holds true in $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$, adding zero and using the product rule yields

$$
\begin{aligned}
&\partial_t\xi^c + (v\cdot\nabla)\xi^c + (\mathrm{Id}-\xi^c\otimes\xi^c)(\nabla v)^{\mathsf{T}}\xi^c \\
&= \partial_t\xi^c + (v\cdot\nabla)\xi^c + (\mathrm{Id}-\widehat{\xi}^c\otimes\widehat{\xi}^c)(\nabla v)^{\mathsf{T}}\xi^c - (1-|\widehat{\xi}^c|^2)(\xi^c\otimes\xi^c)(\nabla v)^{\mathsf{T}}\xi^c \\
&= \frac{1}{|\widehat{\xi}^c|}\Big(\partial_t\widehat{\xi}^c + (v\cdot\nabla)\widehat{\xi}^c + (\mathrm{Id}-\widehat{\xi}^c\otimes\widehat{\xi}^c)(\nabla v)^{\mathsf{T}}\widehat{\xi}^c\Big) - \frac{\widehat{\xi}^c}{2|\widehat{\xi}^c|^3}(\partial_t|\widehat{\xi}^c|^2 + (v\cdot\nabla)|\widehat{\xi}^c|^2) \\
&\quad - (1-|\widehat{\xi}^c|^2)(\xi^c\otimes\xi^c)(\nabla v)^{\mathsf{T}}\xi^c
\end{aligned}
$$

throughout $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$. Observe that the first right hand side term is estimated by (2.159), the second by (2.163), and the third by a Lipschitz estimate based on the fact $|\widehat{\xi}^c(\cdot,t)| = 1$ along $\mathcal{T}_i(t)\cap B_{\widehat{r}_c}(\mathcal{T}_c(t))$ for all $t\in[0,T]$. Hence, (2.113) holds true.

Step 5: Contact point extensions as perturbations of bulk extensions. As a preparation for the proof of the compatability estimates, we claim that

$$
|\xi^c - \widehat{\xi}^c| \leq C\,\mathrm{dist}^2(\cdot,\mathcal{T}_i). \tag{2.167}
$$

Note that because of the definition (2.157), the compatibility conditions (2.139) at the contact point, the regularity estimates (2.137)–(2.138) for the local building blocks, the controlled blow-up (2.154), the coercivity estimate (2.144), and the estimate (2.128), it holds

$$
\begin{aligned}
\nabla\frac{1}{|\widehat{\xi}^c|} = -\frac{(\widehat{\xi}^c\cdot\nabla)\widehat{\xi}^c}{|\widehat{\xi}^c|^3} &= -\frac{(\xi^c_{\mathcal{T}_i}\cdot\nabla)\widehat{\xi}^c}{|\widehat{\xi}^c|^3} + O(\mathrm{dist}(\cdot,\mathcal{T}_i)) \\
&= -\frac{(\xi^c_{\mathcal{T}_i}\cdot\nabla)\xi^c_{\mathcal{T}_i}}{|\widehat{\xi}^c|^3} + O(\mathrm{dist}(\cdot,\mathcal{T}_i)) = O(\mathrm{dist}(\cdot,\mathcal{T}_i)).
\end{aligned}
$$

Hence, the asserted estimate (2.167) follows from $\xi^c - \widehat{\xi}^c = (|\widehat{\xi}^c|^{-1}-1)\widehat{\xi}^c$, the fact that $\xi^c(\cdot,t) = \widehat{\xi}^c(\cdot,t) \equiv n_{I_v}(\cdot,t)$ along the local interface patch $\mathcal{T}_i(t)\cap B_{\widehat{r}^c}(\mathcal{T}_c(t))$ for all $t\in[0,T]$, and the previous display.

We exploit (2.167) as follows. Within the interface wedge $W^c_{\mathcal{T}_i}$, it now follows from the definitions (2.109), (2.135) and (2.157) that

$$
\xi^c - \xi^i = \xi^c_{\mathcal{T}_i} - \xi^i + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)) = \alpha_{\mathcal{T}_i}s_{\mathcal{T}_i}\tau_{I_v} - \frac{1}{2}\alpha^2_{\mathcal{T}_i}s^2_{\mathcal{T}_i}n_{I_v} + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)).
$$

Within interpolation wedges, we have the same representation thanks to the first-order compatibility (2.139) in form of

$$\begin{aligned}\xi^c - \xi^i &= \widehat{\xi}^c - \xi^i + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)) \\ &= (\xi^c_{\mathcal{T}_i} - \xi^i) + (1-\lambda_c^{\pm})(\xi^c_{\partial\Omega} - \xi^c_{\mathcal{T}_i}) + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)) \\ &= \alpha_{\mathcal{T}_i} s_{\mathcal{T}_i} \tau_{I_v} - \frac{1}{2}\alpha^2_{\mathcal{T}_i} s^2_{\mathcal{T}_i} n_{I_v} + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)).\end{aligned}$$

In particular, the compatibility bounds (2.115) and (2.116) are satisfied within interface and interpolation wedges, respectively. □

2.6 Existence of boundary adapted extensions of the unit normal

2.6.1 From local to global extensions

The idea for proving Proposition 7 consists of stitching together the local extensions from the previous two sections by means of a suitable partition of unity on the interface I_v. For a construction of the latter, recall first the decomposition of the interface I_v into its topological features, namely, the connected components of $I_v \cap \Omega$ and the connected components of $I_v \cap \partial\Omega$. Denoting by $N \in \mathbb{N}$ the total number of such topological features present in the interface I_v we split $\{1,\ldots,N\} =: \mathcal{I} \cup \mathcal{C}$ by means of two disjoint subsets. Here, the subset $\mathcal{I}$ enumerates the space-time connected components of $I_v \cap \Omega$ (being time-evolving connected *interfaces*), whereas the subset $\mathcal{C}$ enumerates the space-time connected components of $I_v \cap \partial\Omega$ (being time-evolving *contact points*). If $i \in \mathcal{I}$, we let $\mathcal{T}_i \subset I_v$ denote the space-time trajectory in Ω of the corresponding connected interface. Furthermore, for every $c \in \mathcal{C}$ we write $\mathcal{T}_c$ representing the space-time trajectory in $\partial\Omega$ of the corresponding contact point. Finally, let us write $i \sim c$ for $i \in \mathcal{I}$ and $c \in \mathcal{C}$ if and only if $\mathcal{T}_i$ ends at $\mathcal{T}_c$; otherwise $i \not\sim c$.

Lemma 26 (Construction of a partition of unity)**.** *Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. For each $i \in \mathcal{I}$ let r_i be the localization radius of Definition 13, and for each $c \in \mathcal{C}$ denote by $\widehat{r}_c$ the localization radius of Proposition 16. There then exists a family $(\eta_1,\ldots,\eta_N)$ of cutoff functions*

$$\eta_n \colon \mathbb{R}^2 \times [0,T] \to [0,1], \quad n \in \{1,\ldots,N\},$$
$$\textit{with the regularity } \eta_n \in (C^0_t C^2_x \cap C^1_t C^0_x)\Big(\mathbb{R}^2 \times [0,T] \setminus \bigcup_{c\in\mathcal{C}} \mathcal{T}_c\Big), \tag{2.168}$$

and a localization radius $\widehat{r} \in (0, \min_{i\in\mathcal{I}} r_i \wedge \min_{c\in\mathcal{C}} \widehat{r}_c)$, which together are subject to the following list of conditions:

- *The family $(\eta_1,\ldots,\eta_N)$ is a partition of unity along the interface I_v. Defining a bulk cutoff by means of $\eta_{\mathrm{bulk}} := 1 - \sum_{n=1}^N \eta_n$, it holds $\eta_{\mathrm{bulk}} \in [0,1]$. On top we have coercivity estimates in form of*

$$\frac{1}{C}(\mathrm{dist}^2(\cdot,I_v)\wedge 1) \le \eta_{\mathrm{bulk}} \le C(\mathrm{dist}^2(\cdot,I_v)\wedge 1) \qquad \textit{in } \mathbb{R}^2 \times [0,T], \tag{2.169}$$
$$|\nabla \eta_{\mathrm{bulk}}| \le C(\mathrm{dist}(\cdot,I_v)\wedge 1) \qquad \textit{in } \mathbb{R}^2 \times [0,T], \tag{2.170}$$

- *For all two-phase interfaces $i \in \mathcal{I}$ it holds*

$$\operatorname{supp} \eta_i(\cdot,t) \subset \Psi_{\mathcal{T}_i}(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]) \quad \textit{for all } t \in [0,T], \tag{2.171}$$

with $\Psi_{\mathcal{T}_i}$ denoting the change of variables from Definition 13. For contact points $c \in \mathcal{C}$, it is required that

$$\operatorname{supp} \eta_c(\cdot,t) \subset B_{\widehat{r}}\big(\mathcal{T}_c(t)\big) \quad \textit{for all } t \in [0,T]. \tag{2.172}$$

- *For all distinct two-phase interfaces $i, i' \in \mathcal{I}$ it holds*

$$\operatorname{supp} \eta_i(\cdot,t) \cap \operatorname{supp} \eta_{i'}(\cdot,t) = \emptyset \quad \textit{for all } t \in [0,T]. \tag{2.173}$$

The same is required for all distinct contact points $c, c' \in \mathcal{I}$

$$\operatorname{supp} \eta_c(\cdot,t) \cap \operatorname{supp} \eta_{c'}(\cdot,t) = \emptyset \quad \textit{for all } t \in [0,T]. \tag{2.174}$$

- *Let a two-phase interface $i \in \mathcal{I}$ and a contact point $c \in \mathcal{C}$ be fixed. Then* $\operatorname{supp} \eta_i \cap \operatorname{supp} \eta_c \neq \emptyset$ *if and only if $i \sim c$, and in that case it holds*

$$\operatorname{supp} \eta_i(\cdot,t) \cap \operatorname{supp} \eta_c(\cdot,t) \subset B_{\widehat{r}}(\mathcal{T}_c(t)) \cap \Big(W^c_{\mathcal{T}_i}(t) \cup W^c_{\Omega^\pm_v}(t)\Big) \tag{2.175}$$

for all $t \in [0,T]$, with the wedges $W^c_{\mathcal{T}_i}$ and $W^c_{\Omega^\pm_v}$ introduced in Definition 17.

Proof. The proof proceeds in several steps.

Step 1: (Definition of auxiliary cutoff functions) Fix a smooth cutoff function $\theta\colon \mathbb{R} \to [0,1]$ with the properties that $\theta(r) = 1$ for $|r| \leq \frac{1}{2}$ and $\theta(r) = 0$ for $|r| \geq 1$. Define

$$\zeta(r) := (1-r^2)\theta(r^2), \quad r \in \mathbb{R}. \tag{2.176}$$

Based on this quadratic profile, we may introduce two classes of cutoff functions associated to the two different natures of topological features present in the interface I_v. To this end, let $\widehat{r} \in (0, \min_{i\in\mathcal{I}} r_i \wedge \min_{c\in\mathcal{C}} \widehat{r}_c)$. Moreover, let $\delta \in (0,1]$ be a constant. Both constants $\widehat{r}$ and δ will be determined in the course of the proof.

For two-phase interfaces $\mathcal{T}_i \subset I_v$, $i \in \mathcal{I}$, we may then define

$$\zeta_i(x,t) := \zeta\Big(\frac{\operatorname{sdist}(x,\mathcal{T}_i(t))}{\delta\widehat{r}}\Big),\ (x,t) \in \operatorname{im}(\Psi_{\mathcal{T}_i}) := \Psi_{\mathcal{T}_i}\Big(\mathcal{T}_i\times(-2r_i,2r_i)\Big) \tag{2.177}$$

where the change of variables $\Psi_{\mathcal{T}_i}$ and the associated signed distance $\operatorname{sdist}(\cdot,\mathcal{T}_i)$ are from Definition 13 of the admissible localization radius r_i. Furthermore, for contact points $\mathcal{T}_c$, $c \in \mathcal{C}$, we define

$$\zeta_c(x,t) := \zeta\Big(\frac{\operatorname{dist}(x,\mathcal{T}_c(t))}{\delta\widehat{r}}\Big), \quad (x,t) \in \mathbb{R}^2 \times [0,T]. \tag{2.178}$$

Step 2: (Choice of the constant $\widehat{r} \in (0, \min_{i\in\mathcal{I}} r_i \wedge \min_{c\in\mathcal{C}} \widehat{r}_c)$) It is a consequence of the uniform regularity of the interface I_v in space-time that one may choose $\widehat{r} \in (0, \min_{i\in\mathcal{I}} r_i \wedge \min_{c\in\mathcal{C}} \widehat{r}_c)$ small enough such that the following localization properties hold true

$$\Psi_{\mathcal{T}_i}(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]) \cap \Psi_{\mathcal{T}_{i'}}(\mathcal{T}_{i'}(t)\times\{t\}\times[-\widehat{r},\widehat{r}]) = \emptyset \quad \forall i' \in \mathcal{I},\ i' \neq i, \tag{2.179}$$

$$\Psi_{\mathcal{T}_i}(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]) \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \neq \emptyset \;\Leftrightarrow\; \exists c\in\mathcal{C}\colon i\sim c, \tag{2.180}$$
$$B_{\widehat{r}}(\mathcal{T}_c(t)) \cap B_{\widehat{r}}(\mathcal{T}_{c'}(t)) = \emptyset \quad \forall c,c'\in\mathcal{C},\, c'\neq c. \tag{2.181}$$

for all $t\in[0,T]$ and all $i\in\mathcal{I}$.

Step 3: (Construction of the partition of unity, part I) We start with the construction of the cutoffs η_i for two-phase interfaces $i\in\mathcal{I}$. Away from contact points, we set

$$\eta_i(x,t) := \zeta_i(x,t), \quad (x,t)\in \mathrm{im}(\Psi_{\mathcal{T}_i}) \setminus \bigcup_{c\in\mathcal{C}}\bigcup_{t'\in[0,T]} B_{\widehat{r}}\Big(\mathcal{T}_c(t')\Big)\times\{t'\}, \tag{2.182}$$

which is well-defined due to the choice of $\widehat{r}$.

Assume now there exists $c\in\mathcal{C}$ such that $i\sim c$. Recall from Definition 17 of the admissible localization radius r_c that for all $t\in[0,T]$ we decomposed $\Omega\cap B_{r_c}(\mathcal{T}_c(t))$ by means of five pairwise disjoint open wedges $W^{\pm,c}_{\partial\Omega}(t), W^c_{\mathcal{T}_i}(t), W^c_{\Omega^\pm_v}(t)\subset\mathbb{R}^2$. In the wedge $W^c_{\mathcal{T}_i}$ containing the two-phase interface $\mathcal{T}_i\subset I_v$, we define

$$\eta_i(x,t) := (1-\zeta_c(x,t))\zeta_i(x,t),\; (x,t)\in \bigcup_{t'\in[0,T]}\Big(B_{\widehat{r}}\Big(\mathcal{T}_c(t')\Big)\cap W^c_{\mathcal{T}_i}(t')\Big)\times\{t'\}. \tag{2.183}$$

This is indeed well-defined by the choice of $\widehat{r}$ and having

$$B_{r_c}(\mathcal{T}_c(t))\cap W^c_{\mathcal{T}_i}(t) \subset \Psi_{\mathcal{T}_i}(\mathcal{T}_i(t)\times\{t\}\times(-2r_c,2r_c))$$

for all $t\in[0,T]$; the latter in turn being a consequence of Definition 17 of the admissible localization radius r_c.

Within the ball $B_{\widehat{r}}(\mathcal{T}_c(t))$, we aim to restrict the support of $\eta_i(\cdot,t)$ to the region $B_{\widehat{r}}(\mathcal{T}_c(t))\cap \Big(W^c_{\mathcal{T}_i}(t)\cup W^c_{\Omega^\pm_v}(t)\Big)$ for all $t\in[0,T]$. This will be done by means of the interpolation functions $\lambda^\pm_c$ of Lemma 24. Recall in this context the convention that $\lambda^\pm_c(\cdot,t)$ was set equal to one on $\Big(\partial W^c_{\Omega^\pm_v}(t)\cap\partial W^c_{\mathcal{T}_i}(t)\Big)\setminus\mathcal{T}_c(t)$ and set equal to zero on $\Big(\partial W^c_{\Omega^\pm_v}(t)\cap\partial W^{\pm,c}_{\partial\Omega}(t)\Big)\setminus\mathcal{T}_c(t)$ for all $t\in[0,T]$. In particular, we may define in the interpolation wedges $W^c_{\Omega^\pm_v}$

$$\begin{aligned}&\eta_i(x,t) := \lambda^\pm_c(x,t)(1-\zeta_c(x,t))\zeta_i(x,t),\\ &(x,t)\in \bigcup_{t'\in[0,T]}\Big(B_{\widehat{r}}\Big(\mathcal{T}_c(t')\Big)\cap W^c_{\Omega^\pm_v}(t)\Big)\times\{t'\}.\end{aligned} \tag{2.184}$$

Again, this is well-defined because of the choice of $\widehat{r}$ and the fact that

$$B_{r_c}(\mathcal{T}_c(t))\cap W^c_{\Omega^\pm_v}(t) \subset \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times\{t\}\times(-2r_c,2r_c))$$

for all $t\in[0,T]$ due to Definition 17 of the admissible localization radius r_c.

Outside of the space-time domains appearing in the definitions (2.182)–(2.184), we simply set η_i equal to zero.

In view of the definitions (2.176)–(2.178) and the definitions (2.182)–(2.184), it now suffices to choose $\delta\in(0,1]$ sufficiently small such that (2.171) holds true, and in case there exists $c\in\mathcal{C}$ such that $i\sim c$ one may on top achieve

$$\operatorname{supp}\eta_i(\cdot,t)\cap B_{\widehat{r}}(\mathcal{T}_c(t)) \subset B_{\widehat{r}}(\mathcal{T}_c(t))\cap\Big(W^c_{\mathcal{T}_i}(t)\cup W^c_{\Omega^\pm_v}(t)\Big) \tag{2.185}$$

for all $t\in[0,T]$. Moreover, in light of (2.171) and (2.179) we also obtain (2.173).

Step 4: (Construction of the partition of unity, part II) We proceed with the construction of the cutoffs η_c for contact points $c \in \mathcal{C}$. To this end, let $i \in \mathcal{I}$ be the unique two-phase interface such that $i \sim c$. In the wedge $W^c_{\mathcal{T}_i}$ containing the two-phase interface $\mathcal{T}_i \subset I_v$ we set

$$\eta_c(x,t) := \zeta_c(x,t)\zeta_i(x,t),\ (x,t) \in \bigcup_{t'\in[0,T]} \Big(B_{\widehat{r}}\big(\mathcal{T}_c(t')\big) \cap W^c_{\mathcal{T}_i}(t')\Big)\times\{t'\}, \tag{2.186}$$

which is well-defined based on the same reason as for (2.183).

Moreover, in the interpolation wedges $W^c_{\Omega_v^\pm}$ we define

$$\begin{aligned}&\eta_c(x,t) := \lambda_c^\pm(x,t)\zeta_c(x,t)\zeta_i(x,t) + (1-\lambda_c^\pm(x,t))\zeta_c(x,t),\\ &(x,t) \in \bigcup_{t'\in[0,T]} \Big(B_{\widehat{r}}\big(\mathcal{T}_c(t')\big) \cap W^c_{\Omega_v^\pm}(t)\Big)\times\{t'\}.\end{aligned} \tag{2.187}$$

By the same argument as for (2.184), this is again well-defined.

Outside of the space-time domains appearing in the previous two definitions we simply set $\eta_c := \zeta_c$. In particular, we register for reference purposes that

$$\eta_c(x,t) := \zeta_c(x,t),\ (x,t) \in \bigcup_{t'\in[0,T]} \Big(B_{\hat{r}}\big(\mathcal{T}_c(t')\big) \setminus \Big(W^c_{\mathcal{T}_i}(t') \cup W^c_{\Omega_v^\pm}(t)\Big)\Big)\times\{t'\}. \tag{2.188}$$

It now immediately follows from the definition (2.178) that (2.172) is satisfied. In particular, for pairs $i \in \mathcal{I}$ and $c \in \mathcal{C}$ such that $i \sim c$, $\operatorname{supp}\eta_i \cap \operatorname{supp}\eta_c \neq \emptyset$ and we obtain (2.175) as an update of (2.185). Moreover, by (2.172) and (2.181) we deduce the validity of (2.174). In the case of pairs $i \in \mathcal{I}$ and $c \in \mathcal{C}$ with $i \not\sim c$, due to (2.180), (2.171) and (2.172), we can conclude that $\operatorname{supp}\eta_i \cap \operatorname{supp}\eta_c = \emptyset$.

Step 5: (Partition of unity property along the interface) Fix $t \in [0,T]$, and consider first the case of $x \in I_v(t) \setminus \bigcup_{c\in\mathcal{C}} B_{\widehat{r}}(\mathcal{T}_c(t))$. The combination of the support properties (2.171) and (2.172) with the localization property (2.179) implies there exists a unique two-phase interface $i_* = i_*(x) \in \mathcal{I}$ such that $\sum_{n=1}^N \eta_n(x,t) = \eta_{i_*}(x,t)$. Hence, we may deduce from (2.182) that $\sum_{n=1}^N \eta_n(x,t) = 1$ for all $t \in [0,T]$ and all $x \in I_v(t) \setminus \bigcup_{c\in\mathcal{C}} B_{\widehat{r}}(\mathcal{T}_c(t))$.

Fix a contact point $c \in \mathcal{C}$ and a point $x \in I_v(t) \cap B_{\widehat{r}}(\mathcal{T}_c(t))$. Let $i \in \mathcal{I}$ be the unique two-phase interface such that $i \sim c$. By the support properties (2.171) and (2.172) in combination with the localization properties (2.179)–(2.181) it follows that $\sum_{n=1}^N \eta_n(x,t) = \eta_c(x,t) + \eta_i(x,t)$. In particular $\sum_{n=1}^N \eta_n(x,t) = 1$ due to the definitions (2.183) and (2.186). The two discussed cases thus imply that

$$\sum_{n=1}^N \eta_n(x,t) = 1, \quad (x,t) \in \bigcup_{t'\in[0,T]} I_v(t') \times \{t'\}. \tag{2.189}$$

Step 6: (Regularity) Outside of interpolation wedges, the required regularity is an immediate consequence of the uniform regularity of the interface I_v and the definitions (2.182), (2.183), (2.186) and (2.187).

In interpolation wedges, one has to argue based on the definitions (2.184) and (2.187). In terms of regularity, the critical cases originating from an application of the product rule consist of those when derivatives hit the interpolation parameter. However, the by (2.154)–(2.155) controlled blow-up of the derivatives of the interpolation parameter is always counteracted by the presence of the term $1-\zeta_c$ (cf. (2.184) and (2.187)) which is of second order in the distance to the contact point due to (2.176) and (2.178). In other words, the required regularity also holds true within interpolation wedges.

The two considered cases taken together entail the asserted regularity.

Step 7: (Estimate for the bulk cutoff) In the course of establishing the desired coercivity estimates (2.169) and (2.170), we also convince ourselves of the fact that

$$\eta_{\mathrm{bulk}} = 1 - \sum_{n=1}^{N} \eta_n \in [0,1] \tag{2.190}$$

throughout $\mathbb{R}^2 \times [0,T]$. By the support properties (2.171) and (2.172), in both cases it suffices to argue for points contained in $\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big) \setminus \bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)$ or $B_{\widehat{r}}(\mathcal{T}_c(t))$ for all $i\in\mathcal{I}$, all $c\in\mathcal{C}$ and all $t\in[0,T]$.

We start with the latter and fix $i\in\mathcal{I}$ as well as $t\in[0,T]$. Due to the localization property (2.179) and subsequently plugging in (2.182), we get

$$\eta_{\mathrm{bulk}}(\cdot,t) = 1-\eta_i(\cdot,t) = 1-\zeta_i(\cdot,t) \text{ in } \Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big) \setminus \bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big). \tag{2.191}$$

The validity of (2.169), (2.170) and (2.190) in $\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big)\setminus\bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)$ thus follows immediately from definition (2.177).

Fix $c\in\mathcal{C}$, and let $i\in\mathcal{I}$ be the unique two-phase interface with $i\sim c$. Due to (2.171), (2.172) as well as (2.179)–(2.181) we have

$$\eta_{\mathrm{bulk}}(\cdot,t) = 1-\eta_c(\cdot,t)-\eta_i(\cdot,t) \quad \text{in } B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)\cap\Big(W^c_{\mathcal{T}_i}(t)\cup W^c_{\Omega^\pm_v}(t)\Big). \tag{2.192}$$

Plugging in (2.183) and (2.186) or (2.184) and (2.187), respectively, yields

$$\eta_{\mathrm{bulk}}(\cdot,t) = 1-\zeta_i(\cdot,t) \quad \text{in } B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)\cap W^c_{\mathcal{T}_i}(t), \tag{2.193}$$

as well as

$$\eta_{\mathrm{bulk}}(\cdot,t) = \lambda^\pm_c(\cdot,t)(1-\zeta_i(\cdot,t)) + (1-\lambda^\pm_c(\cdot,t))(1-\zeta_c(\cdot,t)) \qquad \text{in } B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)\cap W^c_{\Omega^\pm_v}(t). \tag{2.194}$$

Hence, we can infer by means of (2.177) and (2.178) that (2.169), (2.170) and (2.190) hold true in the domain $B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)\cap\Big(W^c_{\mathcal{T}_i}(t)\cup W^c_{\Omega^\pm_v}(t)\Big)$. Finally, we have

$$\eta_{\mathrm{bulk}}(\cdot,t) = 1-\eta_c(\cdot,t) = 1-\zeta_c(\cdot,t) \quad \text{in } B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)\setminus\Big(W^c_{\mathcal{T}_i}(t)\cup W^c_{\Omega^\pm_v}(t)\Big) \tag{2.195}$$

as a consequence of (2.171), (2.172), (2.179)–(2.181) and (2.188). The previous display in turn implies (2.169), (2.170) and (2.190) in $B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)\setminus\Big(W^c_{\mathcal{T}_i}(t)\cup W^c_{\Omega^\pm_v}(t)\Big)$ because of (2.178). This eventually concludes the proof of Lemma 26. □

Construction 27 (From local to global extensions)**.** *Let $d=2$, and let $\Omega\subset\mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v,v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Let $(\eta_1,\ldots,\eta_N)$ be a partition of unity along the interface I_v as given by* *the proof of Lemma 26. For each two-phase interface $i \in \mathcal{I}$ denote by ξ^i the bulk extension of Proposition 15, and for each contact point $c \in \mathcal{C}$ let ξ^c be the contact point extension of Proposition 16.*

We then define a vector field $\xi\colon \overline{\Omega} \times [0,T] \to \mathbb{R}^2$ with regularity

$$\xi \in \Big(C^0_t C^2_x \cap C^1_t C^0_x\Big)\Big(\overline{\Omega \times [0,T]} \setminus (I_v \cap (\partial\Omega \times [0,T]))\Big) \tag{2.196}$$

by means of the formula

$$\xi := \sum_{n=1}^{N} \eta_n \xi^n. \tag{2.197}$$

Before we proceed on with a proof of Proposition 7, we first deduce that the bulk cutoff η_{bulk} of Lemma 26 is transported by the fluid velocity v up to an admissible error in the distance to the interface of the strong solution.

Lemma 28 (Transport equation for bulk cutoff)**.** *Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Let $(\eta_1, \ldots, \eta_N)$ be a partition of unity along the interface I_v as given by the proof of Lemma 26.*

The bulk cutoff $\eta_{\mathrm{bulk}} = 1 - \sum_{n=1}^{N} \eta_n$ is then transported by the fluid velocity v to second order in form of

$$|\partial_t \eta_{\mathrm{bulk}} + (v \cdot \nabla)\eta_{\mathrm{bulk}}| \leq C(1 \wedge \mathrm{dist}^2(\cdot, I_v)) \quad \textit{in } \Omega \times [0,T]. \tag{2.198}$$

Proof. Let $\widehat{r} \in (0, \frac{1}{2}]$ be the localization radius of Lemma 26. In view of the regularity estimate (2.168) and the fact that

$$\Omega \setminus \bigg(\bigcup_{c \in \mathcal{C}} B_{\widehat{r}}(\mathcal{T}_c(t)) \cup \bigcup_{i \in \mathcal{I}} \mathrm{im}(\Psi_{\mathcal{T}_i})\bigg) \subset \Omega \cap \Big\{x \in \mathbb{R}^2\colon\ \mathrm{dist}(x, I_v(t)) > \widehat{r}\Big\}$$

for all $t \in [0,T]$, it suffices to establish (2.198) within $\Omega \cap \Psi_{\mathcal{T}_i}\Big(\mathcal{T}_i(t) \times \{t\} \times [-\widehat{r}, \widehat{r}]\Big) \setminus \bigcup_{c \in \mathcal{C}} B_{\widehat{r}}\Big(\mathcal{T}_c(t)\Big)$ or $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t))$ for all $i \in \mathcal{I}$, all $c \in \mathcal{C}$ and all $t \in [0,T]$.

Step 1: (Estimate near the interface but away from contact points) Fix a two-phase interface $i \in \mathcal{I}$. As a consequence of the two identities in (2.191), we may compute

$$\partial_t \eta_{\mathrm{bulk}} + (v \cdot \nabla)\eta_{\mathrm{bulk}} = -\Big(\partial_t \zeta_i + (v \cdot \nabla)\zeta_i\Big) + \eta_{\mathrm{bulk}}(v \cdot \nabla)\zeta_i \tag{2.199}$$

in $\Omega \cap \Psi_{\mathcal{T}_i}\Big(\mathcal{T}_i(t) \times \{t\} \times [-\widehat{r}, \widehat{r}]\Big) \setminus \bigcup_{c \in \mathcal{C}} B_{\widehat{r}}\Big(\mathcal{T}_c(t)\Big)$ for all $t \in [0,T]$. Recall that the signed distance to the two-phase interface $\mathcal{T}_i \subset I_v$ is transported to first order by the fluid velocity v, and that the profile ζ from (2.176) is quadratic around the origin. Hence, by the chain rule and the definition (2.177) we obtain

$$\Big|\partial_t \zeta_i + (v \cdot \nabla)\zeta_i\Big| \leq C\, \mathrm{dist}^2(\cdot, I_v) \quad \text{in } \Omega \cap \Psi_{\mathcal{T}_i}\Big(\mathcal{T}_i(t) \times \{t\} \times [-\widehat{r}, \widehat{r}]\Big) \tag{2.200}$$

for all $t \in [0,T]$. Since we also have the coercivity estimate (2.169) for the bulk cutoff at our disposal, we may thus upgrade (2.199) to (2.198) in $\Omega \cap \Psi_{\mathcal{T}_i}\Big(\mathcal{T}_i(t) \times \{t\} \times [-\widehat{r}, \widehat{r}]\Big) \setminus \bigcup_{c \in \mathcal{C}} B_{\widehat{r}}\Big(\mathcal{T}_c(t)\Big)$ for all $t \in [0,T]$.

Step 2: (Estimate near contact points, part I) Fix $c \in \mathcal{C}$, and denote by $i \in \mathcal{I}$ the unique two-phase interface such that $i \sim c$. This step is devoted to the proof of (2.198) in the wedge $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\mathcal{T}_i}(t)$ containing the interface $\mathcal{T}_i(t) \subset I_v(t)$, $t \in [0,T]$. Because of (2.192), (2.193) and (2.197) we have

$$\partial_t \eta_{\mathrm{bulk}} + (v\cdot\nabla)\eta_{\mathrm{bulk}} = -\Big(\partial_t\zeta_i + (v\cdot\nabla)\zeta_i\Big) + \eta_{\mathrm{bulk}}(v\cdot\nabla)\zeta_i \tag{2.201}$$

in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\mathcal{T}_i}(t)$ for all $t \in [0,T]$. Due to Definition 17 of the admissible localization radius r_c and $\widehat{r} \leq r_c$ by Lemma 26, it holds $B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\mathcal{T}_i}(t) \subset \Psi_{\mathcal{T}_i}\Big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\Big)$ for all $t \in [0,T]$. In particular, the estimate (2.200) is applicable in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\mathcal{T}_i}(t)$ for all $t \in [0,T]$. Hence, the estimate (2.200) in combination with the coercivity estimate (2.169) for the bulk cutoff allow to deduce (2.198) from (2.201) in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\mathcal{T}_i}(t)$ for all $t \in [0,T]$.

Step 3: (Estimate near contact points, part II) Fix a contact point $c \in \mathcal{C}$. The goal of this step is to prove (2.198) in the wedges $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^{\pm,c}_{\partial\Omega}(t)$ containing the boundary $\partial\Omega$ for all $t \in [0,T]$. To this end, it follows from (2.195) and (2.197) that

$$\partial_t \eta_{\mathrm{bulk}} + (v\cdot\nabla)\eta_{\mathrm{bulk}} = -\Big(\partial_t\zeta_c + (v\cdot\nabla)\zeta_c\Big) + \eta_{\mathrm{bulk}}(v\cdot\nabla)\zeta_c \tag{2.202}$$

in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^{\pm,c}_{\partial\Omega}(t)$ for all $t \in [0,T]$. Note that because of (2.176) one can view the profile ζ_c from (2.178) as a smooth function of the contact point $\mathcal{T}_c$. Performing a slight yet convenient abuse of notation $\mathcal{T}_c(t) = \{c(t)\}$, we obtain as a consequence of $\frac{\mathrm{d}}{\mathrm{d}t}c(t) = v(c(t),t)$ and an application of the chain rule that $\partial_t\zeta_c(\cdot,t) + \Big(v(c(t),t)\cdot\nabla\Big)\zeta_c(\cdot,t) = 0$ at $c(t)$ for all $t \in [0,T]$. Furthermore, proceeding similarly as done in the proof of [45, Lemma 11], we can also deduce that $\partial_t\zeta_c(\cdot,t) + \Big(v(c(t),t)\cdot\nabla\Big)\zeta_c(\cdot,t) = 0$ in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t))$ for all $t \in [0,T]$. By the regularity of the fluid velocity v, this in turn implies by adding zero (and exploiting the quadratic behaviour of the profile ζ from (2.176) around the origin) that

$$\Big|\partial_t\zeta_c + (v\cdot\nabla)\zeta_c\Big| \leq C\operatorname{dist}^2(\cdot,\mathcal{T}_c) \quad \text{in } \Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \tag{2.203}$$

for all $t \in [0,T]$. Since $\widehat{r} \leq r_c$ by Lemma 26, we can infer from Definition 17 of the admissible localization radius r_c that $\operatorname{dist}(\cdot,\mathcal{T}_c)$ is dominated by $\operatorname{dist}(\cdot,I_v)$ in $B_{\widehat{r}}(\mathcal{T}_c(t)) \cap \Big(W^{\pm,c}_{\partial\Omega}(t) \cup W^c_{\Omega^\pm_v}(t)\Big)$ for all $t \in [0,T]$. Hence, we deduce from (2.203) that

$$\Big|\partial_t\zeta_c + (v\cdot\nabla)\zeta_c\Big| \leq C\operatorname{dist}^2(\cdot,I_v) \text{ in } \Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap \Big(W^{\pm,c}_{\partial\Omega}(t) \cup W^c_{\Omega^\pm_v}(t)\Big) \tag{2.204}$$

for all $t \in [0,T]$. Inserting the estimate (2.204) and the coercivity estimate (2.169) for the bulk cutoff into (2.202) thus yields (2.198) in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^{\pm,c}_{\partial\Omega}(t)$ for all $t \in [0,T]$.

Step 4: (Estimate near contact points, part III) Fix $c \in \mathcal{C}$, and denote by $i \in \mathcal{I}$ the unique two-phase interface such that $i \sim c$. We aim to verify (2.198) in the interpolation wedges $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\Omega^\pm_v}(t)$ for all $t \in [0,T]$. To this end, we may employ (2.192), (2.194) and (2.197) to argue that

$$\begin{aligned}
&\partial_t \eta_{\mathrm{bulk}} + (v\cdot\nabla)\eta_{\mathrm{bulk}} \\
&= -\lambda^\pm_c\Big\{\Big(\partial_t\zeta_i + (v\cdot\nabla)\zeta_i\Big) - \eta_{\mathrm{bulk}}(v\cdot\nabla)\zeta_i\Big\} \\
&\quad - (1-\lambda^\pm_c)\Big\{\Big(\partial_t\zeta_c + (v\cdot\nabla)\zeta_c\Big) - \eta_{\mathrm{bulk}}(v\cdot\nabla)\zeta_c\Big\} \\
&\quad + \Big(\partial_t\lambda^\pm_c + (v\cdot\nabla)\lambda^\pm_c\Big)(\zeta_c - \zeta_i)
\end{aligned} \tag{2.205}$$

in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\Omega_v^\pm}(t)$ for all $t \in [0,T]$. Due to Definition 17 of the admissible localization radius r_c and $\widehat{r} \leq r_c$ by Lemma 26, it holds $B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\Omega_v^\pm}(t) \subset \Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t) \times \{t\} \times [-\widehat{r}, \widehat{r}]\big)$ for all $t \in [0,T]$. The estimates (2.200) and (2.169) therefore imply that the first term on the right hand side of (2.205) is of required order. For the second term on the right hand side of (2.205), we may instead rely on the estimates (2.204) and (2.169).

Note that in view of the definitions (2.176)–(2.178), the auxiliary cutoffs ζ_i and ζ_c are compatible to second order in the sense that $|\zeta_i - \zeta_c| \leq C \operatorname{dist}^2(\cdot, \mathcal{T}_c)$ in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\Omega_v^\pm}(t)$ for all $t \in [0,T]$. Recall from the previous step that $\operatorname{dist}(\cdot, \mathcal{T}_c)$ is dominated by $\operatorname{dist}(\cdot, I_v)$ in $B_{\widehat{r}}(\mathcal{T}_c(t)) \cap \big(W^{\pm,c}_{\partial\Omega}(t) \cup W^c_{\Omega_v^\pm}(t)\big)$ for all $t \in [0,T]$. Hence,

$$|\zeta_i - \zeta_c| \leq C \operatorname{dist}^2(\cdot, I_v) \tag{2.206}$$

in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\Omega_v^\pm}(t)$ for all $t \in [0,T]$. In particular, together with (2.156) the bound (2.206) allows to upgrade (2.205) to the desired estimate (2.198) in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\Omega_v^\pm}(t)$ for all $t \in [0,T]$.

Step 5: (Conclusion) Recall from Definition 17 of the admissible localization radius r_c that for all $t \in [0,T]$ the set $\Omega \cap B_{r_c}(\mathcal{T}_c(t))$ is decomposed by means of the five pairwise disjoint open wedges $W^{\pm,c}_{\partial\Omega}(t), W^c_{\mathcal{T}_i}(t), W^c_{\Omega_v^\pm}(t) \subset \mathbb{R}^2$. Hence, the previous three steps entail the validity of (2.198) in $\Omega \cap B_{r_c}(\mathcal{T}_c(t))$ for all $t \in [0,T]$. In particular, based on the discussion at the beginning of this proof and the argument in the vicinity of the interface but away from contact points (see *Step 1*), we may conclude the proof of Lemma 26. ☐

2.6.2 Proof of Proposition 7

All ingredients are in place to proceed with the proof of the main result of this section, i.e., that the vector field ξ of Construction 27 gives rise to a boundary adapted extension of the interface unit normal for two-phase fluid flow in the sense of Definition 2 with respect to (χ_v, v).

Proof of (2.16a). This is an easy consequence of the lower bound in the coercivity estimate (2.169) for the bulk cutoff, the definition (2.197) of the global vector field ξ, the fact that the local vector fields $(\xi^n)_{n\in\{1,\ldots,N\}}$ as provided by Proposition 15 and Proposition 16 are of unit length, and the triangle inequality in form of $|\xi| = |\sum_{n=1}^N \eta_n \xi_n| \leq \sum_{n=1}^N \eta_n |\xi^n| = \sum_{n=1}^N \eta_n = 1 - \eta_{\mathrm{bulk}}$ in $\Omega \times [0,T]$. ☐

Proof of (2.16b). By definition (2.197) of the candidate extension ξ and the localization properties (2.171)–(2.175) of the partition of unity $(\eta_1, \ldots, \eta_N)$ from Lemma 26, it suffices to verify (2.16b) in terms of $\xi = \eta_c \xi^c$ in the associated region $B_{\widehat{r}}(\mathcal{T}_c(t)) \cap \partial\Omega$ for all contact points $c \in \mathcal{C}$ and all $t \in [0,T]$. However, this in turn is an immediate consequence of Proposition 16. ☐

Proof of (2.16c). For a proof of (2.16c), we start computing based on the definition (2.197) of the global vector field ξ that $\nabla \cdot \xi = \sum_{n=1}^N \eta_n \nabla \cdot \xi^n + \sum_{n=1}^N (\xi^n \cdot \nabla)\eta_n$. As a consequence of the corresponding local versions of (2.16c) from Proposition 15 and Proposition 16, and the fact that $(\eta_1, \ldots, \eta_n)$ is a partition of unity along the interface I_v by Lemma 26 we obtain $\sum_{n=1}^N \eta_n \nabla \cdot \xi^n = -H_{I_v}$ along $I_v \cap \Omega$. Moreover, by adding zero and subsequently relying on the definition (2.197) of the global vector field ξ, the localization properties (2.171)–(2.175) of the partition of unity $(\eta_1,\ldots,\eta_N)$ from Lemma 26, the compatibility estimate (2.115) and the estimates (2.169) and (2.170) for the bulk cutoff we may infer that

$$\begin{aligned}\sum_{n=1}^{N}(\xi^n\cdot\nabla)\eta_n &= -(\xi\cdot\nabla)\eta_{\mathrm{bulk}} - \sum_{n=1}^{N}((\xi-\xi^n)\cdot\nabla)\eta_n \\ &= -(\xi\cdot\nabla)\eta_{\mathrm{bulk}} + \eta_{\mathrm{bulk}}\sum_{n=1}^{N}(\xi^n\cdot\nabla)\eta_n \\ &\quad + \sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_c\Big((\xi^i-\xi^c)\cdot\nabla\eta_i\Big) + \sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_i\Big((\xi^c-\xi^i)\cdot\nabla\eta_c\Big) \\ &= O(1\wedge\operatorname{dist}(\cdot,I_v)) \quad \text{in } \Omega\times[0,T].\end{aligned}$$

In summary, we thus obtain (2.16c). □

Proof of (2.16d). For a proof of (2.16d), we start estimating based on the definition (2.197) of the global vector field ξ as well as the corresponding local versions of (2.16d) from Proposition 15 and Proposition 16

$$\begin{aligned}\partial_t\xi &= \sum_{n=1}^{N}\eta_n\partial_t\xi^n + \sum_{n=1}^{N}\xi^n\partial_t\eta_n \\ &= -\sum_{n=1}^{N}\eta_n(v\cdot\nabla)\xi^n + \sum_{n=1}^{N}\xi^n\partial_t\eta_n \\ &\quad -\sum_{n=1}^{N}\eta_n(\mathrm{Id}-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n + O(1\wedge\operatorname{dist}(\cdot,I_v)) \quad \text{in } \Omega\times[0,T].\end{aligned} \tag{2.207}$$

Adding zero twice and applying the product rule, we may further rewrite based on the definition (2.197) of the candidate extension ξ and the localization properties (2.171)–(2.175) of the partition of unity $(\eta_1,\ldots,\eta_N)$ from Lemma 26

$$\begin{aligned}&-\sum_{n=1}^{N}\eta_n(v\cdot\nabla)\xi^n + \sum_{n=1}^{N}\xi^n\partial_t\eta_n \\ &= -(v\cdot\nabla)\xi + \sum_{n=1}^{N}\xi^n\Big(\partial_t\eta_n + (v\cdot\nabla)\eta_n\Big) \\ &= -(v\cdot\nabla)\xi - \xi\Big(\partial_t\eta_{\mathrm{bulk}} + (v\cdot\nabla)\eta_{\mathrm{bulk}}\Big) + \sum_{n=1}^{N}(\xi^n-\xi)\Big(\partial_t\eta_n + (v\cdot\nabla)\eta_n\Big) \\ &= -(v\cdot\nabla)\xi - \xi\Big(\partial_t\eta_{\mathrm{bulk}} + (v\cdot\nabla)\eta_{\mathrm{bulk}}\Big) + \eta_{\mathrm{bulk}}\sum_{n=1}^{N}\xi^n\Big(\partial_t\eta_n + (v\cdot\nabla)\eta_n\Big) \\ &\quad + \sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_c(\xi^i-\xi^c)\Big(\partial_t\eta_i+(v\cdot\nabla)\eta_i\Big) + \sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_i(\xi^c-\xi^i)\Big(\partial_t\eta_c+(v\cdot\nabla)\eta_c\Big)\end{aligned}$$

in $\Omega\times[0,T]$. Hence, estimating based on the compatibility estimate (2.115) as well as the estimates (2.169) and (2.198) for the bulk cutoff yields the bound

$$-\sum_{n=1}^{N}\eta_n(v\cdot\nabla)\xi^n + \sum_{n=1}^{N}\xi^n\partial_t\eta_n = -(v\cdot\nabla)\xi + O(1\wedge\operatorname{dist}(\cdot,I_v)) \text{ in } \Omega\times[0,T]. \tag{2.208}$$

Adding zero twice and making use of the definition (2.197) of the candidate extension ξ together with the localization properties (2.171)–(2.175) of the partition of unity $(\eta_1,\ldots,\eta_N)$

from Lemma 26, we next compute

$$
\begin{aligned}
&\mathbb{1}_{\operatorname{supp}\eta_n}\xi^n\otimes\xi^n \\
&=\mathbb{1}_{\operatorname{supp}\eta_n}\xi\otimes\xi+\mathbb{1}_{\operatorname{supp}\eta_n}(\xi^n-\xi)\otimes\xi^n+\mathbb{1}_{\operatorname{supp}\eta_n}\xi\otimes(\xi^n-\xi) \\
&=\mathbb{1}_{\operatorname{supp}\eta_n}\xi\otimes\xi \\
&\quad+\mathbb{1}_{\operatorname{supp}\eta_n}\eta_{\mathrm{bulk}}\xi^n\otimes\xi^n+\mathbb{1}_{\operatorname{supp}\eta_n}\eta_{\mathrm{bulk}}\xi\otimes\xi^n \\
&\quad+\mathbb{1}_{n=i\in\mathcal{I}}\mathbb{1}_{\operatorname{supp}\eta_i}\sum_{c\in\mathcal{C},i\sim c}\eta_c(\xi^i-\xi^c)\otimes\xi^i+\mathbb{1}_{n=c\in\mathcal{C}}\mathbb{1}_{\operatorname{supp}\eta_c}\sum_{i\in\mathcal{I},i\sim c}\eta_i(\xi^c-\xi^i)\otimes\xi^c \\
&\quad+\mathbb{1}_{n=i\in\mathcal{I}}\mathbb{1}_{\operatorname{supp}\eta_i}\sum_{c\in\mathcal{C},i\sim c}\eta_c\xi\otimes(\xi^i-\xi^c)+\mathbb{1}_{n=c\in\mathcal{C}}\mathbb{1}_{\operatorname{supp}\eta_c}\sum_{i\in\mathcal{I},i\sim c}\eta_i\xi\otimes(\xi^c-\xi^i)
\end{aligned}
\tag{2.209}
$$

in $\Omega\times[0,T]$. Relying on the same ingredients as for the previous computation we also have

$$
\begin{aligned}
-\sum_{n=1}^N\eta_n(\nabla v)^{\mathsf{T}}\xi^n&=-(\nabla v)^{\mathsf{T}}\xi-\sum_{n=1}^N\eta_n(\nabla v)^{\mathsf{T}}(\xi^n-\xi)+\eta_{\mathrm{bulk}}(\nabla v)^{\mathsf{T}}\xi \\
&=-(\nabla v)^{\mathsf{T}}\xi+\eta_{\mathrm{bulk}}(\nabla v)^{\mathsf{T}}\xi-\eta_{\mathrm{bulk}}\sum_{n=1}^N\eta_n(\nabla v)^{\mathsf{T}}\xi^n \\
&\quad-\sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_i\eta_c(\nabla v)^{\mathsf{T}}(\xi^i-\xi^c)-\sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_c\eta_i(\nabla v)^{\mathsf{T}}(\xi^c-\xi^i)
\end{aligned}
$$

in $\Omega\times[0,T]$. The compatibility estimate (2.115) as well as the estimates (2.169) and (2.198) therefore imply in view of the previous two displays that

$$
\begin{aligned}
&-\sum_{n=1}^N\eta_n(\mathrm{Id}-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n \\
&=-(\mathrm{Id}-\xi\otimes\xi)(\nabla v)^{\mathsf{T}}\xi+O(1\wedge\operatorname{dist}(\cdot,I_v))\quad\text{in }\Omega\times[0,T].
\end{aligned}
\tag{2.210}
$$

The combination of the bounds (2.207)–(2.210) now immediately entails the desired estimate (2.16d) on the time evolution of the global vector field ξ. □

Proof of (2.16e). We get as a consequence of the product rule and inserting the local versions of (2.16e) from Proposition 15 and Proposition 16

$$
\begin{aligned}
\xi\cdot\partial_t\xi&=\sum_{n=1}^N\eta_n\xi\cdot\partial_t\xi^n+\sum_{n=1}^N(\xi\cdot\xi^n)\partial_t\eta_n \\
&=-\sum_{n=1}^N\eta_n\xi^n\cdot(v\cdot\nabla)\xi^n+\sum_{n=1}^N\eta_n(\xi-\xi^n)\cdot\partial_t\xi^n \\
&\quad+\sum_{n=1}^N(\xi\cdot\xi^n)\partial_t\eta_n+O(\operatorname{dist}(\cdot,I_v)^2\wedge 1)\quad\text{in }\Omega\times[0,T].
\end{aligned}
$$

Adding zero to produce the left hand sides of the local versions of (2.16d) from Proposition 15 and Proposition 16 further updates the previous display to

$$
\begin{aligned}
\xi\cdot\partial_t\xi=&-\sum_{n=1}^N\eta_n\xi\cdot(v\cdot\nabla)\xi^n+\sum_{n=1}^N(\xi\cdot\xi^n)\partial_t\eta_n \\
&-\sum_{n=1}^N\eta_n(\xi-\xi^n)\cdot(\mathrm{Id}-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=1}^{N}\eta_n(\xi-\xi^n)\cdot\Big(\partial_t\xi^n+(v\cdot\nabla)\xi^n+(\mathrm{Id}-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n\Big)\\
&+O(\mathrm{dist}(\cdot,I_v)^2\wedge 1)\quad\text{in }\Omega\times[0,T].
\end{aligned}
$$

We then continue with adding zeros to obtain

$$
\begin{aligned}
\xi\cdot\partial_t\xi &= -\xi\cdot(v\cdot\nabla)\xi\\
&\quad+\sum_{n=1}^{N}\Big(\xi\cdot(\xi^n-\xi)\Big)\Big(\partial_t\eta_n+(v\cdot\nabla)\eta_n\Big)-|\xi|^2\Big(\partial_t\eta_{\mathrm{bulk}}+(v\cdot\nabla)\eta_{\mathrm{bulk}}\Big)\\
&\quad-\sum_{n=1}^{N}\eta_n(\xi-\xi^n)\cdot(\xi\otimes\xi-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n\\
&\quad-\sum_{n=1}^{N}\eta_n(\xi-\xi^n)\cdot(\mathrm{Id}-\xi\otimes\xi)(\nabla v)^{\mathsf{T}}(\xi^n-\xi)\\
&\quad+\sum_{n=1}^{N}\eta_n(\xi-\xi^n)\cdot\Big(\partial_t\xi^n+(v\cdot\nabla)\xi^n+(\mathrm{Id}-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n\Big)\\
&\quad+O(\mathrm{dist}(\cdot,I_v)^2\wedge 1)\quad\text{in }\Omega\times[0,T].
\end{aligned}
\tag{2.211}
$$

As it is by now routine, we may employ the localization properties (2.171)–(2.175) of the partition of unity $(\eta_1,\ldots,\eta_N)$ from Lemma 26 and the estimates (2.169) and (2.198) for the bulk cutoff to reduce the task of estimating the right hand side terms of (2.211) to an application of the compatibility estimates (2.115)–(2.116). More precisely, we obtain by straightforward applications of these two ingredients that

$$
\begin{aligned}
&\sum_{n=1}^{N}\Big(\xi\cdot(\xi-\xi^n)\Big)\Big(\partial_t\eta_n+(v\cdot\nabla)\eta_n\Big)\\
&=\sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_c^2\Big(\big(\xi^c-\xi^i\big)\cdot\big(\xi^c-\xi^i\big)\Big)\Big(\partial_t\eta_i+(v\cdot\nabla)\eta_i\Big)\\
&\quad+\sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_c\eta_i\Big(\big(\xi^c-\xi^i\big)\cdot\big(\xi^i-\xi^c\big)\Big)\Big(\partial_t\eta_c+(v\cdot\nabla)\eta_c\Big)\\
&\quad+\sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_c^2\Big(\xi^i\cdot(\xi^c-\xi^i)\Big)\Big(\partial_t\eta_i+(v\cdot\nabla)\eta_i\Big)\\
&\quad+\sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_c\eta_i\Big(\xi^i\cdot(\xi^i-\xi^c)\Big)\Big(\partial_t\eta_c+(v\cdot\nabla)\eta_c\Big)\\
&\quad+\sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_i\eta_c\Big(\xi^i\cdot(\xi^c-\xi^i)\Big)\Big(\partial_t\eta_i+(v\cdot\nabla)\eta_i\Big)\\
&\quad+\sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_i^2\Big(\xi^i\cdot(\xi^i-\xi^c)\Big)\Big(\partial_t\eta_c+(v\cdot\nabla)\eta_c\Big)\\
&\quad+O(\mathrm{dist}(\cdot,I_v)^2\wedge 1)\quad\text{in }\Omega\times[0,T],
\end{aligned}
\tag{2.212}
$$

$$
\begin{aligned}
&\sum_{n=1}^{N}\eta_n(\xi-\xi^n)\cdot(\mathrm{Id}-\xi\otimes\xi)(\nabla v)^{\mathsf{T}}(\xi-\xi^n)\\
&=\sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_i\eta_c^2(\xi^c-\xi^i)\cdot(\mathrm{Id}-\xi\otimes\xi)(\nabla v)^{\mathsf{T}}(\xi^c-\xi^i)\\
&\quad+\sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_c\eta_i^2(\xi^i-\xi^c)\cdot(\mathrm{Id}-\xi\otimes\xi)(\nabla v)^{\mathsf{T}}(\xi^i-\xi^c)\\
&\quad+O(\mathrm{dist}(\cdot,I_v)^2\wedge 1)\quad\text{in }\Omega\times[0,T],
\end{aligned}
\tag{2.213}
$$

$$\begin{aligned}&\sum_{n=1}^{N}\eta_n(\xi-\xi^n)\cdot\Big(\partial_t\xi^n+(v\cdot\nabla)\xi^n+(\mathrm{Id}-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n\Big)\\&=\sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C}}\eta_i\eta_c(\xi^c-\xi^i)\cdot\Big(\partial_t\xi^i+(v\cdot\nabla)\xi^i+(\mathrm{Id}-\xi^i\otimes\xi^i)(\nabla v)^{\mathsf{T}}\xi^i\Big)\\&\quad+\sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_c\eta_i(\xi^i-\xi^c)\cdot\Big(\partial_t\xi^c+(v\cdot\nabla)\xi^c+(\mathrm{Id}-\xi^c\otimes\xi^c)(\nabla v)^{\mathsf{T}}\xi^c\Big)\\&\quad+O(\mathrm{dist}(\cdot,I_v)^2\wedge 1)\quad\text{in }\Omega\times[0,T],\end{aligned}\tag{2.214}$$

and finally

$$\begin{aligned}&\sum_{n=1}^{N}\eta_n(\xi-\xi^n)\cdot(\xi\otimes\xi-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n\\&=\sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_c(\xi^c-\xi^i)\cdot(\xi\otimes\xi-\xi^i\otimes\xi^i)(\nabla v)^{\mathsf{T}}\xi^i\\&\quad+\sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_i(\xi^i-\xi^c)\cdot(\xi\otimes\xi-\xi^c\otimes\xi^c)(\nabla v)^{\mathsf{T}}\xi^c\\&\quad+O(\mathrm{dist}(\cdot,I_v)^2\wedge 1)\quad\text{in }\Omega\times[0,T].\end{aligned}\tag{2.215}$$

We then exploit the compatibility estimates (2.115) and (2.116) for an estimate of (2.212), the compatibility estimate (2.115) for an estimate of (2.213), the local versions of (2.16d) from Proposition 15 and Proposition 16 in combination with the compatibility estimate (2.115) for an estimate of (2.214), and finally (2.209) together with the estimate for the bulk cutoff (2.169) and the compatibility estimate (2.115) to estimate (2.215). In summary, using also the bound on the advection derivative (2.198) as well as the coercivity estimate (2.169), we may upgrade (2.211) to the desired estimate (2.16e). □

2.7 Existence of transported weights: Proof of Lemma 8

We decompose the argument for the construction of a transported weight ϑ in the sense of Definition 3 in several steps.

Step 1: (Choice of suitable profiles) Let $\bar\vartheta\colon\mathbb{R}\to\mathbb{R}$ be chosen such that it represents a smooth truncation of the identity in the sense that $\bar\vartheta(r)=r$ for $|r|\leq\frac{1}{2}$, $\bar\vartheta(r)=-1$ for $r\leq-1$, $\bar\vartheta(r)=1$ for $r\geq 1$, $0\leq\bar\vartheta'\leq 2$ as well as $|\bar\vartheta''|\leq C$.

For each two-phase interface $i\in\mathcal{I}$ present in the interface I_v of the strong solution, we then define an auxiliary weight

$$\bar\vartheta_i(x,t):=-\bar\vartheta\Big(\frac{\mathrm{sdist}(x,\mathcal{T}_i(t))}{\delta\widehat{r}}\Big),\quad (x,t)\in\mathrm{im}(\Psi_{\mathcal{T}_i})\tag{2.216}$$

where the change of variables $\Psi_{\mathcal{T}_i}$ and the associated signed distance $\mathrm{sdist}(\cdot,\mathcal{T}_i)$ are the ones from Definition 13 of the admissible localization radius r_i. Moreover, $\widehat{r}$ represents the localization scale of Lemma 26 and $\delta\in(0,1]$ denotes a constant to be chosen in the course of the proof.

Recalling also from Definition 17 of the admissible localization radii $(r_c)_{c\in\mathcal{C}}$ the definition of the change of variables $\Psi_{\partial\Omega}$ with associated signed distance $\mathrm{sdist}(\cdot,\partial\Omega)$ we define another two auxiliary weights by means of

$$\bar\vartheta^{\pm}_{\partial\Omega}(x,t):=\mp\bar\vartheta\Big(\frac{\mathrm{sdist}(x,\partial\Omega)}{\delta\widehat{r}}\Big),\tag{2.217}$$

$$(x,t) \in \bigcup_{t'\in[0,T]} \Big(\Omega_v^{\pm}(t') \cap \Psi_{\partial\Omega}\Big(\partial\Omega\times(-2\widehat{r},2\widehat{r})\Big)\Big)\times\{t'\}.$$

Step 2: (Construction of the transported weight) Away from contact points and the interface but in the vicinity of the domain boundary, we introduce the following notational shorthand

$$\mathcal{U}_{\widehat{r}}(t) := \bigcup_{i\in\mathcal{I}} \Psi_{\mathcal{T}_i}\Big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\Big) \cup \bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\Big(\mathcal{T}_c(t)\Big), \quad t\in[0,T], \tag{2.218}$$

and then define

$$\begin{aligned}&\vartheta(x,t) := \bar{\vartheta}^{\pm}_{\partial\Omega}(x,t), \\ &(x,t) \in \bigcup_{t'\in[0,T]} \Big(\Omega_v^{\pm}(t') \cap \Psi_{\partial\Omega}\Big(\partial\Omega\times[-\widehat{r},\widehat{r}]\Big) \setminus \mathcal{U}_{\widehat{r}}(t')\Big)\times\{t'\}.\end{aligned} \tag{2.219}$$

Fix next a two-phase interface $i\in\mathcal{I}$. Away from contact points but in the vicinity of the interface, we then define

$$\begin{aligned}&\vartheta(x,t) := \bar{\vartheta}_i(x,t), \\ &(x,t) \in \bigcup_{t'\in[0,T]} \bigg(\Omega \cap \Psi_{\mathcal{T}_i}\Big(\mathcal{T}_i(t')\times\{t'\}\times[-\widehat{r},\widehat{r}]\Big) \setminus \bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\Big(\mathcal{T}_c(t')\Big)\bigg)\times\{t'\}.\end{aligned} \tag{2.220}$$

Let now a contact point $c\in\mathcal{C}$ be fixed, and denote by $i\in\mathcal{I}$ the unique two-phase interface with $i\sim c$. Recall from Definition 17 of the admissible localization radius r_c that for all $t\in[0,T]$ we decomposed $\Omega\cap B_{r_c}(\mathcal{T}_c(t))$ by means of five pairwise disjoint open wedges $W^{\pm,c}_{\partial\Omega}(t), W^c_{\mathcal{T}_i}(t), W^c_{\Omega^\pm_v}(t) \subset \mathbb{R}^2$. In the wedge $W^c_{\mathcal{T}_i}$ containing the two-phase interface $\mathcal{T}_i\subset I_v$, we still define

$$\vartheta(x,t) := \bar{\vartheta}_i(x,t), \quad (x,t)\in \bigcup_{t'\in[0,T]} \Big(\Omega\cap B_{\widehat{r}}\Big(\mathcal{T}_c(t')\Big)\cap W^c_{\mathcal{T}_i}(t')\Big)\times\{t'\}. \tag{2.221}$$

In the wedges $W^{\pm,c}_{\partial\Omega}$ containing the domain boundary $\partial\Omega$, we instead set

$$\vartheta(x,t) := \bar{\vartheta}^{\pm}_{\partial\Omega}(x,t), \quad (x,t)\in \bigcup_{t'\in[0,T]} \Big(\Omega\cap B_{\widehat{r}}\Big(\mathcal{T}_c(t')\Big)\cap W^{\pm,c}_{\partial\Omega}(t')\Big)\times\{t'\}. \tag{2.222}$$

In the interpolation wedges $W^c_{\Omega^\pm_v}$, we make use of the interpolation parameter $\lambda^\pm_c$ of Lemma 24 to interpolate between the two constructions near the interface (2.221) and near the domain boundary (2.222). Recall in this context the convention that $\lambda^\pm_c(\cdot,t)$ was set equal to one on $\Big(\partial W^c_{\Omega^\pm_v}(t)\cap\partial W^c_{\mathcal{T}_i}(t)\Big)\setminus\mathcal{T}_c(t)$ and set equal to zero on $\Big(\partial W^c_{\Omega^\pm_v}(t)\cap\partial W^{\pm,c}_{\partial\Omega}(t)\Big)\setminus\mathcal{T}_c(t)$ for all $t\in[0,T]$. With this notation in place, we define on the interpolation wedges

$$\begin{aligned}&\vartheta(x,t) := \lambda^\pm_c(x,t)\bar{\vartheta}_i(x,t) + (1-\lambda^\pm_c(x,t))\bar{\vartheta}^{\pm}_{\partial\Omega}(x,t), \\ &(x,t)\in \bigcup_{t'\in[0,T]} \Big(\Omega\cap B_{\widehat{r}}\Big(\mathcal{T}_c(t')\Big)\cap W^c_{\Omega^\pm_v}(t')\Big)\times\{t'\}.\end{aligned} \tag{2.223}$$

Finally, choosing δ small enough in the definition (2.216) of the auxiliary weights $(\vartheta_i)_{i\in\mathcal{I}}$ and recalling the localization properties (2.179)–(2.181) of the scale $\widehat{r}$, it is safe to define in the space-time domain not captured by the definitions (2.219)–(2.223)

$$\vartheta(x,t) := \mp 1, \tag{2.224}$$

$$(x,t) \in \bigcup_{t'\in[0,T]} \Big(\Omega_v^\pm(t') \setminus \Big(\mathcal{U}_{\widehat{r}}(t') \cup \Psi_{\partial\Omega}(\partial\Omega\times[-\widehat{r},\widehat{r}])\Big)\Big)\times\{t'\}.$$

Recall for this definition also the notation (2.218).

Step 3: (Regularity and coercivity) The validity of the asserted sign conditions in Definition 3 are immediate from (2.219)–(2.224). Since the first-order derivatives of the interpolation parameter $\lambda_c^\pm$ feature controlled blow-up (2.154), it is also a direct consequence of the definitions (2.219)–(2.224) that $\vartheta \in W^{1,\infty}_{x,t}(\Omega \times [0,T])$ as asserted.

In view of the definition (2.224) of the weight in the bulk it suffices to establish (2.27) in the regions $\Omega \cap \Psi_{\partial\Omega}\big(\partial\Omega\times[-\widehat{r},\widehat{r}]\big) \setminus \mathcal{U}_{\widehat{r}}(t)$, $\Omega \cap \Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big) \setminus \bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)$ and $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t))$ for all $i \in \mathcal{I}$, all $c \in \mathcal{C}$ and all $t \in [0,T]$. However, in these regions the asserted estimate (2.27) is immediately implied by the properties of the truncation of unity $\bar\vartheta$ from *Step 1* of this proof and the definitions (2.219)–(2.223).

Step 4: (Advection equation) Because of the definition (2.224) of the weight ϑ in the bulk, it suffices to establish (2.28) in the regions $\Omega \cap \Psi_{\partial\Omega}\big(\partial\Omega\times[-\widehat{r},\widehat{r}]\big) \setminus \mathcal{U}_{\widehat{r}}(t)$, $\Omega \cap \Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big) \setminus \bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)$ and $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t))$ for all $i \in \mathcal{I}$, all $c \in \mathcal{C}$ and all $t \in [0,T]$.

Observe first that it follows from the definitions (2.217), (2.219) and (2.222) as well as the boundary condition for the fluid velocity $(v \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$ that

$$\partial_t\vartheta + (v\cdot\nabla)\vartheta = 0 \quad \text{along } \partial\Omega \setminus \bigcup_{c\in\mathcal{C}} \mathcal{T}_c(t) \tag{2.225}$$

for all $t \in [0,T]$. By a Lipschitz estimate together with the coercivity estimate (2.27), the desired estimate (2.28) follows in $\Omega \cap \Psi_{\partial\Omega}\big(\partial\Omega\times[-\widehat{r},\widehat{r}]\big) \setminus \mathcal{U}_{\widehat{r}}(t)$ for all $t \in [0,T]$.

Fix next a two-phase interface $i \in \mathcal{I}$. We then claim that

$$\Big|\partial_t\bar\vartheta_i + (v\cdot\nabla)\bar\vartheta_i\Big| \leq C\,\mathrm{dist}(\cdot, I_v) \quad \text{in } \Omega \cap \Psi_{\mathcal{T}_i(t)}\Big(\mathcal{T}_i(t)\times[-\widehat{r},\widehat{r}]\Big) \tag{2.226}$$

for all $t \in [0,T]$. Indeed, one only needs to recall that the signed distance to the two-phase interface $\mathcal{T}_i \subset I_v$ is transported by the fluid velocity v to first order in the distance to the interface. In particular, combining (2.226) with the definition (2.220) and the coercivity estimate (2.27) entails (2.28) in $\Omega\cap\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big)\setminus\bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)$ for all $t \in [0,T]$.

Let now a contact point $c \in \mathcal{C}$ be given, and let $i \in \mathcal{I}$ be the unique two-phase interface such that $i \sim c$. The desired estimate (2.28) follows immediately from (2.226) and (2.221) in the wedge $\Omega \cap B_{\widehat{r}}\big(\mathcal{T}_c(t)\big) \cap W^c_{\mathcal{T}_i}(t)$ for all $t \in [0,T]$. For the wedges containing the domain boundary $\partial\Omega$, the estimate (2.28) in form of

$$\Big|\partial_t\bar\vartheta^\pm_{\partial\Omega} + (v\cdot\nabla)\bar\vartheta^\pm_{\partial\Omega}\Big| \leq C\,\mathrm{dist}(\cdot,\partial\Omega) \text{ in } \Omega \cap B_{\widehat{r}}\Big(\mathcal{T}_c(t)\Big) \cap \Big(W^c_{\Omega_v^\pm}(t) \cup W^{\pm,c}_{\partial\Omega}(t)\Big) \tag{2.227}$$

for all $t \in [0,T]$, is satisfied because of the analogue of (2.225) and a Lipschitz estimate. Finally, in the interpolation wedges one may estimate

$$\begin{aligned}|\partial_t\vartheta{+}(v\cdot\nabla)\vartheta| \leq{}& |\bar\vartheta_i - \bar\vartheta^\pm_{\partial\Omega}||\partial_t\lambda_c^\pm{+}(v\cdot\nabla)\lambda_c^\pm| \\ &+ \lambda_c^\pm|\partial_t\bar\vartheta_i{+}(v\cdot\nabla)\bar\vartheta_i| + (1{-}\lambda_c^\pm)|\partial_t\bar\vartheta^\pm_{\partial\Omega}{+}(v\cdot\nabla)\bar\vartheta^\pm_{\partial\Omega}|.\end{aligned}$$

The desired bound thus follows from the estimate (2.156) for the advective derivative of the interpolation parameter $\lambda_c^\pm$, the estimates (2.226) and (2.227), and the fact that the auxiliary weights from (2.216) and (2.217) are compatible in the sense

$$|\bar{\vartheta}_i - \bar{\vartheta}^\pm_{\partial\Omega}| \leq C(\mathrm{dist}(\cdot,\partial\Omega) \wedge \mathrm{dist}(\cdot, I_v))$$

in $\Omega \cap B_{\widehat{r}}\big(\mathcal{T}_c(t)\big) \cap W^c_{\Omega_v^\pm}(t)$ for all $t \in [0,T]$. This concludes the proof of Lemma 8. □

2.8 Existence of varifold solutions to two-phase fluid flow with surface tension

The aim of this last section is to give a sketch of a proof regarding existence of varifold solutions to two-phase fluid flow with surface tension and with ninety degree contact angle (see Definition 11). Note that this is not treated by the work of Abels [1] in which the existence of a varifold solution in the presence of surface tension is only established in a full space setting. However, in principle it still suggests itself to follow, where possible, the structure of the proof for the case of an unbounded domain by Abels [1]. In this regard, we first discuss two tools which are needed due to the different setting of the present work, i.e., geometric evolution with a ninety degree contact angle condition and the associated boundary conditions for the solenoidal fluid velocity. These tools concern an existence result for weak solutions to the required transport equation (for sufficiently regular transport velocities) and elliptic regularity estimates for the Helmholtz decomposition associated with the bounded and smooth domain Ω. In a second step, we present the corresponding approximate problem, focusing again on the key steps of the proof which differ with respect to the case of an unbounded domain studied by Abels [1]. Note that analogous to the existence theory of [1], we will assume some regularity for the geometry of the initial data and, for simplicity, that the densities of the two fluids coincide and are normalized to 1.

Transport equation. In order to construct approximate solutions of the two-phase flow with surface tension and with ninety degree contact angle, one first needs an existence result for weak solutions to the transport equation in a bounded domain. In particular, it suffices to motivate the validity of [1, Lemma 2.3, $\Omega \equiv \mathbb{R}^d$] in case of a smooth and bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$.

To this aim, let the open subset $\Omega_0^+ \subset \Omega$ be subject to the regularity conditions in Definition 9, let $\chi_0 := \chi_{\Omega_0^+} \in \mathrm{BV}(\Omega;\{0,1\})$, let $T \in (0,\infty)$, and consider a sufficiently regular fluid velocity $v \in C([0,T];C_b^2(\Omega)) \cap C(\overline{\Omega}\times[0,T])$ such that $\mathrm{div}\, v = 0$ in Ω and $(n_{\partial\Omega}\cdot v)|_{\partial\Omega} = 0$. Consider any $C([0,T];C_b^2(\mathbb{R}^d))$ extension of v which we denote by $\widetilde{v}$. Then, a solution $\widetilde{\chi}$ to the transport equation associated with $\widetilde{v}$ can be constructed on $\mathbb{R}^d$ by the usual method of characteristics (see, e.g., [1, Proof of Lemma 2.3]). The associated flow map is a C^1-diffeomorphism at any time $t \in [0,T]$. However, note that it maps $\partial\Omega$ onto itself, due to $v|_{\partial\Omega} = \widetilde{v}|_{\partial\Omega}$ being tangential along $\partial\Omega$. Moreover, since the flow map is a global diffeomorphism (and since continuous images of connected sets are connected), it also maps Ω onto itself. Then, one can conclude by means of the same computations as in the proof of [1, Lemma 2.3] — using in the process the fact that $\mathrm{div}\, v = 0$ in Ω — that the restriction $\chi := \widetilde{\chi}|_{\Omega\times[0,T]} \in L^\infty(0,T;\mathrm{BV}(\Omega;\{0,1\}))$ is a weak solution of the transport equation associated with v in the sense of

$$\int_0^T \int_\Omega \chi\,(\partial_t\varphi + v\cdot\nabla\varphi)\,\mathrm{d}x\mathrm{d}t + \int_\Omega \chi_0\varphi(x,0)\mathrm{d}x = 0 \tag{2.228}$$

for any $\varphi \in C_c^1([0,T); C(\overline{\Omega})) \cap C_c([0,T); C^1(\overline{\Omega}))$. Moreover, we have

$$\|\chi\|_{L^\infty(0,T;BV(\Omega))} \leqslant M\left(\|v\|_{C\left([0,T];C_b^2(\Omega)\right)}\right)\|\chi_0\|_{BV(\Omega)}, \tag{2.229}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\chi(\cdot,t)|(\Omega) = -\left\langle H_{\chi(\cdot,t)}, v(\cdot,t)\right\rangle \quad \text{for all } t \in (0,T) \tag{2.230}$$

for some continuous function M. Note that the latter holds because the 90 degree contact angle condition is preserved by sufficiently regular transport velocities (see, e.g., the remark after Definition 10).

Helmholtz decomposition associated with bounded domains. We recall properties of the Helmholtz projection P_Ω associated with the smooth bounded domain Ω, referring the reader to [101, Corollaries 7.4.4-5] (see also [116]).

Define $W_p(\Omega) := \{g \in W^{1,p}(\Omega;\mathbb{R}^d) : \operatorname{div} g = 0, (g \cdot n_{\partial\Omega})|_{\partial\Omega} = 0\}$. Given $f \in W^{1,p}(\Omega;\mathbb{R}^d)$, $2 \le p < \infty$, there are unique functions $\phi \in W^{2,p}(\Omega)$ and $w \in W_p(\Omega)$ such that $f = \nabla\phi + w$. The bounded linear operator $P_\Omega \in \mathcal{B}(W^{1,p}(\Omega;\mathbb{R}^d), W_p(\Omega))$ defined by $P_\Omega f := w$ is a projection, which is the Helmholtz projection associated with the smooth bounded domain Ω. Moreover, if $f \in W^{2,p}(\Omega;\mathbb{R}^d)$ it holds $\phi \in W^{3,p}(\Omega)$ and

$$\|P_\Omega f\|_{W^{2,p}(\Omega;\mathbb{R}^d)} \le C\|f\|_{W^{2,p}(\Omega;\mathbb{R}^d)}, \tag{2.231}$$

and if $f \in W^{k,2}(\Omega;\mathbb{R}^d)$, $k \ge 2$, then $\phi \in W^{k,2}(\Omega)$ and

$$\|P_\Omega f\|_{W^{k,2}(\Omega;\mathbb{R}^d)} \le C\|f\|_{W^{k,2}(\Omega;\mathbb{R}^d)}. \tag{2.232}$$

This follows from existence and regularity theory of the associated Neumann problem (see for the case $p > 2$ the result of [101, Corollary 7.4.5])

$$\begin{aligned} \Delta\phi &= \operatorname{div} f && \text{in } \Omega,\\ (n_{\partial\Omega}\cdot\nabla)\phi &= f\cdot n_{\partial\Omega} && \text{on } \partial\Omega. \end{aligned}$$

Solutions to approximate two-phase fluid flow. In order to formulate the approximate equations, let ψ be a standard mollifier, for every $k \in \mathbb{N}$ we denote by $\psi_k := k^d\psi(k\cdot)$ its usual rescaling, and by P_Ω the Helmholtz projection associated with the smooth domain Ω. Moreover, let $\Psi_k\cdot = P_\Omega(\psi_k * \cdot)$. Consider the initial data $v_0 \in L^2(\Omega)$ with $\operatorname{div} v_0 = 0$ and $(n_{\partial\Omega}\cdot v_0)|_{\partial\Omega} = 0$, and let $\chi_0 := \chi_{\Omega_0^+} \in \mathrm{BV}(\Omega;\{0,1\})$, where $\Omega_0^+ \subset \Omega$ is subject to the regularity conditions in Definition 9. Let $\mu,\sigma > 0$. Then, we consider an approximate two-phase flow on $(0,T_w)$, $T_w \in (0,\infty)$. This is a pair (v_k,χ_k) consisting on one side of a fluid velocity field $v_k \in L^\infty([0,T_w];L^2(\Omega)) \cap L^2([0,T_w];W_2(\Omega))$ solving

$$\begin{aligned} &\int_\Omega v_k(\cdot,T)\cdot\eta(\cdot,T)\,\mathrm{d}x - \int_\Omega v_0\cdot\eta(\cdot,0)\,\mathrm{d}x - \int_0^T\int_\Omega v_k\cdot\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\int_\Omega \Psi_k v_k\otimes(\psi_k * v_k) : \nabla(\psi_k*\eta)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \mu(\nabla v_k + \nabla v_k^\mathsf{T}) : \nabla\eta\,\mathrm{d}x\,\mathrm{d}t\\ &= \sigma\int_0^T\int_{\partial^*\{\chi_k=1\}\cap\Omega} \mathrm{H}_{\chi_k}\cdot\Psi_k\eta\,\mathrm{d}S\,\mathrm{d}t \end{aligned} \tag{2.233}$$

for a.e. $T \in [0,T_w)$ and every $\eta \in C^\infty([0,T_w);C^1(\overline{\Omega};\mathbb{R}^d)\cap\bigcap_{p\ge2}W^{2,p}(\Omega;\mathbb{R}^d))$ with $\operatorname{div}\eta = 0$ and $(n_{\partial\Omega}\cdot\eta)_{\partial\Omega} = 0$, and on the other side an evolving phase indicator $\chi_k \in L^\infty([0,T_w];$

$\mathrm{BV}(\Omega;\{0,1\}))$ which is the unique weak solution — in the sense of (2.228) — to the transport equation

$$\begin{aligned} \partial_t\chi_k + (\Psi_k v_k)\cdot\nabla\chi_k &= 0 && \text{in } (0,T_w)\times\Omega, \\ \chi_k|_{t=0} &= \chi_0 && \text{in } \Omega. \end{aligned}$$

The existence of approximate solutions (v_k,χ_k) satisfying the energy equality

$$\begin{aligned} &\frac{1}{2}\|v_k(\cdot,T)\|^2_{L^2(\Omega)} + \sigma|\nabla\chi_k(\cdot,T)|(\Omega) + \frac{\mu}{2}\|\nabla v_k\|^2_{L^2(\Omega\times(0,T))} \\ &= \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} + \sigma|\nabla\chi_0|(\Omega), \quad T\in(0,T_w), \end{aligned} \tag{2.234}$$

and satisfying

$$\text{the map} \quad (0,T_w)\ni t\mapsto|\nabla\chi_k(\cdot,t)|(\Omega) \quad \text{is absolutely continuous,} \tag{2.235}$$

can then be proved by means of a fixed-point argument as done in [1, Proof of Theorem 4.2], relying in the process on the above two ingredients corresponding to the different setting of the present work: the existence result for weak solutions to the transport equation (2.228) with sufficiently regular transport velocity, and the elliptic regularity estimates (2.232) for the Helmholtz projection associated with Ω. In particular, one obtains uniform bounds

$$\sup_{k\in\mathbb{N}}\sup_{t\in(0,T_w)}\|v_k(\cdot,t)\|^2_{L^2(\Omega)} + \sup_{k\in\mathbb{N}}\|\nabla v_k\|^2_{L^2(\Omega\times(0,T_w))} < \infty, \tag{2.236}$$

$$\sup_{k\in\mathbb{N}}\sup_{t\in(0,T_w)}|\nabla\chi_k(\cdot,t)|(\Omega) < \infty. \tag{2.237}$$

Limit passage in the approximation scheme to a varifold solution. As for the passage to the limit, we only discuss the surface tension term on the right hand side of the approximate problem (2.233) as well as the validity of the energy inequality (2.41). The other terms as well as the passage to the limit in the transport equation can be treated as in [1]. First, we define a varifold $V_k\in\mathcal{M}((0,T_w)\times\overline{\Omega}\times\mathbb{S}^{d-1})$ by

$$V_k := \mathcal{L}^1\llcorner(0,T_w)\otimes(V_k(t))_{t\in(0,T_w)}, \tag{2.238}$$

where

$$V_k(t) := |\nabla\chi_k(\cdot,t)|\llcorner\Omega\otimes\Big(\delta_{\frac{\nabla\chi_k(\cdot,t)}{|\nabla\chi_k(\cdot,t)|}}\Big)_{x\in\Omega}\in\mathcal{M}(\overline{\Omega}\times\mathbb{S}^{d-1}) \quad \text{for any } t\in(0,T_w).$$

Since $\chi_k\in L^\infty([0,T_w];\mathrm{BV}(\Omega;\{0,1\}))$ is uniformly bounded in the sense of (2.237), there then exists $\chi\in L^\infty([0,T_w];\mathrm{BV}(\Omega;\{0,1\}))$ such that, up to taking a subsequence,

$$\chi_k \rightharpoonup^* \chi \qquad \text{in } L^\infty(\Omega\times(0,T_w)), \tag{2.239}$$

$$\nabla\chi_k \rightharpoonup^* \nabla\chi \qquad \text{in } L^\infty([0,T_w];\mathcal{M}(\Omega)). \tag{2.240}$$

Moreover, we have $\sup_k\|V_k\|_{\mathcal{M}}<\infty$ due to (2.237) and the definition of V_k. In particular, there exists $V\in\mathcal{M}((0,T_w)\times\overline{\Omega}\times\mathbb{S}^{d-1})$ such that, up to taking a subsequence,

$$V_k \rightharpoonup^* V \qquad \text{in } \mathcal{M}((0,T_w)\times\overline{\Omega}\times\mathbb{S}^{d-1}). \tag{2.241}$$

Note that the compatibility condition (2.42) then simply follows from exploiting (2.240) and (2.241). As a preparation for the remaining arguments, note also that thanks to the condition (2.235) a careful inspection of the argument of [56, Lemma 2] reveals that one may disintegrate the limit varifold V in form of

$$V = \mathcal{L}^1 \llcorner (0,T_w) \otimes (V_t)_{t\in(0,T_w)}, \quad V_t \in \mathcal{M}(\overline{\Omega}\times\mathbb{S}^{d-1}),\ t\in(0,T_w), \tag{2.242}$$

and that the limit interface energy satisfies

$$|V_t|_{\mathbb{S}^{d-1}}(\overline{\Omega}) \leq \liminf_k |\nabla\chi_k(\cdot,t)|(\Omega) \quad \text{for a.e. } t\in[0,T_w). \tag{2.243}$$

For any $\eta \in C^\infty([0,T_w);C^1(\overline{\Omega};\mathbb{R}^d)\cap\bigcap_{p\geq 2}W^{2,p}(\Omega;\mathbb{R}^d))$ such that $\operatorname{div}\eta = 0$ and $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega} = 0$, we discuss the limit of

$$\int_0^T\int_\Omega \Big(\mathrm{Id} - \frac{\nabla\chi_k}{|\nabla\chi_k|}\otimes\frac{\nabla\chi_k}{|\nabla\chi_k|}\Big) : \nabla(\Psi_k\eta)\,\mathrm{d}|\nabla\chi_k|\,\mathrm{d}t \quad \text{for } k\to\infty,$$

for almost every $T\in[0,T_w)$. By adding a zero, we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega \Big(\mathrm{Id} - \frac{\nabla\chi_k}{|\nabla\chi_k|}\otimes\frac{\nabla\chi_k}{|\nabla\chi_k|}\Big) : \nabla(\Psi_k\eta - \eta)\,\mathrm{d}|\nabla\chi_k|\,\mathrm{d}t \\
&+ \int_0^T\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s\otimes s) : \nabla\eta\,\mathrm{d}V_k(t,x,s)\,,
\end{aligned}$$

where the second term converges to $\int_0^T\int_{\overline{\Omega}\times\mathbb{S}^{d-1}}(\mathrm{Id} - s\otimes s) : \nabla\eta\,\mathrm{d}V_t(x,s)$ for $k\to\infty$ for any $\eta \in C_0^\infty([0,T_w);C^1(\overline{\Omega};\mathbb{R}^d)\cap\bigcap_{p\geq 2}W^{2,p}(\Omega;\mathbb{R}^d))$. Indeed, the latter guarantees $(\mathrm{Id}-s\otimes s) : \nabla\eta \in C_0((0,T_w)\times\overline{\Omega}\times\mathbb{S}^{d-1})$ so that one may use (2.241) for such η. However, the additional support assumption on the time variable can be removed by means of a standard truncation argument relying on the disintegration formulas (2.238) and (2.242), respectively, and the uniform bound $\sup_k \|V_k\|_{\mathcal{M}} < \infty$. As for the first term, we exploit the regularity properties of the Helmholtz projection. More precisely, we may estimate for any $p>3$ based on (2.231) and the Sobolev embedding $W^{1,p}(\Omega)\hookrightarrow C(\overline{\Omega})$, $d\in\{2,3\}$,

$$\begin{aligned}
&\Big|\int_0^T\int_\Omega \Big(\mathrm{Id} - \frac{\nabla\chi_k}{|\nabla\chi_k|}\otimes\frac{\nabla\chi_k}{|\nabla\chi_k|}\Big) : \nabla(\Psi_k\eta - \eta)\,\mathrm{d}|\nabla\chi_k|\,\mathrm{d}t\Big| \\
&\leq C\int_0^T \|\nabla(\Psi_k\eta-\eta)\|_{C(\overline{\Omega};\mathbb{R}^{d\times d})}\,\mathrm{d}t \\
&\leq C\int_0^T \|\nabla P_\Omega(\psi_k * \eta-\eta)\|_{C(\overline{\Omega};\mathbb{R}^{d\times d})}\,\mathrm{d}t \\
&\leq C\int_0^T \|\psi_k * \eta-\eta\|_{W^{2,p}(\Omega;\mathbb{R}^{d})}\,\mathrm{d}t.
\end{aligned}$$

The right hand side obviously goes to zero by letting $k\to\infty$. In summary, we obtain as desired

$$\begin{aligned}
&\int_0^T\int_\Omega \Big(\mathrm{Id} - \frac{\nabla\chi_k}{|\nabla\chi_k|}\otimes\frac{\nabla\chi_k}{|\nabla\chi_k|}\Big) : \nabla(\Psi_k\eta)\,\mathrm{d}|\nabla\chi_k|\,\mathrm{d}t \\
&\to \int_0^T\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s\otimes s) : \nabla\eta\,\mathrm{d}V_t(x,s) \quad \text{for } k\to\infty,
\end{aligned}$$

for almost every $T\in[0,T_w)$ and all $\eta \in C^\infty([0,T_w);C^1(\overline{\Omega};\mathbb{R}^d)\cap\bigcap_{p\geq 2}W^{2,p}(\Omega;\mathbb{R}^d))$ such that $\operatorname{div}\eta=0$ and $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega}=0$.

At last, we comment how to recover the energy inequality (2.41). This can be obtained from combining the energy equality (2.234) with the lower-semicontinuity property (2.243) and the convergence properties of v_k to its limit v (i.e., up to a subsequence, $v_k \rightharpoonup v$ in $L^2(0,T_w;H^1(\Omega))$ and $v_k \rightharpoonup^* v$ in $L^\infty(0,T_w;L^2(\Omega))$ due to the uniform bound (2.236)).

CHAPTER 3

Quantitative convergence of the vectorial Allen-Cahn equation towards multiphase mean curvature flow

This chapter contains an edited and revised version of the paper "Quantitative convergence of the vectorial Allen-Cahn equation towards multiphase mean curvature flow"[49], which is a joint work with Julian Fischer and accepted for publication at *Ann. Inst. H. Poincaré Anal. Non Linéaire*. The preprint can be found on the arXiv (identifier 2203.17143).

Abstract. Phase-field models such as the Allen-Cahn equation may give rise to the formation and evolution of geometric shapes, a phenomenon that may be analyzed rigorously in suitable scaling regimes. In its sharp-interface limit, the vectorial Allen-Cahn equation with a potential with $N \geq 3$ distinct minima has been conjectured to describe the evolution of branched interfaces by multiphase mean curvature flow.

In the present work, we give a rigorous proof for this statement in two and three ambient dimensions and for a suitable class of potentials: As long as a strong solution to multiphase mean curvature flow exists, solutions to the vectorial Allen-Cahn equation with well-prepared initial data converge towards multiphase mean curvature flow in the limit of vanishing interface width parameter $\varepsilon \searrow 0$. We even establish the rate of convergence $O(\varepsilon^{1/2})$.

Our approach is based on the gradient flow structure of the Allen-Cahn equation and its limiting motion: Building on the recent concept of "gradient flow calibrations" for multiphase mean curvature flow, we introduce a notion of relative entropy for the vectorial Allen-Cahn equation with multi-well potential. This enables us to overcome the limitations of other approaches, e. g. avoiding the need for a stability analysis of the Allen-Cahn operator or additional convergence hypotheses for the energy at positive times.

3.1 Introduction

In the present work, we study the behavior of solutions to the vector-valued Allen-Cahn equation

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2}\partial_u W(u_\varepsilon) \tag{3.1}$$

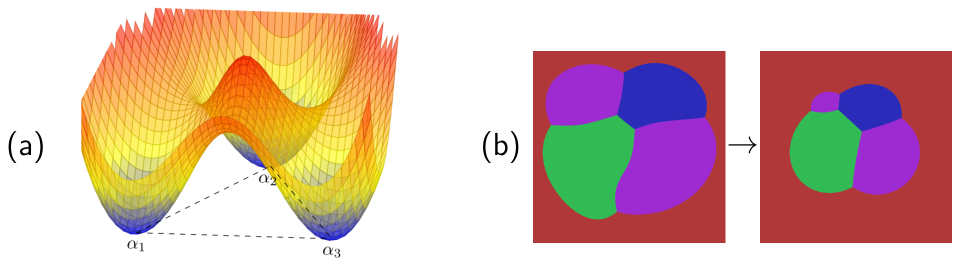


Figure 3.1: (a) A triple-well potential that attains its minimum at the three points α_1, α_2, α_3. (b) A partition of $\mathbb{R}^2$ evolving by multiphase mean curvature flow, corresponding to the sharp-interface limit $\varepsilon \to 0$ of the vectorial Allen-Cahn equation (3.1) with N-well potential W.

(with W being an N-well potential, see e. g. Figure 3.1a, and $u_\varepsilon : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{N-1}$) in the limit of vanishing interface width $\varepsilon \to 0$. We prove that for a suitable class of N-well potentials W, in the limit $\varepsilon \to 0$ the solutions u_ε describe a branched interface evolving by multiphase mean curvature flow (see Figure 3.1b), provided that a classical solution to the latter exists and provided that one starts with a sequence of well-prepared initial data $u_\varepsilon(\cdot,0)$. For quantitatively well-prepared initial data $u_\varepsilon(\cdot,0)$, we even establish a rate of convergence $O(\varepsilon^{1/2})$ towards the multiphase mean curvature flow limit.

The Allen-Cahn equation (3.1) with N-well potential is an important example of a phase-field model, an evolution equation for an order parameter u_ε that may vary in space and time. Phase-field models may give rise to the formation and evolution of geometric shapes, a phenomenon that becomes amenable to a rigorous mathematical analysis in suitable scaling regimes. For several important structural classes of potentials W, such a rigorous analysis has long been available for the Allen-Cahn equation: For instance, for the scalar Allen-Cahn equation with two-well potential W – that is, for (3.1) with $N=2$ – the convergence towards (two-phase) mean curvature flow in the limit $\varepsilon \to 0$ has been established by De Mottoni and Schatzman [36], Bronsard and Kohn [23], Chen [28], Ilmanen [62], and Evans, Soner, and Souganidis [40] in the context of three different notions of solutions to mean curvature flow (namely, strong solutions, Brakke solutions, respectively viscosity solutions). In such two-phase situations, sharp-interface limits have also been established for more complex phase-field models [29, 12, 2, 42, 3], typically based on an approach that relies on matched asymptotic expansions and a stability analysis of the PDE linearized around a transition profile. Beyond the case of two-well potentials, results have been much more scarce. One of the few well-understood settings is the case of the Ginzburg-Landau equation, which corresponds to the Allen-Cahn equation (3.1) with a Sombrero-type potential $W(u) = (1-|u|^2)^2$ and $N=3$, i. e. with a potential that features a continuum of minima at $\{u \in \mathbb{R}^2 : |u| = 1\}$. In this case, the convergence of solutions to (codimension two) vortex filaments evolving by mean curvature has been shown in dimensions $d \geq 3$ by Jerrard and Soner [67], Lin [79], and Bethuel, Orlandi, and Smets [18].

In contrast, for the (vectorial) Allen-Cahn equation (3.1) with a potential W with $N \geq 3$ distinct minima, the only previous results on the sharp-interface limit have been a formal expansion analysis by Bronsard and Reitich [24] and a convergence result that is conditional on the convergence of the Allen-Cahn energy

$$E[u_\varepsilon] := \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}\,dx$$

at positive times (more precisely, in $L^1([0,T])$) by Laux and Simon [74]. In particular, to the best of our knowledge not even an unconditional proof of qualitative convergence for well-prepared initial data has been available so far. One of the main challenges that has prevented a full analysis is the emergence of "branching" interfaces in the (conjectured) limit of multiphase mean curvature flow (see Figure 3.1b), corresponding to a geometric singularity in the limiting motion.

In the present work, we introduce a relative energy approach for the problem of the sharp-interface limit of the vectorial Allen-Cahn equation in a multiphase setting: Building on the concept of "gradient flow calibrations" that has been introduced by Hensel, Laux, Simon, and the first author [46] precisely for the purpose of handling these branching singularities in multiphase mean curvature flow and combining it with ideas from [48], we introduce a notion of relative energy for the Allen-Cahn equation

$$E[u_\varepsilon|\xi] := \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon} + \sum_{i=1}^{N} \xi_i \cdot \nabla\psi_i(u_\varepsilon)\,dx.$$

Here, the ξ_i denote a "gradient flow calibration" for the strong solution to multiphase mean curvature flow; in particular, $\xi_{i,j}(x,t) := \xi_i - \xi_j$ is an extension of the unit normal vector field of the interface between phases i and j in the strong solution to mean curvature flow at time t. The $\psi_i : \mathbb{R}^{N-1} \to [0,1]$ are suitable $C^{1,1}$ functions that serve as phase indicator functions; in particular, denoting the N minima of the N-well potential W by α_j $(1 \leq j \leq N)$, the functions ψ_i satisfy $\psi_i(\alpha_j) = \delta_{ij}$. Note that the functions $\psi_j - \psi_i$ will play a role that is somewhat similar to the role of the functions $\psi(u) = \int_0^u \sqrt{2W(s)}\,ds$ in the Modica-Mortola trick for a two-well potential $W : \mathbb{R} \to \mathbb{R}_0^+$ like $W(u) = \frac{9}{8}(1-u^2)^2$.

The properties of the gradient flow calibration ξ_i and the assumptions on the functions $\psi_i : \mathbb{R}^{N-1} \to [0,1]$ will ensure that the estimate $\left|\sum_{i=1}^N \xi_i \cdot \nabla\psi_i(u_\varepsilon)\right| \leq \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)$ holds, thereby guaranteeing coercivity of the relative energy $E[u_\varepsilon|\xi]$. In our main result, we prove that for suitable initial data $u_\varepsilon(\cdot,0)$ we have $||\psi_i(u_\varepsilon(\cdot,t)) - \bar\chi_i(\cdot,t)||_{L^1(\mathbb{R}^d)} \leq C\varepsilon^{1/2}$ for all $t \leq T$, where the $\bar\chi_i$ denote the phase indicator functions from the strong solution to multiphase mean curvature flow.

Rigorous results on sharp-interface limits for phase-field models – such as our result – are also of particular interest from a numerical perspective: In evolution equations for interfaces like e. g. mean curvature flow, the occurrence of topological changes typically poses a challenge for numerical simulations. One approach to the simulation of evolving interfaces is to construct a mesh that discretizes the initial interface and to numerically evolve the resulting mesh over time; however, it is then a highly nontrivial (and still widely open) question how to continue the numerical mesh beyond a topology change in a numerically consistent way. An alternative approach to the simulation of evolving interfaces that avoids this issue are phase-field models, in which the geometric evolution equation for the interface is replaced by an evolution equation for an order parameter posed on the entire space, allowing also for "mixtures" of the phases at the transition regions. The natural diffuse-interface approximation for multiphase mean curvature flow is given by the vector-valued Allen-Cahn equation with N-well potential (3.1). The advantage of phase-field approximations for geometric motions such as (3.1) is that one may solve them numerically using standard discretization schemes for parabolic PDEs; however, to establish convergence of the overall scheme towards the original interface evolution problem, it is necessary to rigorously justify the sharp-interface limit for the diffuse-interface model.

3.1.1 Notation

Throughout the paper, we use standard notation for parabolic PDEs. By $\dot{H}^1(\mathbb{R}^d)$ we denote the space of functions have a weak derivative $\nabla u \in L^2(\mathbb{R}^d)$ and (in case $d \geq 3$) decay at infinity. In particular, for a function $u \in L^2([0,T]; \dot{H}^1(\mathbb{R}^d))$ we denote by ∇u its (weak) spatial gradient and by $\partial_t u$ its (weak) time derivative. For functions defined on phase space, like our potential $W : \mathbb{R}^{N-1} \to [0,\infty)$ or the approximate phase indicator functions $\psi_i : \mathbb{R}^{N-1} \to [0,\infty)$, we denote their gradient by $\partial_u W$ respectively $\partial_u \psi_i$. For a smooth interface $I_{i,j}$, we denote its mean curvature vector by $\mathrm{H}_{i,j}$.

3.2 Main results

Our main result identifies the sharp-interface limit $\varepsilon \to 0$ for the vectorial Allen-Cahn equation (3.1) for a sufficiently broad class of N-well potentials W characterized by the following conditions.

(A1) Let $W : \mathbb{R}^{N-1} \to [0,\infty)$ be an N-well potential of class $C^{1,1}_{loc}(\mathbb{R}^{N-1})$ that attains its minimum $W(u) = 0$ precisely in N distinct points $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^{N-1}$. Assume that there exists an integer $q \geq 2$ and constants $C, c > 0$ such that in a neighborhood of each α_i we have

$$c|u - \alpha_i|^q \leq W(u) \leq C|u - \alpha_i|^q.$$

(A2) Let $U \subset \mathbb{R}^{N-1}$ be a bounded convex open set with piecewise C^1 boundary and $\{\alpha_1, \ldots, \alpha_N\} \subset \overline{U}$. Suppose that $\partial_u W(u)$ points towards U for any $u \in \partial U$.

(A3) Suppose that for any two distinct $i, j \in \{1, \ldots, N\}$, there exists a unique minimizing path $\gamma_{i,j}$ connecting α_i to α_j in the sense $\int_{\gamma_{i,j}} \sqrt{2W(\gamma_{i,j})}\,\mathrm{d}\gamma_{i,j} = \inf_\gamma \int_\gamma \sqrt{2W(\gamma)}\,\mathrm{d}\gamma = 1$, where the infimum is taken over all continuously differentiable paths γ connecting α_i to α_j.

(A4) Suppose that there exist continuously differentiable functions $\psi_i : \overline{U} \to [0,1]$, $1 \leq i \leq N$, and a disjoint partition of $\overline{U}$ into sets $\mathcal{T}_{i,j}$, $i < j \in \{1, \ldots, N\}$, subject to the following properties:

- For any $i \in \{1, \ldots, N\}$, we have $\psi_i(\alpha_i) = 1$ and $\psi_i(u) < 1$ for $u \neq \alpha_i$.
- Suppose that on $\mathcal{T}_{i,j}$, all ψ_k with $k \notin \{i,j\}$ vanish.
- Set $\psi_0 := 1 - \sum_{i=1}^N \psi_i$ to achieve $\sum_{i=0}^N \psi_i \equiv 1$ and define $\psi_{i,j} := \psi_j - \psi_i$. Suppose that there exists $\delta > 0$ such that for any distinct $i, j \in \{1, \ldots, N\}$ and any $u \in \mathcal{T}_{i,j}$ we have

$$\begin{aligned} &\left|\tfrac{1}{2}\partial_u \psi_{i,j}(u)\right|^2 + \left(\tfrac{5}{4} + \delta\right)\left|\tfrac{1}{2}\partial_u \psi_0(u)\right|^2 + \delta\,|\partial_u \psi_{i,j}(u) \cdot \partial_u \psi_0(u)| \\ &\leq 2W(u). \end{aligned}$$

Additionally, suppose there exists a constant $C > 0$ such that for any distinct $i, j \in \{1, \ldots, N\}$ and any $u \in \mathcal{T}_{i,j}$ it holds that $|\partial_u \psi_i(u)| \leq C\sqrt{2W(u)}$.

The assumption that our potential W has a finite set of minima as stated in (A1) is fundamental for the scaling limit we consider, as a different structure of the potential would give rise to a different limiting motion – recall that for instance a Sombrero-type potential would lead to (codimension two) vortex filaments structures [18, 67]. The assumption (A2) is rather mild, ensuring the existence of bounded weak solutions to the vectorial Allen-Cahn equation by a maximum principle (see Remark 32). The condition (A3) ensures that for each pair of minima, there is a unique optimal profile connecting the two phases; furthermore, it fixes the surface energy density for an interface between any pair of phases i and j to be 1. We expect that it would be possible to generalize our results to more general classes of surface tensions as considered in [46]; to avoid even more complex notation, we refrain from doing so in the present manuscript.

The assumption (A4) is the only truly restrictive condition in our assumptions; in fact, it does not include potentials which at the same time feature quadratic growth at the minima α_i (i. e., with $q = 2$ in (A1)) and regularity of class C^2. Nevertheless, as we shall see in Proposition 36 below, there exists a broad class of N-well potentials – including in particular potentials of class $C^{1,1}$ with quadratic growth at the minima α_i – that satisfy all of our assumptions.

Our main result on the quantitative convergence of the vectorial Allen-Cahn equation towards multiphase mean curvature flow reads as follows.

Theorem 29. *Let $d \in \{2,3\}$. In case $d = 2$, let $(\bar\chi_1, \dots, \bar\chi_N)$ be a classical solution to multiphase mean curvature flow on $\mathbb{R}^d$ on a time interval $[0,T]$ in the sense of Definition 33 below; in case $d = 3$, let $(\bar\chi_1, \dots, \bar\chi_N)$ be a classical solution to multiphase mean curvature flow of double bubble type in the sense of [57, Definition 10]. Let ξ be a corresponding gradient flow calibration in the sense of Definition 34 below. Suppose that W is a potential satisfying the assumptions (A1)–(A4). For every $\varepsilon > 0$, let $u_\varepsilon \in L^\infty([0,T]; \dot{H}^1(\mathbb{R}^d; \overline{U}))$ be a bounded weak solution to the vectorial Allen-Cahn equation* (3.1).

Assume furthermore that the initial data $u_\varepsilon(\cdot, 0)$ are well-prepared in the sense that

$$E[u_\varepsilon|\xi](0) \le C\varepsilon,$$

$$\max_{i\in\{1,\dots,N\}} \int_{\mathbb{R}^d} \Big|\psi_i(u_\varepsilon(\cdot,0)) - \bar\chi_i(\cdot,0)\Big| \min\{\operatorname{dist}(x, \partial \operatorname{supp} \bar\chi_i(\cdot,0)), 1\}\, dx \le C\varepsilon,$$

where $E[u_\varepsilon|\xi]$ denotes the relative entropy given as

$$E[u_\varepsilon|\xi] := \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) + \sum_{i=1}^N \xi_i \cdot \nabla(\psi_i \circ u_\varepsilon)\, dx. \tag{3.2}$$

Then the solutions u_ε to the vectorial Allen-Cahn equation converge towards multiphase mean curvature flow with the rate $O(\varepsilon^{1/2})$ in the sense that

$$\sup_{t\in[0,T]} E[u_\varepsilon|\xi] \le C\varepsilon,$$

$$\sup_{t\in[0,T]} \max_{i\in\{1,\dots,N\}} ||\psi_i(u_\varepsilon(\cdot,t)) - \bar\chi_i(\cdot,t)||_{L^1(\mathbb{R}^d)} \le C\varepsilon^{1/2}.$$

First, let us remark that in the planar case strong solutions to multiphase mean curvature flow are known to exist prior to the first topology change for quite general initial data [24, 85]. Beyond topology changes, in general the evolution by multiphase mean curvature flow may become unstable and uniqueness of solutions may fail, see e. g. the discussion in [85] or [46].

Thus, quantitative approximation results for multiphase mean curvature flow of the form of our Theorem 29 should not be expected to hold beyond the first topology change. In this sense, our result is optimal.

Second, let us emphasize that by [46] and [57] the existence of a gradient flow calibration is ensured in the following situations:

- In the planar case $d = 2$, gradient flow calibrations exist as long as a strong solution exists.
- In the three-dimensional case $d = 3$, gradient flow calibrations exist as long as a strong solution of double bubble type (i. e. in particular with at most 3 phases meeting at each point) exists.

Note that more generally we expect gradient flow calibrations to exist as long as a classical solution to multiphase mean curvature flow exists. Since the construction becomes increasingly technical when the geometrical features become more complex, the construction has not yet been carried out in these more general situations. Nevertheless, as soon as gradient flow calibration becomes available, our results below apply and yield the convergence of the vectorial Allen-Cahn equation to multiphase mean curvature flow in the corresponding setting.

Next, let us remark that we may weaken the assumptions on the sequence of initial data if we are content with lower rates of convergence or merely qualitative convergence statements.

Remark 30. *As an inspection of the proof of Theorem 29 readily reveals, the assumption of quantitative well-preparedness of the initial data in our theorem can be relaxed, even to a qualitative one. For instance, by merely assuming the qualitative convergences* $\lim_{\varepsilon\to 0} E[u_\varepsilon|\xi](0) = 0$ *and* $\lim_{\varepsilon\to 0} \max_{i\in\{1,\ldots,N\}} ||\psi_i(u_\varepsilon(\cdot,0)) - \bar\chi_i(\cdot,0)||_{L^1(\mathbb{R}^d)} = 0$ *at initial time, from Theorem 40 and Proposition 41 we are able to obtain the qualitative convergence statement*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} E[u_\varepsilon|\xi] = 0 = \lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \max_{i\in\{1,\ldots,N\}} ||\psi_i(u_\varepsilon(\cdot,t)) - \bar\chi_i(\cdot,t)||_{L^1(\mathbb{R}^d)}.$$

Observe furthermore that by the definition of the relative entropy, the convergence $\lim_{\varepsilon\to 0} E[u_\varepsilon|\xi](0) = 0$ *is in fact implied by the convergence of the initial energies* $E[u_\varepsilon](0) \to E[\bar\chi](0) = \frac{1}{2}\sum_i |\nabla\bar\chi_i(\cdot,0)|(\mathbb{R}^d)$ *and the convergence of the initial data* $u_\varepsilon(\cdot,0) \to \sum_{i=1}^N \alpha_i\bar\chi_i(\cdot,0)$ *in* $L^1(\mathbb{R}^d)$.

To summarize, under the assumptions of Theorem 29 but given now a sequence of solutions $(u_\varepsilon)_\varepsilon$ *to the Allen-Cahn equation* (3.1) *satisfying only the qualitative converge properties at initial time*

$$\begin{aligned} u_\varepsilon(\cdot,0) &\underset{\varepsilon\to 0}{\longrightarrow} \sum_{i=1}^N \alpha_i\bar\chi_i(\cdot,0) && \text{in } L^1(\mathbb{R}^d),\\ E[u_\varepsilon](0) &\underset{\varepsilon\to 0}{\longrightarrow} E[\bar\chi](0), \end{aligned}$$

the solutions u_ε *converge to multiphase mean curvature flow in the sense that*

$$u_\varepsilon(\cdot,t) \underset{\varepsilon\to 0}{\longrightarrow} \sum_{i=1}^N \alpha_i\bar\chi_i(\cdot,t) \qquad \text{in } L^1(\mathbb{R}^d) \text{ for all } t\in[0,T].$$

As the next proposition (and its rather straightforward proof, proceeding by glueing together one-dimensional Modica-Mortola profiles) shows, well-prepared initial data satisfying the upper bound $O(\varepsilon)$ for the relative energy actually exist.

Proposition 31. *Let assumptions (A1)–(A4) be in place. Let $d = 2$ and let $(\bar\chi_1(\cdot, 0), \ldots, \bar\chi_N(\cdot, 0))$ be any initial data whose interfaces consist of finitely many C^1 curves that meet at finitely many triple junctions at angles of $120°$. Alternatively, let $d = 3$ and let $(\bar\chi_1(\cdot, 0), \ldots, \bar\chi_N(\cdot, 0))$ be any initial data whose interfaces consist of finitely many C^1 interfaces that meet at finitely many triple lines of class C^1 at angles of $120°$.*

Then for any $\varepsilon > 0$ there exists initial data $u_\varepsilon(\cdot, 0)$ that is well-prepared in the sense that

$$E_\varepsilon[u_\varepsilon|\xi](0) \leq C\varepsilon,$$

$$\max_{i\in\{1,\ldots,N\}} \int_{\mathbb{R}^d} |\psi_i(u_\varepsilon(\cdot, 0)) - \bar\chi_i(\cdot, 0)| \operatorname{dist}(x, \partial \operatorname{supp} \bar\chi_i(\cdot, 0))\, dx \leq C\varepsilon,$$

where the constant C depends on the initial data $(\bar\chi_1(\cdot, 0), \ldots, \bar\chi_N(\cdot, 0))$ and on the potential W.

Nevertheless, note that in the presence of triple junctions this rate of convergence $O(\varepsilon)$ for the relative entropy cannot be improved without modifying either the definition of the relative entropy (3.7) or our assumptions (A1)–(A4), as it may be impossible to construct initial data $u_\varepsilon(\cdot, 0)$ with $E_\varepsilon[u_\varepsilon|\xi] \ll \varepsilon$. Let us illustrate the reason for this limitation in the case $d = 2$: Suppose that the initial data $\bar\chi(\cdot, 0)$ for the strong solution contain at least one triple junction. By virtue of the term $\int \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\, dx$ in the energy and the pointwise nonnegativity of the integrand in the relative entropy, if we were to have $E[u_\varepsilon|\xi](0) \ll \varepsilon$, the approximating initial data $u_\varepsilon(\cdot, 0)$ would have to contain a true mixture of three phases in an ε-ball $B_\varepsilon(y)$ somewhere. At the same time, our assumptions (A1)–(A4) allow the potential W to be arbitrarily large for a true mixture of three phases (i. e., away from the boundary of the triangle in Figure 3.1a for a three-well potential as in Definition 45), independently of the functions ψ_i. If W is large enough, on $B_\varepsilon(y)$ the energy density $\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)$ then cannot be compensated by the term involving $\nabla\psi_i(u_\varepsilon)$ in the relative entropy, resulting in a lower bound for the relative entropy of the order of $\int_{B_\varepsilon(y)} \frac{1}{2\varepsilon}W(u_\varepsilon)\, dx \geq c\varepsilon^{-1} \times \varepsilon^2 = c\varepsilon$. This limits the overall convergence rate for our method to $O(\varepsilon^{1/2})$ when measured e. g. in the L^1 norm. We expect this to be a limitation of our method, caused by an insufficient control of the precise dynamics of the diffuse-interface model at triple junctions by the relative entropy $E[u_\varepsilon|\xi]$; for suitably prepared initial data, we would anticipate a convergence rate $O(\varepsilon)$. Whether such an improved convergence rate can be deduced by a more refined relative entropy approach is an open question.

Observe that the assumptions (A1) and (A2) are indeed sufficient to deduce global existence of bounded solutions to the Allen-Cahn equation (3.1), starting from any measurable initial data taking values in $\overline{U}$.

Remark 32. *Let W be any potential of class $C^{1,1}_{loc}$ satisfying our assumption (A2). Given any measurable initial data $u_\varepsilon(\cdot, 0)$ taking values in $\overline{U}$, for any $T > 0$ there exists a unique bounded weak solution u_ε to the Allen-Cahn equation* (3.1) *on the time interval $[0, T]$. To see this, one may first show existence of a weak solution for a slightly modified PDE obtained by replacing $\partial_u W$ outside of $\overline{U}$ by a Lipschitz extension. For this modified PDE, existence of a weak solution can be shown in a standard way. A comparison argument (using (A2) and in particular the convexity of $\overline{U}$) then ensures that the weak solution to this modified equation* *may only take values in $\overline{U}$, proving both that it is bounded and that it actually solves the original equation. Uniqueness is shown via the standard argument of a Gronwall-type estimate for the squared $L^2(\mathbb{R}^d)$ norm of the difference between two solutions.*

We next recall the definition of strong solutions to multiphase mean curvature flow in the case of two dimensions. For intuitive but technical-to-state geometric notions, we will refer to the precise definitions in [46].

Definition 33 (Strong solution for multiphase mean curvature flow)**.** *Let $d = 2$, let $P \geq 2$ be an integer, and let $T > 0$ be a finite time horizon. Let $\bar{\chi}_0 = (\bar{\chi}_1^0, \ldots, \bar{\chi}_P^0)$ be an initial regular partition of $\mathbb{R}^2$ with finite interface energy in the sense of [46, Definition 14].*

A measurable map

$$\bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P) \colon \mathbb{R}^d \times [0, T] \to \{0, 1\}^P,$$

is called a strong solution for multiphase mean curvature flow with initial data $\bar{\chi}_0$ *if it satisfies the following conditions:*

i) (Smoothly evolving regular partition with finite interface energy) Denote by $I_{i,j} := \operatorname{supp} \bar{\chi}_i \cap \operatorname{supp} \bar{\chi}_j$ for $i \neq j$ the interface between phases i and j. The map $\bar{\chi}$ is a smoothly evolving regular partition of $\mathbb{R}^d \times [0, T]$ and $\mathcal{I} := \bigcup_{i,j \in \{1,\ldots,P\}, i \neq j} I_{i,j}$ is a smoothly evolving regular network of interfaces in $\mathbb{R}^d \times [0, T]$ in the sense of [46, Definition 15]. In particular, for every $t \in [0, T]$, $\bar{\chi}(\cdot, t)$ is a regular partition of $\mathbb{R}^d$ and $\bigcup_{i \neq j} I_{i,j}(t)$ is a regular network of interfaces in $\mathbb{R}^d$ in the sense of [46, Definition 14] such that

$$\sup_{t \in [0,T]} E[\bar{\chi}(\cdot, t)] = \sup_{t \in [0,T]} \sum_{i,j=1, i<j}^{P} \int_{I_{i,j}(t)} 1 \, \mathrm{d}S < \infty. \tag{3.3a}$$

ii) (Evolution by mean curvature) For $i, j = 1, \ldots, P$ with $i \neq j$ and $(x, t) \in I_{i,j}$ let $\bar{V}_{i,j}(x, t)$ denote the normal speed of the interface at the point $x \in I_{i,j}(t)$. Denoting by $\mathrm{H}_{i,j}(x, t)$ and $\mathrm{n}_{i,j}(x, t)$ the mean curvature vector and the normal vector of $I_{i,j}(t)$ at $x \in I_{i,j}(t)$, the interfaces $I_{i,j}$ evolve by mean curvature in the sense

$$\bar{V}_{i,j}(x, t)\mathrm{n}_{i,j}(x, t) = \mathrm{H}_{i,j}(x, t), \quad \text{for all } t \in [0, T],\ x \in I_{i,j}(t). \tag{3.3b}$$

iii) (Initial conditions) We have $\bar{\chi}_i(x, 0) = \bar{\chi}_i^0(x)$ for all points $x \in \mathbb{R}^d$ and each phase $i \in \{1, \ldots, P\}$.

Our main results centrally rely on the concept of gradient flow calibrations introduced in [46], whose definition we next recall.

Definition 34. *Let $d \geq 2$. Let $(\bar{\chi}_1, \ldots, \bar{\chi}_N)$ be a smoothly evolving partition of $\mathbb{R}^d$ on a time interval $[0, T)$. Denote by $I_{i,j} := \operatorname{supp} \bar{\chi}_i \cap \operatorname{supp} \bar{\chi}_j$, $1 \leq i, j \leq N$, $i \neq j$, the corresponding interfaces. We say that a collection of $C^{1,1}$ vector fields $\xi_i : \mathbb{R}^d \times [0, T) \to \mathbb{R}^d$, $1 \leq i \leq N$, and $B : \mathbb{R}^d \times [0, T) \to \mathbb{R}^d$ is a gradient flow calibration if the following conditions are satisfied:*

$$\partial_t \xi_{i,j} + (B \cdot \nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}} \xi_{i,j} = O(\operatorname{dist}(\cdot, I_{i,j})), \tag{3.4a}$$

$$\frac{1}{2}\xi_{i,j} \cdot (\partial_t \xi_{i,j} + (B \cdot \nabla)\xi_{i,j}) = O(\operatorname{dist}^2(\cdot, I_{i,j})), \tag{3.4b}$$

$$(B \cdot \xi_{i,j})\xi_{i,j} + (\nabla \cdot \xi_{i,j})\xi_{i,j}, = O(\operatorname{dist}(\cdot, I_{i,j})) \tag{3.4c}$$

$$\nabla B : \xi_{i,j} \otimes \xi_{i,j} = O(\operatorname{dist}(\cdot, I_{i,j})), \tag{3.4d}$$

$$\nabla B : (\xi_{i,j}^{\perp} \otimes \xi_{i,j} + \xi_{i,j} \otimes \xi_{i,j}^{\perp}) = O(\operatorname{dist}(\cdot, I_{i,j})), \tag{3.4e}$$

$$1 - C_{len} \operatorname{dist}^2(\cdot, I_{i,j}) \leq |\xi_{i,j}|^2 \leq 1 - c_{len} \min\{\operatorname{dist}^2(\cdot, I_{i,j}), 1\}, \tag{3.4f}$$

$$\xi_{i,j} = \mathrm{n}_{i,j} \qquad \textit{on } I_{i,j}, \tag{3.4g}$$

$$|\sqrt{3}\xi_i| \leq 1 \qquad \textit{and} \qquad \sum_{i=1}^{N} \xi_i = 0, \tag{3.4h}$$

$$|\xi_{i,j}|^2 + (4 - \delta_{cal}) \sum_{\substack{k=1 \\ k \notin \{i,j\}}}^{N} |\sqrt{3}\xi_{i,j} \cdot \xi_k|^2 \leq 1, \tag{3.4i}$$

for some constants $C_{len} > 0$, $c_{len} \in (0,1)$, an arbitrarily small $\delta_{cal} > 0$ and any distinct $i, j \in \{1, ..N\}$.

Moreover, we call a family of $C^{1,1}$ functions ϑ_i a family of evolving distance weights if they satisfy

$$\begin{aligned} \vartheta_i(\cdot, t) &\leq -c \min\{\operatorname{dist}(\cdot, I_{i,j}(t)), 1\} && \textit{in } \{\bar{\chi}_i(\cdot, t) = 1\}, && (3.5a)\\ \vartheta_i(\cdot, t) &\geq c \min\{\operatorname{dist}(\cdot, I_{i,j}(t)), 1\} && \textit{outside of } \{\bar{\chi}_i(\cdot, t) = 1\}, && (3.5b)\\ |\vartheta_i(\cdot, t)| &\leq C \min\{\operatorname{dist}(\cdot, I_{i,j}(t)), 1\} && \textit{globally}, && (3.5c) \end{aligned}$$

and

$$|\partial_t \vartheta_i + B \cdot \nabla \vartheta_i| \leq C|\vartheta_i|. \tag{3.6}$$

Note that the existence of a calibration for a given smoothly evolving partition entails that the partition must evolve by multiphase mean curvature flow (i. e., the partition must be a strong solution to multiphase mean curvature flow). In fact, the conditions (3.4a), (3.4c), (3.4g), and (3.4f) are sufficient to deduce the property (3.3b). Observe that the condition (3.4i) is not stated in [46], however it follows from the construction of the gradient flow calibration provided in [46] (for more details see Section 3.7).

For many geometries, $(\bar{\chi}_1, \ldots, \bar{\chi}_N)$ being a strong solution to multiphase mean curvature flow is also sufficient to construct a gradient flow calibration.

Theorem 35 (Existence of gradient flow calibrations, [46, Theorem 6] and [57, Theorem 1])**.** *Let $d \in \{2,3\}$ and let $\bar\chi_0$ be a regular partition of $\mathbb{R}^d$ with finite surface energy; for $d=3$, assume furthermore that the partition corresponds to a double bubble type geometry. Let $\bar\chi$ be a strong solution to multiphase mean curvature flow on the time interval $[0,T]$ in the sense of Definition 33 (for $d=2$) respectively in the sense of [57, Definition 10] (for $d=3$). Then for any $\delta_{cal}>0$ and any $c_{len}\geq 1$ there exists a gradient flow calibration in the sense of Definition 34 up to time T. Furthermore, there also exists a family of evolving distance weights.*

Finally, we conclude this section by showing that the class of potentials W satisfying the assumptions (A1)–(A4) is indeed sufficiently broad. In fact, given

- a prescribed set of N minima $\alpha_i \in \mathbb{R}^{N-1}$, $1\leq i\leq N$,
- a prescribed set of non-intersecting minimal paths $\gamma_{i,j}$, $1\leq i<j\leq N$, that meet at the α_i at positive angles, and
- a potential $\tilde W : \cup_{i,j:i<j}\gamma_{i,j} \to [0,\infty)$ defined on the minimal paths $\gamma_{i,j}$ and subject to (A1) and (A3), i. e. in particular with $\int_{\gamma_{i,j}} \sqrt{2\tilde W(u)}\,d\gamma(u) = 1$,

it is always possible to extend the potential $\tilde W$ to a potential $W:\mathbb{R}^{N-1}\to[0,\infty)$ that satisfies condition (A4). More precisely, to satisfy (A4) it is sufficient to require $W(u) \geq (1+M|u-\alpha_i|^{-4}\operatorname{dist}(u,\gamma)^4)W(P_\gamma u)$ in some neighborhood $\mathcal{U}_i$ of α_i (with P_γ denoting the projection onto the nearest point among all paths $\gamma := \cup_{j,k}\gamma_{j,k}$) as well as $W(u)\geq M\operatorname{dist}(u,\cup_{i<j}\gamma_{i,j})^2$ in $\mathbb{R}^{N-1}\setminus\cup_i\mathcal{U}_i$. Here, M is a constant depending only on $\tilde W$, the paths $\gamma_{i,j}$, and the neighborhoods $\mathcal{U}_i$.

For the sake of simplicity, we limit ourselves in our rigorous statement to the study of potentials defined on a simplex $\triangle^{N-1}$; however, it is not too difficult to see that our construction would generalize to the aforementioned situation.

Proposition 36. *Let $N\geq 3$. Let $\triangle^{N-1}$ be an $(N-1)$-simplex with edges of unit length in $\mathbb{R}^{N-1}$. Let $W:\triangle^{N-1}\to[0,\infty)$ be a strongly coercive symmetric N-well potential on the simplex $\triangle^{N-1}$ in the sense of Definition 45 below. Then, the assumptions (A1)–(A4) (see Section 3.2) are satisfied. In particular, (A4) holds true for the set of functions $\psi_i:\triangle^{N-1}\to[0,1]$, $1\leq i\leq N$, provided by Construction 47 below.*

3.3 Strategy of the proof

The key idea for our proof is the notion of relative entropy (or, more accurately, relative energy) given by

$$\begin{aligned} E[u_\varepsilon|\xi] &:= E[u_\varepsilon] + \sum_{i=1}^N \int_{\mathbb{R}^d} \xi_i\cdot\nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) + \sum_{i=1}^N \xi_i\cdot\nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x. \end{aligned} \tag{3.7}$$

The form of the ansatz for the relative entropy is inspired by two earlier approaches:

- The concept of gradient flow calibrations introduced in [46] by the first author, Hensel, Laux, and Simon to derive weak-strong uniqueness and stability results for distributional solutions to multiphase mean curvature flow. Gradient flow calibrations provide a lower bound of the form $-\sum_i \int \xi_i \cdot d\nabla\chi_i$ on the interface energy functional $\frac{1}{2}\sum_i \int 1\, d|\nabla\chi_i|$, thereby facilitating a relative entropy approach to weak-strong uniqueness principles for multiphase mean curvature flow. We emphasize that gradient flow calibrations are specifically designed to handle the (singular) geometries at triple junctions in the strong solution. We refer to [66, 45] for earlier uses of relative entropy techniques for weak-strong uniqueness for geometric evolution problems with smooth geometries (in the strong solution) (see also [58] for a further development of the relative entropy argument in order to incorporate the constant ninety degree contact angle condition).

- The relative entropy approach to the sharp-interface limit of the scalar Allen-Cahn equation by the first author, Laux, and Simon [48], relying on the Modica-Mortola trick to obtain a lower bound of the form $\int \xi \cdot \nabla\psi(u_\varepsilon)\, dx$ for the Ginzburg-Landau energy $E[u_\varepsilon] = \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\, dx$ (see also [59] for an adaptation of this approach in order to encode the constant contact angle condition and [72] for a subsequent application of the relative energy method to a problem in the context of liquid crystals).

The two key steps towards establishing our main results are as follows:

- Establishing a number of coercivity properties of the relative entropy $E[u_\varepsilon|\xi]$, including for example

$$E[u_\varepsilon|\xi] \geq c \int \min\{\mathrm{dist}^2(\cdot, \cup_{i\neq j} I_{i,j}), 1\}(\tfrac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \tfrac{1}{\varepsilon}W(u_\varepsilon))\, dx. \tag{3.8}$$

- Deriving a Gronwall-type estimate for the time evolution of the relative energy of the type

$$\partial_t E[u_\varepsilon|\xi] \leq C E[u_\varepsilon|\xi].$$

We shall illustrate this strategy by stating the main intermediate results in the present section below.

As it central for our strategy, let us first give the main argument for the coercivity of the relative entropy (3.7) (despite it being slightly technical). It makes use of the following elementary lemma.

Lemma 37. *Let ξ_i, $1 \leq i \leq N$, be vector fields of class C^1 satisfying $\sum_{i=1}^N \xi_i = 0$; suppose that at any point $(x,t) \in \mathbb{R}^d \times [0,T]$ at most three of the ξ_i do not vanish. Let $\psi_i : \mathbb{R}^{N-1} \to [0,1]$, $1 \leq i \leq N$, be functions as in assumption (A4). In particular, set $\psi_0 := 1 - \sum_{i=1}^N \psi_i$. Let $u_\varepsilon \in L^\infty([0,T]; \dot{H}^1(\mathbb{R}^d))$. Defining $\psi_{i,j} := \psi_j - \psi_i$ and $\xi_{i,j} := \xi_i - \xi_j$, we have for any distinct $i, j, k \in \{1, \ldots, N\}$*

$$\sum_{\ell=1}^N \xi_\ell \otimes \nabla(\psi_\ell \circ u_\varepsilon) = -\frac{1}{2}\xi_{i,j} \otimes \nabla(\psi_{i,j} \circ u_\varepsilon) + \sum_{\substack{k=1 \\ k\notin\{i,j\}}}^N \frac{1}{2}\xi_k \otimes \nabla(\psi_0 \circ u_\varepsilon) \tag{3.9}$$

almost everywhere in $\{u_\varepsilon \in \mathcal{T}_{i,j}\}$ *as well as*

$$\sum_{\ell=1}^{N} \nabla\xi_\ell \otimes \nabla(\psi_\ell \circ u_\varepsilon) = -\frac{1}{2}\nabla\xi_{i,j} \otimes \nabla(\psi_{i,j} \circ u_\varepsilon) + \sum_{\substack{k=1\\k\notin\{i,j\}}}^{N} \frac{1}{2}\nabla\xi_k \otimes \nabla(\psi_0 \circ u_\varepsilon), \tag{3.10}$$

$$\sum_{\ell=1}^{N} \nabla\xi_\ell \otimes \partial_u\psi_\ell(u_\varepsilon) = -\frac{1}{2}\nabla\xi_{i,j} \otimes \partial_u\psi_{i,j}(u_\varepsilon) + \sum_{\substack{k=1\\k\notin\{i,j\}}}^{N} \frac{1}{2}\nabla\xi_k \otimes \partial_u\psi_0(u_\varepsilon) \tag{3.11}$$

almost everywhere in $\{u_\varepsilon \in \mathcal{T}_{i,j}\}$.

Proof. By adding zeros, using the definitions $\xi_{i,j} := \xi_i - \xi_j$ and $\psi_{i,j} := \psi_j - \psi_i$, we obtain

$$\begin{aligned}\sum_{\ell=1}^{N} \xi_\ell \cdot \nabla(\psi_\ell \circ u_\varepsilon) = &- \frac{1}{2}\xi_{i,j} \cdot \nabla(\psi_{i,j} \circ u_\varepsilon) + \frac{1}{2}\xi_i \cdot \nabla((\psi_i + \psi_j) \circ u_\varepsilon)\\ &+ \frac{1}{2}\xi_j \cdot \nabla((\psi_i + \psi_j) \circ u_\varepsilon) + \sum_{\substack{k=1\\k\notin\{i,j\}}}^{N} \xi_k \cdot \nabla(\psi_k \circ u_\varepsilon),\end{aligned}$$

almost everywhere in $\mathbb{R}^d \times (0,T)$. The equation (3.9) now follows by exploiting that $\partial_u\psi_k = 0$ on $\mathcal{T}_{i,j}$ for $k \notin \{i,j\}$, inserting the definition of ψ_0, and using $\sum_{\ell=1}^{N} \xi_\ell = 0$. The proof of the other properties is analogous. □

With the previous lemma and our assumptions (A1)–(A4), it becomes rather straightforward to establish coercivity of our relative energy: Observe that we may compute for (x,t) with $u_\varepsilon(x,t) \in \mathcal{T}_{i,j}$

$$\begin{aligned}&\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) + \sum_{\ell=1}^{N} \xi_\ell \cdot \nabla(\psi_\ell \circ u_\varepsilon)\\ &= \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) - \Bigg(\tfrac{1}{2}\partial_u\psi_{i,j}(u_\varepsilon) \otimes \xi_{i,j} - \sum_{\substack{k=1\\k\notin\{i,j\}}}^{N} \tfrac{1}{2\sqrt{3}}\partial_u\psi_0(u_\varepsilon) \otimes \sqrt{3}\xi_k\Bigg) : \nabla u_\varepsilon\\ &= \frac{1}{2}\Bigg|\sqrt{\varepsilon}\nabla u_\varepsilon - \frac{1}{\sqrt{\varepsilon}}\Bigg(\tfrac{1}{2}\partial_u\psi_{i,j}(u_\varepsilon) \otimes \xi_{i,j} - \sum_{\substack{k=1\\k\notin\{i,j\}}}^{N} \tfrac{1}{2\sqrt{3}}\partial_u\psi_0(u_\varepsilon) \otimes \sqrt{3}\xi_k\Bigg)\Bigg|^2\\ &\quad + \frac{1}{2\varepsilon}\Bigg[2W(u_\varepsilon) - \Bigg|\tfrac{1}{2}\partial_u\psi_{i,j}(u_\varepsilon) \otimes \xi_{i,j} - \sum_{\substack{k=1\\k\notin\{i,j\}}}^{N} \tfrac{1}{2\sqrt{3}}\partial_u\psi_0(u_\varepsilon) \otimes \sqrt{3}\xi_k\Bigg|^2\Bigg]\end{aligned} \tag{3.12}$$

due to the fact that $\psi_k \equiv 0$ on $\mathcal{T}_{i,j}$ for any $k \in \{1,...,N\} \setminus \{i,j\}$. This will be the starting point to prove the coercivity properties satisfied by the relative energy functional (3.7); note in particular that

$$\begin{aligned}&\Bigg|\tfrac{1}{2}\partial_u\psi_{i,j}(u_\varepsilon) \otimes \xi_{i,j} - \sum_{\substack{k=1\\k\notin\{i,j\}}}^{N} \tfrac{1}{2\sqrt{3}}\partial_u\psi_0(u_\varepsilon) \otimes \sqrt{3}\xi_k\Bigg|^2\\ &= |\xi_{i,j}|^2\Big|\tfrac{1}{2}\partial_u\psi_{i,j}(u_\varepsilon)\Big|^2 + \sum_{\substack{k=1\\k\notin\{i,j\}}}^{N} |\sqrt{3}\xi_k|^2\Big|\tfrac{1}{2\sqrt{3}}\partial_u\psi_0(u_\varepsilon)\Big|^2\end{aligned}$$

$$- \sum_{\substack{k=1 \\ k\notin\{i,j\}}}^{N} \frac{1}{2}(\xi_{i,j}\cdot\xi_k)\partial_u\psi_{i,j}(u_\varepsilon)\cdot\partial_u\psi_0(u_\varepsilon).$$

Using our assumption (A4) and the properties of the gradient flow calibration $|\xi_i| \leq \frac{1}{\sqrt{3}}$, $|\xi_{i,j}| \leq 1$ and (3.4i), this establishes a first coercivity bound like (3.8). Going substantially beyond this simple estimate, we shall see that in fact we have the following coercivity properties.

Proposition 38. *Let W and ψ_i be functions subject to assumption (A4). Let ξ_i, $1 \leq i \leq N$, be any collection of C^1 vector fields satisfying $\sum_{i=1}^N \xi_i = 0$, $|\sqrt{3}\xi_i| \leq 1$ for all i, as well as with the notation $\xi_{i,j} := \xi_i - \xi_j$*

$$|\xi_{i,j}|^2 + (4-\delta_{\textit{cal}}) \sum_{\substack{k=1 \\ k\notin\{i,j\}}}^{N} |\sqrt{3}\xi_{i,j}\cdot\xi_k|^2 \leq 1 \tag{3.13}$$

for some arbitrarily small $\delta_{\textit{cal}} > 0$. Furthermore, suppose that at each point at most three of the vector fields ξ_i do not vanish. For any function $u_\varepsilon \in \dot{H}^1(\mathbb{R}^d;\overline{U})$ with $E[u_\varepsilon] < \infty$, we then have the estimates

$$\int_{\mathbb{R}^d} \left(\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right)^2 \mathrm{d}x \leq CE[u_\varepsilon|\xi]\,, \tag{3.14a}$$

$$\sum_{\substack{i,j=1 \\ i<j}}^{N} \int_{\mathbb{R}^d} \left|\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} - \xi_{i,j}\right|^2 |\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \leq CE[u_\varepsilon|\xi], \tag{3.14b}$$

$$\sum_{\substack{i,j=1 \\ i<j}}^{N} \int_{\mathbb{R}^d} \min\{\mathrm{dist}^2(x,I_{i,j}),1\}|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \leq CE[u_\varepsilon|\xi], \tag{3.14c}$$

$$\sum_{\substack{i,j=1 \\ i<j}}^{N} \int_{\mathbb{R}^d} \min\{\mathrm{dist}^2(x,I_{i,j}),1\}\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \leq CE[u_\varepsilon|\xi]\,, \tag{3.14d}$$

$$\sum_{\substack{i,j=1 \\ i<j}}^{N} \int_{\mathbb{R}^d} \varepsilon|(\mathrm{Id}-\xi_{i,j}\otimes\xi_{i,j})\nabla u_\varepsilon^{\mathsf{T}}|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \leq CE[u_\varepsilon|\xi]. \tag{3.14e}$$

These coercivity estimates will be derived as a consequence of the computation (3.12) and the following coercivity properties.

Proposition 39. *Let W and ψ_i be functions subject to assumption (A4). Let ξ_i, $1 \leq i \leq N$, be as in Proposition 38. For any function $u_\varepsilon \in \dot{H}^1(\mathbb{R}^d;\overline{U})$ with $E[u_\varepsilon] < \infty$, we then have the estimates*

$$\sum_{\substack{i,j=1 \\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{\varepsilon}\left|\partial_u\psi_0(u_\varepsilon)\right|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \leq CE[u_\varepsilon|\xi], \tag{3.15a}$$

$$\sum_{\substack{i,j=1 \\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{\varepsilon}\left|\partial_u\psi_{i,j}(u_\varepsilon)\cdot\partial_u\psi_0(u_\varepsilon)\right| \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \leq CE[u_\varepsilon|\xi]\,, \tag{3.15b}$$

$$\sum_{\substack{i,j=1 \\ i<j}}^{N} \int_{\mathbb{R}^d} |\nabla(\psi_0\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \leq CE[u_\varepsilon|\xi]\,. \tag{3.15c}$$

To introduce a proxy at the level of the Allen-Cahn equation for the limiting mean curvature (or, more precisely, a quantity H_ε such that $|\mathrm{H}_\varepsilon|^2$ is a proxy for the dissipation in mean curvature flow), we introduce the abbreviation

$$\mathrm{H}_\varepsilon := -\varepsilon\left(\Delta u_\varepsilon - \frac{1}{\varepsilon^2}\partial_u W(u_\varepsilon)\right)\cdot\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}. \tag{3.16}$$

The key step in our proof is to establish the following estimate for the relative energy using a Gronwall-type argument.

Theorem 40 (Relative energy inequality)**.** *Let $\bar\chi = (\bar\chi_1, \ldots, \bar\chi_N)$ be a smoothly evolving partition of $\mathbb{R}^d$; let $((\xi_i)_i, B)$ be an associated gradient flow calibration in the sense of Definition 34. Let W be a potential subject to assumptions (A1)–(A4). Let u_ε be a bounded solution to the vector-valued Allen-Cahn equation* (3.1) *with initial data $u_\varepsilon(\cdot,0) \in \dot{H}^1(\mathbb{R}^d;\overline{U})$ with finite energy $E[u_\varepsilon(\cdot,0)] < \infty$. Then for any $t \in [0,T]$ the estimate*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E[u_\varepsilon|\xi] + \sum_{\substack{i,j=1\\ i<j}}^{N}\int_{\mathbb{R}^d}\frac{1}{2\varepsilon}|\mathrm{H}_\varepsilon - \varepsilon(B\cdot\xi_{i,j})\xi_{i,j}|\nabla u_\varepsilon||^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&+\int_{\mathbb{R}^d}\frac{1}{2\varepsilon}\left(\left|\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\right|^2 - |\mathrm{H}_\varepsilon|^2\right)\mathrm{d}x\\
&+\int_{\mathbb{R}^d}\frac{1}{4\varepsilon}\left|\left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\right) + \sum_{i=1}^{N}(\nabla\cdot\xi_i)\partial_u\psi_i(u_\varepsilon)\right|^2\mathrm{d}x\\
&\leq C(d,\bar\chi)E[u_\varepsilon|\xi]
\end{aligned} \tag{3.17}$$

holds true, with H_ε as defined in (3.16) *and $E[u_\varepsilon|\xi]$ as defined in* (3.2)*.*

Building on the previous estimate and the coercivity properties of the relative entropy, we will show the following error estimate at the level of the indicator functions.

Proposition 41. *Let the assumptions of Theorem 40 be in place. In addition, let ϑ_i be a family of evolving distance weights as defined in Definition 34. We then have for all $i \in \{1,\ldots,N\}$*

$$\begin{aligned}
&\sup_{t\in[0,T]}\int_{\mathbb{R}^d}|\psi_i(u_\varepsilon) - \bar\chi_i|\min\{\mathrm{dist}(\cdot,\partial\operatorname{supp}\bar\chi_i(\cdot,t)),1\}\,\mathrm{d}x\\
&\leq C(d,T,(\bar\chi(t))_{t\in[0,T]})E[u_\varepsilon|\xi](0)\\
&\quad + C(d,T,(\bar\chi(t))_{t\in[0,T]})\int_{\mathbb{R}^d}|\psi_i(u_\varepsilon(\cdot,0)) - \bar\chi_i(\cdot,0)|\min\{\mathrm{dist}(\cdot,\partial\operatorname{supp}\bar\chi_i(\cdot,0)),1\}\,\mathrm{d}x.
\end{aligned}$$

The proof of Theorem 40 crucially relies on the coercivity properties of Proposition 39 and 38 and the following simplification of the evolution equation for the relative entropy.

Lemma 42. *Let W be a potential of class $C^{1,1}_{loc}(\mathbb{R}^{N-1})$ subject to assumptions (A1)–(A4). Let u_ε be a solution to the vector-valued Allen-Cahn equation* (3.1) *with initial data $u_\varepsilon(\cdot,0) \in \dot{H}^1(\mathbb{R}^d;\overline{U})$ with finite energy $E[u_\varepsilon(\cdot,0)] < \infty$. Let (ξ_i, B) be a gradient flow calibration in the sense of Definition 34. The time evolution of the relative energy* (3.7) *is then given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}E[u_\varepsilon|\xi] \tag{3.18}$$

$$
\begin{aligned}
&= -\sum_{\substack{i,j=1\\i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2\varepsilon}\Big|\,\mathrm{H}_\varepsilon - \varepsilon(B\cdot\xi_{i,j})\xi_{i,j}|\nabla u_\varepsilon|\Big|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^d} \frac{1}{2\varepsilon}\left|\Big(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\Big) + \sum_{i=1}^{N}(\nabla\cdot\xi_i)\partial_u\psi_i(u_\varepsilon)\right|^2 \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^d} \frac{1}{2\varepsilon}\left(\Big|\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\Big|^2 - |\,\mathrm{H}_\varepsilon|^2\right)\mathrm{d}x \\
&\quad + \mathrm{Err}_{AllenCahn} + \mathrm{Err}_{instab} + \mathrm{Err}_{dt\xi} + \mathrm{Err}_{MC\xi} + \mathrm{Err}_{OtherPhases}
\end{aligned}
$$

where we have abbreviated

$$
\begin{aligned}
&\mathrm{Err}_{instab} := \\
&\int_{\mathbb{R}^d}(\nabla\cdot B)\Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) + \sum_{i=1}^{N}\xi_i\cdot\nabla(\psi_i\circ u_\varepsilon)\Big)\,\mathrm{d}x \\
&- \sum_{\substack{i,j=1\\i<j}}^{N}\int_{\mathbb{R}^d}\frac{1}{2}\nabla B : \left(\xi_{i,j} - \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\right)\otimes\left(\xi_{i,j} - \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\right) \\
&\qquad\qquad \times |\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x
\end{aligned}
\tag{3.19a}
$$

and

$$
\begin{aligned}
&\mathrm{Err}_{AllenCahn} := \\
&\quad \sum_{\substack{i,j=1\\i<j}}^{N}\int_{\mathbb{R}^d}\nabla B : \left(\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\otimes\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\frac{1}{2}|\nabla(\psi_{i,j}\circ u_\varepsilon)|\right. \\
&\qquad\qquad \left. - \varepsilon\nabla u_\varepsilon^{\mathsf{T}}\nabla u_\varepsilon\right)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x
\end{aligned}
\tag{3.19b}
$$

and

$$
\begin{aligned}
\mathrm{Err}_{dt\xi} &:= \sum_{\substack{i,j=1\\i<j}}^{N}\int_{\mathbb{R}^d}\frac{1}{2}(\partial_t\xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}}\xi_{i,j}) \\
&\qquad \cdot\left(\xi_{i,j} - \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\right)|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\quad - \sum_{\substack{i,j=1\\i<j}}^{N}\int_{\mathbb{R}^d}\frac{1}{2}\xi_{i,j}\cdot(\partial_t\xi_{i,j} + (B\cdot\nabla)\xi_{i,j})|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x
\end{aligned}
\tag{3.19c}
$$

as well as

$$
\begin{aligned}
\mathrm{Err}_{MC\xi} &:= \int_{\mathbb{R}^d}\frac{1}{2\varepsilon}\left|\sum_{i=1}^{N}(\nabla\cdot\xi_i)\partial_u\psi_i(u_\varepsilon)\right|^2\mathrm{d}x \\
&\quad - \sum_{i=1}^{N}\int_{\mathbb{R}^d}(\nabla\cdot\xi_i)B\cdot\nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x \\
&\quad + \sum_{\substack{i,j=1\\i<j}}^{N}\int_{\mathbb{R}^d}\frac{\varepsilon}{2}|B\cdot\xi_{i,j}|^2|\xi_{i,j}|^2|\nabla u_\varepsilon|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x
\end{aligned}
\tag{3.19d}
$$

$$+\sum_{\substack{i,j=1\\i<j}}^{N}\int_{\mathbb{R}^d}(\mathrm{Id}-\xi_{i,j}\otimes\xi_{i,j}):\mathrm{H}_\varepsilon\otimes B|\nabla u_\varepsilon|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x$$

and

$$\begin{aligned}&\mathrm{Err}_{OtherPhases}:=\\&\sum_{\substack{i,j,k=1\\i<j,k\notin\{i,j\}}}^{N}\int_{\mathbb{R}^d}\frac{1}{2}(\partial_t\xi_k+(B\cdot\nabla)\xi_k+(\nabla B)^{\mathsf{T}}\xi_k)\cdot\nabla(\psi_0\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\&-\sum_{\substack{i,j,k=1\\i<j,k\notin\{i,j\}}}^{N}\int_{\mathbb{R}^d}\frac{1}{2}\nabla B:\nabla(\psi_0\circ u_\varepsilon)\otimes\xi_k\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\&-\sum_{\substack{i,j,k=1\\i<j,k\notin\{i,j\}}}^{N}\int_{\mathbb{R}^d}\frac{1}{2}\nabla B:\xi_k\otimes\nabla(\psi_0\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x.\end{aligned}\tag{3.19e}$$

3.4 The relative energy argument

3.4.1 Derivation of the Gronwall inequality for the relative entropy

We first show how the evolution estimate for the relative entropy from Lemma 42 and the coercivity properties of our relative entropy together imply a Gronwall-type estimate for the evolution of the relative entropy.

Proof of Theorem 40. We proceed by estimating the terms on the right-hand side of the equation (3.18) for the time evolution of the relative energy. Note that it will be sufficient to prove

$$\begin{aligned}&\mathrm{Err}_{AllenCahn}+\mathrm{Err}_{instab}+\mathrm{Err}_{dt\xi}+\mathrm{Err}_{MC\xi}+\mathrm{Err}_{OtherPhases}\\&\leq C(\xi(t),B(t),\bar\delta)E[u_\varepsilon|\xi]\\&\quad+\bar\delta\sum_{\substack{i,j=1\\i<j}}^{N}\int_{\mathbb{R}^d}\frac{1}{2\varepsilon}\Big|\,\mathrm{H}_\varepsilon-\varepsilon(B\cdot\xi_{i,j})\xi_{i,j}|\nabla u_\varepsilon|\Big|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\&\quad+\bar\delta\int_{\mathbb{R}^d}\frac{1}{2\varepsilon}\left|\Big(\varepsilon\Delta u_\varepsilon-\frac{1}{\varepsilon}\partial_uW(u_\varepsilon)\Big)+\sum_{i=1}^{N}(\nabla\cdot\xi_i)\partial_u\psi_i(u_\varepsilon)\right|^2\mathrm{d}x\end{aligned}$$

for any $\bar\delta>0$, as then an absorption argument applied to (3.18) (for $\bar\delta<\frac{1}{4}$) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E[u_\varepsilon|\xi]\leq C(\xi(t),B(t),\bar\delta)E[u_\varepsilon|\xi].$$

The Gronwall inequality then implies our conclusion.

Step 1: Estimates for $\mathrm{Err}_{OtherPhases}$, $\mathrm{Err}_{dt\xi}$, *and* Err_{instab}. We first show that

$$\mathrm{Err}_{instab}+\mathrm{Err}_{dt\xi}+\mathrm{Err}_{OtherPhases}\leq C(\xi(t),B(t))E[u_\varepsilon|\xi].$$

Indeed, it is immediate by the definition (3.19e) and the coercivity property (3.15c) of our relative energy that the inequality

$$\begin{aligned}\mathrm{Err}_{OtherPhases} &\leq \sum_{\substack{i,j=1\\ i<j}}^{N} C(\xi(t),B(t)) \int_{\mathbb{R}^d} |\nabla\psi_0|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\ &\leq C(\xi(t),B(t))E[u_\varepsilon|\xi]\end{aligned}$$

holds. Using the defining properties (3.4a) and (3.4b) of the calibration ξ and the coercivity properties (3.14b) and (3.14c) of our relative energy, we likewise deduce from the definition (3.19c) that $\mathrm{Err}_{dt\xi} \leq C(\xi(t),B(t))E[u_\varepsilon|\xi]$, using for instance the estimate

$$\begin{aligned}&\sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}(\partial_t\xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}}\xi_{i,j}) \\ &\qquad\qquad \cdot \left(\xi_{i,j} - \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\right)|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\ &\leq \sum_{\substack{i,j=1\\ i<j}}^{N} \left(\int_{\mathbb{R}^d} \left|\partial_t\xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}}\xi_{i,j}\right|^2 |\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\right)^{1/2} \\ &\qquad\qquad \times \left(\int_{\mathbb{R}^d} \left|\xi_{i,j} - \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\right|^2 |\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\right)^{1/2} \\ &\overset{\text{(3.4a),(3.14b),(3.14c)}}{\leq} C(\xi(t),B(t))E[u_\varepsilon|\xi].\end{aligned}$$

Similarly, recalling the definition (3.19a) and (3.14b) as well as (3.2), we immediately get $\mathrm{Err}_{instab} \leq C(\xi(t),B(t))E[u_\varepsilon|\xi]$. It therefore only remains to estimate $\mathrm{Err}_{AllenCahn}$ and $\mathrm{Err}_{MC\xi}$.

Step 2: Estimate for $\mathrm{Err}_{AllenCahn}$. By adding zeroes, we may rewrite

$$\begin{aligned}&\mathrm{Err}_{AllenCahn} \\ &= \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \nabla B : \left(\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} \otimes \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} - \xi_{i,j}\otimes\xi_{i,j}\right) \\ &\qquad\qquad \times \left(\frac{1}{2}|\nabla(\psi_{i,j}\circ u_\varepsilon)| - \varepsilon|\nabla u_\varepsilon|^2\right)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\ &\quad + \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \nabla B : \xi_{i,j}\otimes\xi_{i,j}\left(\frac{1}{2}|\nabla(\psi_{i,j}\circ u_\varepsilon)| - \varepsilon|\nabla u_\varepsilon|^2\right)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\ &\quad + \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \nabla B : \left(\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} \otimes \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} - \frac{\nabla u_\varepsilon^{\mathsf{T}}\nabla u_\varepsilon}{|\nabla u_\varepsilon|^2}\right)\varepsilon|\nabla u_\varepsilon|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

The first term on the right-hand side can be bounded by $C(\xi(t),B(t))E[u_\varepsilon|\xi]$ by writing

$$\begin{aligned}&\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} \otimes \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} - \xi_{i,j}\otimes\xi_{i,j} \\ &= \left(\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} - \xi_{i,j}\right) \otimes \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} + \xi_{i,j}\otimes\left(\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} - \xi_{i,j}\right)\end{aligned}$$

and using Young's inequality together with the coercivity estimates (3.23) and (3.24) for our relative energy. The second term on the right-hand side in the above formula can be estimated similarly by exploiting Young's inequality as well as the gradient flow calibration property (3.4d) and the coercivity estimates (3.14d) and (3.24).

It remains to bound the third term on the right-hand side. To this aim, we note that for any symmetric matrix A we have

$$
\begin{aligned}
\nabla B : A = {} & \nabla B : (\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) A (\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) \\
& + (\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) \Big(\nabla B + (\nabla B)^T \Big) \xi_{i,j} \cdot (\xi_{i,j} \cdot A (\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j})) \\
& + \xi_{i,j} \cdot (\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) \Big(\nabla B + (\nabla B)^T \Big) \xi_{i,j} (\xi_{i,j} \cdot A \xi_{i,j}) \\
& + \xi_{i,j} \cdot \nabla B \, \xi_{i,j} (\xi_{i,j} \cdot A \xi_{i,j}).
\end{aligned}
$$

This entails by (3.4e) and $|\xi_{i,j}(\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j})| \leq C \operatorname{dist}^2(\cdot, I_{i,j})$ (the latter being a consequence of (3.4f))

$$
\begin{aligned}
& \sum_{\substack{i,j=1 \\ i<j}}^N \int_{\mathbb{R}^d} \nabla B : \left(\frac{\nabla(\psi_{i,j} \circ u_\varepsilon)}{|\nabla(\psi_{i,j} \circ u_\varepsilon)|} \otimes \frac{\nabla(\psi_{i,j} \circ u_\varepsilon)}{|\nabla(\psi_{i,j} \circ u_\varepsilon)|} - \frac{\nabla u_\varepsilon^\mathsf{T} \nabla u_\varepsilon}{|\nabla u_\varepsilon|^2} \right) \varepsilon |\nabla u_\varepsilon|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \\
& \leq C(\xi(t), B(t)) \sum_{\substack{i,j=1 \\ i<j}}^N \int_{\mathbb{R}^d} \left(\left| (\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) \frac{\nabla(\psi_{i,j} \circ u_\varepsilon)}{|\nabla(\psi_{i,j} \circ u_\varepsilon)|} \right|^2 \varepsilon |\nabla u_\varepsilon|^2 \right. \\
& \qquad \left. + \varepsilon |(\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) \nabla u_\varepsilon^\mathsf{T}|^2 \right) \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \\
& \quad + C(\xi(t), B(t)) \sum_{\substack{i,j=1 \\ i<j}}^N \int_{\mathbb{R}^d} \min\{\operatorname{dist}(x, I_{i,j}), 1\} \left(\left| (\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) \frac{\nabla(\psi_{i,j} \circ u_\varepsilon)}{|\nabla(\psi_{i,j} \circ u_\varepsilon)|} \right| \varepsilon |\nabla u_\varepsilon|^2 \right. \\
& \qquad \left. + \varepsilon |(\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) \nabla u_\varepsilon^\mathsf{T}| |\nabla u_\varepsilon| \right) \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \\
& \quad + C(\xi(t), B(t)) \sum_{\substack{i,j=1 \\ i<j}}^N \int_{\mathbb{R}^d} \min\{\operatorname{dist}^2(x, I_{i,j}), 1\} \varepsilon |\nabla u_\varepsilon|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \\
& \quad + C(\xi(t), B(t)) \sum_{\substack{i,j=1 \\ i<j}}^N \int_{\mathbb{R}^d} \left| \left| \xi_{i,j} \cdot \frac{\nabla(\psi_{i,j} \circ u_\varepsilon)}{|\nabla(\psi_{i,j} \circ u_\varepsilon)|} \right|^2 \varepsilon |\nabla u_\varepsilon|^2 - \varepsilon |(\xi_{i,j} \cdot \nabla) u_\varepsilon|^2 \right| \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \\
& \leq C(\xi(t), B(t)) \sum_{\substack{i,j=1 \\ i<j}}^N \int_{\mathbb{R}^d} \left(\left| (\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) \frac{\nabla(\psi_{i,j} \circ u_\varepsilon)}{|\nabla(\psi_{i,j} \circ u_\varepsilon)|} \right|^2 \varepsilon |\nabla u_\varepsilon|^2 \right. \\
& \qquad \left. + \varepsilon |(\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) \nabla u_\varepsilon^\mathsf{T}|^2 \right) \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \\
& \quad + C(\xi(t), B(t)) \sum_{\substack{i,j=1 \\ i<j}}^N \int_{\mathbb{R}^d} \left(\left(1 - \left| \xi_{i,j} \cdot \frac{\nabla(\psi_{i,j} \circ u_\varepsilon)}{|\nabla(\psi_{i,j} \circ u_\varepsilon)|} \right|^2 \right) \varepsilon |\nabla u_\varepsilon|^2 \right. \\
& \qquad \left. + \varepsilon \Big(|\nabla u_\varepsilon|^2 - |(\xi_{i,j} \cdot \nabla) u_\varepsilon|^2 \Big) \right) \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x
\end{aligned}
$$

$$+ C(\xi(t),B(t)) \sum_{\substack{i,j=1\\i<j}}^{N} \int_{\mathbb{R}^d} \min\{\operatorname{dist}^2(x,I_{i,j}),1\}\varepsilon|\nabla u_\varepsilon|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x,$$

where in the last step we have used Young's inequality. By the coercivity properties (3.14d), (3.23), (3.26), and (3.14e), we conclude that

$$\mathrm{Err}_{AllenCahn} \le C(\xi(t),B(t))E[u_\varepsilon|\xi].$$

Step 3: Estimate for $\mathrm{Err}_{MC\xi}$. For the estimate on $\mathrm{Err}_{MC\xi}$, we have to work a bit more. We begin by adding zeroes and using (3.11) to obtain

$$\begin{aligned}
&\mathrm{Err}_{MC\xi}\\
&\le \sum_{\substack{i,j=1\\i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2\varepsilon}\Big| -\frac{1}{2}(\nabla\cdot\xi_{i,j})\partial_u\psi_{i,j}(u_\varepsilon) + \sum_{\substack{k=1\\k\notin\{i,j\}}}^{N} \frac{1}{2}(\nabla\cdot\xi_k)\partial_u\psi_0(u_\varepsilon)\Big|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad + \sum_{\substack{i,j=1\\i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}(\nabla\cdot\xi_{i,j})B\cdot\nabla(\psi_{i,j}\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad + \sum_{\substack{i,j=1\\i<j}}^{N} \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|B\cdot\xi_{i,j}|^2|\nabla u_\varepsilon|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad + \sum_{\substack{i,j=1\\i<j}}^{N} \int_{\mathbb{R}^d} (\mathrm{Id}-\xi_{i,j}\otimes\xi_{i,j}) : \mathrm{H}_\varepsilon\otimes B|\nabla u_\varepsilon|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad - \sum_{\substack{i,j,k=1\\i<j,k\notin\{i,j\}}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}(\nabla\cdot\xi_k)B\cdot\nabla(\psi_0\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&= \sum_{\substack{i,j=1\\i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\Big| \frac{1}{\sqrt{\varepsilon}}(\nabla\cdot\xi_{i,j})\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} \otimes \frac{1}{2}\partial_u\psi_{i,j}(u_\varepsilon) \otimes \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\\
&\qquad\qquad + \sqrt{\varepsilon}(B\cdot\xi_{i,j})\xi_{i,j}\otimes\nabla u_\varepsilon\Big|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad + \sum_{\substack{i,j=1\\i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2\varepsilon}\Big| \sum_{\substack{k=1\\k\notin\{i,j\}}}^{N} \frac{1}{2}(\nabla\cdot\xi_k)\partial_u\psi_0(u_\varepsilon)\Big|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad - \sum_{\substack{i,j,k=1\\i<j,k\notin\{i,j\}}}^{N} \int_{\mathbb{R}^d} \frac{1}{4\varepsilon}(\nabla\cdot\xi_{i,j})(\nabla\cdot\xi_k)\partial_u\psi_{i,j}(u_\varepsilon)\cdot\partial_u\psi_0(u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad + \sum_{\substack{i,j=1\\i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}(\nabla\cdot\xi_{i,j})(\mathrm{Id}-\xi_{i,j}\otimes\xi_{i,j}) : B\otimes\nabla(\psi_{i,j}\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad + \sum_{\substack{i,j=1\\i<j}}^{N} \int_{\mathbb{R}^d} \frac{\varepsilon}{2}(1-|\xi_{i,j}|^2)|B\cdot\xi_{i,j}|^2|\nabla u_\varepsilon|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad + \sum_{\substack{i,j=1\\i<j}}^{N} \int_{\mathbb{R}^d} (\mathrm{Id}-\xi_{i,j}\otimes\xi_{i,j}) : \mathrm{H}_\varepsilon\otimes B|\nabla u_\varepsilon|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x
\end{aligned}$$

$$- \sum_{\substack{i,j,k=1 \\ i<j, k\notin\{i,j\}}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}(\nabla\cdot\xi_k) B\cdot\nabla(\psi_0\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x, \tag{3.20}$$

where in the second step we have also used $\partial_u\psi_{i,j}(u_\varepsilon)\otimes\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} : \nabla u_\varepsilon = |\nabla(\psi_{i,j}\circ u_\varepsilon)|$.

Now note that the three terms on the right-hand side of (3.20) that involve a $\partial_u\psi_0(u_\varepsilon)$ or $\nabla(\psi_0\circ u_\varepsilon)$ can be directly estimated by $CE[u_\varepsilon|\xi]$ by relying on the coercivity properties (3.15a), (3.15b), and (3.15c). Similarly, the third-to-last term on the right-hand side is estimated by $CE[u_\varepsilon|\xi]$ using (3.4f) and (3.14d). This shows

$$\begin{aligned}
&\mathrm{Err}_{MC\xi} &(3.21)\\
&\leq \sum_{\substack{i,j=1\\i<j}}^{N}\int_{\mathbb{R}^d}\frac{1}{2}\bigg|\frac{1}{\sqrt{\varepsilon}}(\nabla\cdot\xi_{i,j})\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\otimes\frac{1}{2}\partial_u\psi_{i,j}(u_\varepsilon)\otimes\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\\
&\qquad\qquad + \sqrt{\varepsilon}(B\cdot\xi_{i,j})\xi_{i,j}\otimes\nabla u_\varepsilon\bigg|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad+\sum_{\substack{i,j=1\\i<j}}^{N}\int_{\mathbb{R}^d}\frac{1}{2}(\nabla\cdot\xi_{i,j})(\mathrm{Id}-\xi_{i,j}\otimes\xi_{i,j}) : B\otimes\nabla(\psi_{i,j}\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad+\sum_{\substack{i,j=1\\i<j}}^{N}\int_{\mathbb{R}^d}(\mathrm{Id}-\xi_{i,j}\otimes\xi_{i,j}) : \mathrm{H}_\varepsilon\otimes B|\nabla u_\varepsilon|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad+CE[u_\varepsilon|\xi]. &(3.22)
\end{aligned}$$

By adding zeros, the first term on the right-hand side of (3.22) can be rewritten as

$$\begin{aligned}
&\sum_{\substack{i,j=1\\i<j}}^{N}\int_{\mathbb{R}^d}\frac{1}{2}\bigg|[(B\cdot\xi_{i,j})\xi_{i,j}+(\nabla\cdot\xi_{i,j})\xi_{i,j}]\otimes\sqrt{\varepsilon}\nabla u_\varepsilon\\
&\qquad+(\nabla\cdot\xi_{i,j})\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\otimes\left(\frac{1}{2\sqrt{\varepsilon}}\partial_u\psi_{i,j}(u_\varepsilon)\otimes\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}-\sqrt{\varepsilon}\nabla u_\varepsilon\right)\\
&\qquad+(\nabla\cdot\xi_{i,j})\left(\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}-\xi_{i,j}\right)\otimes\sqrt{\varepsilon}\nabla u_\varepsilon\bigg|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\leq\sum_{\substack{i,j=1\\i<j}}^{N}\frac{3}{2}\int_{\mathbb{R}^d}|(B\cdot\xi_{i,j})\xi_{i,j}+(\nabla\cdot\xi_{i,j})\xi_{i,j}|^2\,\varepsilon|\nabla u_\varepsilon|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad+\sum_{\substack{i,j=1\\i<j}}^{N}\frac{3}{2}\|\nabla\cdot\xi_{i,j}\|^2_{L^\infty_x}\int_{\mathbb{R}^d}\left|\frac{1}{2\sqrt{\varepsilon}}\partial_u\psi_{i,j}(u_\varepsilon)\otimes\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}-\sqrt{\varepsilon}\nabla u_\varepsilon\right|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad+\sum_{\substack{i,j=1\\i<j}}^{N}\frac{3}{2}\|\nabla\cdot\xi_{i,j}\|^2_{L^\infty_x}\int_{\mathbb{R}^d}\left|\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}-\xi_{i,j}\right|^2\varepsilon|\nabla u_\varepsilon|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x,
\end{aligned}$$

where in the second step we have used Young's inequality. Using the property (3.4c) of the calibration (ξ, B) and the coercivity propeties (3.14d), (3.25) and (3.23) of the relative energy, we see that the right-hand side is bounded by $CE[u_\varepsilon|\xi]$.

It remains to estimate the second and third term on the right-hand side of (3.22). Adding zero, these terms are seen to be equal to

$$
\begin{aligned}
&\sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} (\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) : B \otimes \nabla u_\varepsilon^{\mathsf{T}} \cdot \bigg(\frac{1}{2} (\nabla \cdot \xi_{i,j}) \partial_u \psi_{i,j}(u_\varepsilon) \\
&\qquad - \sum_{\substack{k=1\\ k\notin\{i,j\}}}^{N} \frac{1}{2} (\nabla \cdot \xi_k) \partial_u \psi_0(u_\varepsilon) - \Big(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \partial_u W(u_\varepsilon) \Big) \bigg) \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \\
&+ \sum_{\substack{i,j,k=1\\ i<j, k\notin\{i,j\}}}^{N} \int_{\mathbb{R}^d} \frac{1}{2} (\nabla \cdot \xi_k) (\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) : B \otimes \nabla u_\varepsilon^{\mathsf{T}} \cdot \partial_u \psi_0(u_\varepsilon) \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \\
&\overset{(3.11)}{\leq} \sum_{\substack{i,j=1\\ i<j}}^{N} C(\bar\delta) \|B\|_{L^\infty_x}^2 \int_{\mathbb{R}^d} \varepsilon |(\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) \nabla u_\varepsilon^{\mathsf{T}}|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \\
&+ \bar\delta \int_{\mathbb{R}^d} \frac{1}{2\varepsilon} \Big| \Big(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \partial_u W(u_\varepsilon) \Big) + \sum_{i=1}^{N} (\nabla \cdot \xi_i) \partial_u \psi_i(u_\varepsilon) \Big|^2 \,\mathrm{d}x \\
&+ \sum_{\substack{i,j,k=1\\ i<j, k\notin\{i,j\}}}^{N} \frac{1}{2} \|\nabla \cdot \xi_k\|_{L^\infty_x} \|B\|_{L^\infty_x} \int_{\mathbb{R}^d} |\nabla(\psi_0 \circ u_\varepsilon)| \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x.
\end{aligned}
$$

Here, in the last step we have used Young's inequality for $\bar\delta > 0$ small enough. Using the coercivity properties (3.14e), (3.14d), and (3.15c) of the relative energy, we see that the first and last term on the right-hand side are bounded by $CE[u_\varepsilon|\xi]$.

Overall, we have shown

$$
\mathrm{Err}_{MC\xi} \leq C(\bar\delta) E[u_\varepsilon|\xi] + \bar\delta \int_{\mathbb{R}^d} \frac{1}{2\varepsilon} \Big| \Big(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \partial_u W(u_\varepsilon) \Big) + \sum_{i=1}^{N} (\nabla \cdot \xi_i) \partial_u \psi_i(u_\varepsilon) \Big|^2 \,\mathrm{d}x,
$$

which was the only missing ingredient for the proof of the theorem. □

In the above estimates, we have used the following additional coercivity properties of the relative entropy. We shall defer their proof to that of the other coercivity properties from Proposition 39 and 38.

Lemma 43. *Let W, ψ_i, $\psi_{i,j}$, ξ_i, and $\xi_{i,j}$ be as in Proposition 39. We then have*

$$
\sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \bigg| \frac{\nabla(\psi_{i,j} \circ u_\varepsilon)}{|\nabla(\psi_{i,j} \circ u_\varepsilon)|} - \xi_{i,j} \bigg|^2 \varepsilon |\nabla u_\varepsilon|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \leq CE[u_\varepsilon|\xi], \tag{3.23}
$$

$$
\sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \bigg| \frac{1}{2\sqrt{\varepsilon}} \frac{|\nabla(\psi_{i,j} \circ u_\varepsilon)|}{|\nabla u_\varepsilon|} - \sqrt{\varepsilon} |\nabla u_\varepsilon| \bigg|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \leq CE[u_\varepsilon|\xi], \tag{3.24}
$$

$$
\sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \bigg| \frac{1}{2\sqrt{\varepsilon}} \partial_u \psi_{i,j}(u_\varepsilon) \otimes \frac{\nabla(\psi_{i,j} \circ u_\varepsilon)}{|\nabla(\psi_{i,j} \circ u_\varepsilon)|} - \sqrt{\varepsilon} \nabla u_\varepsilon \bigg|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \leq CE[u_\varepsilon|\xi], \tag{3.25}
$$

$$
\sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \bigg| \xi_{i,j} \otimes \xi_{i,j} - \frac{\nabla u_\varepsilon^{\mathsf{T}} \nabla u_\varepsilon}{|\nabla u_\varepsilon|^2} \bigg|^2 \varepsilon |\nabla u_\varepsilon|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \leq CE[u_\varepsilon|\xi]. \tag{3.26}
$$

3.4.2 Time evolution of the relative energy

We next give the technical computation that provides the estimate for the evolution of the relative entropy stated in Lemma 42. Although in parts technical, it is at the very heart of the proof of our results.

Proof of Lemma 42. By direct computations, using the definitions (3.2) and (3.1) as well as (an analogue for $\partial_t \xi_i$ of) the relation (3.9), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E[u_\varepsilon|\xi] = &-\int_{\mathbb{R}^d}\Big(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\Big)\,\partial_t u_\varepsilon\,\mathrm{d}x \\
&-\sum_{i=1}^N\int_{\mathbb{R}^d}(\nabla\cdot\xi_i)\partial_u\psi_i(u_\varepsilon)\cdot\partial_t u_\varepsilon\,\mathrm{d}x \\
&+\sum_{i=1}^N\int_{\mathbb{R}^d}\partial_t\xi_i\cdot\nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x \\
= &-\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\Big|\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\Big|^2\,\mathrm{d}x \\
&-\sum_{i=1}^N\int_{\mathbb{R}^d}(\nabla\cdot\xi_i)\partial_u\psi_i(u_\varepsilon)\cdot\Big(\Delta u_\varepsilon - \frac{1}{\varepsilon^2}\partial_u W(u_\varepsilon)\Big)\,\mathrm{d}x \\
&-\sum_{\substack{i,j=1\\ i<j}}^N\int_{\mathbb{R}^d}\frac{1}{2}\partial_t\xi_{i,j}\cdot\nabla(\psi_{i,j}\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&+\sum_{\substack{i,j,k=1\\ i<j,k\notin\{i,j\}}}^N\int_{\mathbb{R}^d}\frac{1}{2}\partial_t\xi_k\cdot\nabla(\psi_0\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x.
\end{aligned}$$

By adding zeros and using again (3.9) as well as (3.10), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E[u_\varepsilon|\xi] \\
&= -\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\Big|\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\Big|^2\,\mathrm{d}x \\
&\quad -\sum_{i=1}^N\int_{\mathbb{R}^d}(\nabla\cdot\xi_i)\partial_u\psi_i(u_\varepsilon)\cdot\Big(\Delta u_\varepsilon - \frac{1}{\varepsilon^2}\partial_u W(u_\varepsilon)\Big)\,\mathrm{d}x \\
&\quad -\sum_{\substack{i,j=1\\ i<j}}^N\int_{\mathbb{R}^d}\frac{1}{2}(\partial_t\xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}}\xi_{i,j})\cdot\nabla(\psi_{i,j}\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\quad +\sum_{\substack{i,j,k=1\\ i<j,k\notin\{i,j\}}}^N\int_{\mathbb{R}^d}\frac{1}{2}(\partial_t\xi_k + (B\cdot\nabla)\xi_k + (\nabla B)^{\mathsf{T}}\xi_k)\cdot\nabla(\psi_0\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\quad -\sum_{i=1}^N\int_{\mathbb{R}^d}\nabla\xi_i:\nabla(\psi_i\circ u_\varepsilon)\otimes B\,\mathrm{d}x - \sum_{i=1}^N\int_{\mathbb{R}^d}\nabla B:\xi_i\otimes\nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x.
\end{aligned} \tag{3.27}$$

Integrating by parts several times and making use of an approximation argument for $((\xi_i)_i, B)$, the last two terms in the equation above can be rewritten as

$$-\sum_{i=1}^N\int_{\mathbb{R}^d}\nabla\xi_i:\nabla(\psi_i\circ u_\varepsilon)\otimes B\,\mathrm{d}x - \sum_{i=1}^N\int_{\mathbb{R}^d}\nabla B:\xi_i\otimes\nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \int_{\mathbb{R}^d} \psi_i(u_\varepsilon)(B\cdot\nabla)(\nabla\cdot\xi_i)\,\mathrm{d}x + \sum_{i=1}^{N} \int_{\mathbb{R}^d} \psi_i(u_\varepsilon)(\nabla B)^\mathsf{T} : \nabla\xi_i\,\mathrm{d}x \\
&\quad - \sum_{i=1}^{N} \int_{\mathbb{R}^d} \nabla B : \xi_i \otimes \nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x \\
&= - \sum_{i=1}^{N} \int_{\mathbb{R}^d} (\nabla\cdot\xi_i) B \cdot \nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x - \sum_{i=1}^{N} \int_{\mathbb{R}^d} \psi_i(u_\varepsilon)(\nabla\cdot B)(\nabla\cdot\xi_i)\,\mathrm{d}x \\
&\quad - \sum_{i=1}^{N} \int_{\mathbb{R}^d} \psi_i(u_\varepsilon)(\xi_i\cdot\nabla)(\nabla\cdot B)\,\mathrm{d}x - \sum_{i=1}^{N} \int_{\mathbb{R}^d} (\nabla B)^\mathsf{T} : \xi_i \otimes \nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x \\
&\quad - \sum_{i=1}^{N} \int_{\mathbb{R}^d} \nabla B : \xi_i \otimes \nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x \\
&= \sum_{i=1}^{N} \int_{\mathbb{R}^d} ((\nabla\cdot B)\xi_i - (\nabla\cdot\xi_i)B)\cdot\nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x \\
&\quad - \sum_{i=1}^{N} \int_{\mathbb{R}^d} \nabla B : \nabla(\psi_i\circ u_\varepsilon)\otimes\xi_i\,\mathrm{d}x - \sum_{i=1}^{N} \int_{\mathbb{R}^d} \nabla B : \xi_i \otimes \nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x \\
&\overset{(3.9)}{=} \sum_{i=1}^{N} \int_{\mathbb{R}^d} ((\nabla\cdot B)\xi_i - (\nabla\cdot\xi_i)B)\cdot\nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x \\
&\quad + \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\nabla B : \nabla(\psi_{i,j}\circ u_\varepsilon)\otimes\xi_{i,j}\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\quad + \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\nabla B : \xi_{i,j}\otimes\nabla(\psi_{i,j}\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\quad - \sum_{\substack{i,j,k=1\\ i<j,k\notin\{i,j\}}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\nabla B : \nabla(\psi_0\circ u_\varepsilon)\otimes\xi_k\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\quad - \sum_{\substack{i,j,k=1\\ i<j,k\notin\{i,j\}}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\nabla B : \xi_k\otimes\nabla(\psi_0\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x.
\end{aligned}
$$

By adding zero, we obtain

$$
\begin{aligned}
&- \sum_{i=1}^{N} \int_{\mathbb{R}^d} \nabla\xi_i : \nabla(\psi_i\circ u_\varepsilon)\otimes B\,\mathrm{d}x - \sum_{i=1}^{N} \int_{\mathbb{R}^d} \nabla B : \xi_i\otimes\nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x \\
&= \sum_{i=1}^{N} \int_{\mathbb{R}^d} ((\nabla\cdot B)\xi_i - (\nabla\cdot\xi_i)B)\cdot\nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x \qquad (3.28) \\
&\quad - \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\nabla B : \left(\xi_{i,j} - \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\right)\otimes\left(\xi_{i,j} - \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\right) \\
&\qquad\qquad \times |\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\quad + \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\nabla B : \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\otimes\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\substack{i,j=1\\i<j}}^N \int_{\mathbb{R}^d} \frac{1}{2}\nabla B : \xi_{i,j}\otimes\xi_{i,j}|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&-\sum_{\substack{i,j,k=1\\i<j,k\notin\{i,j\}}}^N \int_{\mathbb{R}^d} \frac{1}{2}\nabla B : \nabla(\psi_0\circ u_\varepsilon)\otimes\xi_k\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&-\sum_{\substack{i,j,k=1\\i<j,k\notin\{i,j\}}}^N \int_{\mathbb{R}^d} \frac{1}{2}\nabla B : \xi_k\otimes\nabla(\psi_0\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x.
\end{aligned}
$$

Using the relation

$$
\begin{aligned}
&-\int_{\mathbb{R}^d} \mathrm{H}_\varepsilon\cdot B|\nabla u_\varepsilon|\,\mathrm{d}x\\
&\overset{(3.16)}{=} -\int_{\mathbb{R}^d} \varepsilon\nabla B : \nabla u_\varepsilon^\mathsf{T}\nabla u_\varepsilon\,\mathrm{d}x + \int_{\mathbb{R}^d}(\nabla\cdot B)\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2+\frac{1}{\varepsilon}W(u_\varepsilon)\right)\mathrm{d}x,
\end{aligned}
$$

in view of $\sum_{i,j=1:i<j}^N \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)=1$ we can rewrite the third term on the right-hand side as

$$
\begin{aligned}
&\sum_{\substack{i,j=1\\i<j}}^N \int_{\mathbb{R}^d} \frac{1}{2}\nabla B : \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\otimes\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^d} \mathrm{H}_\varepsilon\cdot B|\nabla u_\varepsilon|\,\mathrm{d}x + \int_{\mathbb{R}^d}(\nabla\cdot B)\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2+\frac{1}{\varepsilon}W(u_\varepsilon)\right)\mathrm{d}x\\
&\quad+\sum_{\substack{i,j=1\\i<j}}^N \int_{\mathbb{R}^d} \nabla B : \left(\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\otimes\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\frac{1}{2}|\nabla(\psi_{i,j}\circ u_\varepsilon)|-\varepsilon\nabla u_\varepsilon^\mathsf{T}\nabla u_\varepsilon\right)\\
&\qquad\qquad\times\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x.
\end{aligned}
$$

Inserting this relation into (3.28) and inserting the resulting equation into (3.27), we obtain by collecting terms and adding and subtracting $\sum_{i,j=1:i<j}^N\int_{\mathbb{R}^d}\frac{1}{2}\xi_{i,j}\cdot(\partial_t\xi_{i,j}+(B\cdot\nabla)\xi_{i,j})|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x$

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E[u_\varepsilon|\xi]\\
&=-\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\left|\varepsilon\Delta u_\varepsilon-\frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\right|^2\mathrm{d}x+\int_{\mathbb{R}^d}\mathrm{H}_\varepsilon\cdot B|\nabla u_\varepsilon|\,\mathrm{d}x\\
&\quad-\sum_{i=1}^N\int_{\mathbb{R}^d}(\nabla\cdot\xi_i)\partial_u\psi_i(u_\varepsilon)\cdot\left(\Delta u_\varepsilon-\frac{1}{\varepsilon^2}\partial_u W(u_\varepsilon)\right)\mathrm{d}x\\
&\quad-\sum_{i=1}^N\int_{\mathbb{R}^d}(\nabla\cdot\xi_i)B\cdot\nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x\\
&\quad+\sum_{\substack{i,j=1\\i<j}}^N\int_{\mathbb{R}^d}\frac{1}{2}(\partial_t\xi_{i,j}+(B\cdot\nabla)\xi_{i,j}+(\nabla B)^\mathsf{T}\xi_{i,j})\\
&\qquad\qquad\cdot\left(\xi_{i,j}-\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\right)|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad-\sum_{\substack{i,j=1\\i<j}}^N\int_{\mathbb{R}^d}\frac{1}{2}\xi_{i,j}\cdot(\partial_t\xi_{i,j}+(B\cdot\nabla)\xi_{i,j})|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}^d} (\nabla \cdot B)\Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) + \sum_{i=1}^{N} \xi_i \cdot \nabla(\psi_i \circ u_\varepsilon)\Big)\,\mathrm{d}x \\
&+ \sum_{\substack{i,j,k=1\\ i<j, k\notin\{i,j\}}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}(\partial_t \xi_k + (B\cdot\nabla)\xi_k + (\nabla B)^{\mathsf{T}}\xi_k)\cdot\nabla(\psi_0\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&- \sum_{\substack{i,j,k=1\\ i<j, k\notin\{i,j\}}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\nabla B : \nabla(\psi_0\circ u_\varepsilon)\otimes \xi_k \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&- \sum_{\substack{i,j,k=1\\ i<j, k\notin\{i,j\}}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\nabla B : \xi_k \otimes \nabla(\psi_0\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&- \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\nabla B : \Big(\xi_{i,j} - \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\Big) \otimes \Big(\xi_{i,j} - \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times |\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&+ \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \nabla B : \Big(\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} \otimes \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\frac{1}{2}|\nabla(\psi_{i,j}\circ u_\varepsilon)| - \varepsilon\nabla u_\varepsilon^{\mathsf{T}}\nabla u_\varepsilon\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x.
\end{aligned}
$$

Lemma 42 follows from this equation using the definitions of the errors (3.19a) – (3.19e) and the next formula (whose derivation relies on $\sum_{i,j=1:i<j}^{N} \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) = 1$ and repeated addition of zero).

$$
\begin{aligned}
&- \int_{\mathbb{R}^d} \frac{1}{\varepsilon}\Big|\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\Big|^2 + \int_{\mathbb{R}^d} \mathrm{H}_\varepsilon \cdot B|\nabla u_\varepsilon|\,\mathrm{d}x \\
&- \sum_{i=1}^{N} \int_{\mathbb{R}^d} (\nabla\cdot\xi_i)\partial_u\psi_i(u_\varepsilon)\cdot\Big(\Delta u_\varepsilon - \frac{1}{\varepsilon^2}\partial_u W(u_\varepsilon)\Big)\,\mathrm{d}x \\
&- \sum_{i=1}^{N} \int_{\mathbb{R}^d} (\nabla\cdot\xi_i)B\cdot\nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x \\
&= - \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2\varepsilon}\Big|\mathrm{H}_\varepsilon - \varepsilon(B\cdot\xi_{i,j})\xi_{i,j}|\nabla u_\varepsilon|\Big|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^d} \frac{1}{2\varepsilon}\Big(\Big|\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\Big|^2 - |\mathrm{H}_\varepsilon|^2\Big)\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^d} \frac{1}{2\varepsilon}\Big|\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\Big|^2\,\mathrm{d}x \\
&\quad - \sum_{i=1}^{N} \int_{\mathbb{R}^d} (\nabla\cdot\xi_i)\partial_u\psi_i(u_\varepsilon)\cdot\Big(\Delta u_\varepsilon - \frac{1}{\varepsilon^2}\partial_u W(u_\varepsilon)\Big)\,\mathrm{d}x \\
&\quad - \sum_{i=1}^{N} \int_{\mathbb{R}^d} (\nabla\cdot\xi_i)B\cdot\nabla(\psi_i\circ u_\varepsilon)\,\mathrm{d}x \\
&\quad + \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|B\cdot\xi_{i,j}|^2|\xi_{i,j}|^2|\nabla u_\varepsilon|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} (\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) : \mathrm{H}_\varepsilon \otimes B |\nabla u_\varepsilon| \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \\
= &- \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2\varepsilon} \Big| \mathrm{H}_\varepsilon - \varepsilon (B \cdot \xi_{i,j}) \xi_{i,j} |\nabla u_\varepsilon| \Big|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \\
&- \int_{\mathbb{R}^d} \frac{1}{2\varepsilon} \left| \Big(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \partial_u W(u_\varepsilon) \Big) + \sum_{i=1}^{N} (\nabla \cdot \xi_i) \partial_u \psi_i(u_\varepsilon) \right|^2 \mathrm{d}x \\
&- \int_{\mathbb{R}^d} \frac{1}{2\varepsilon} \left(\Big| \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \partial_u W(u_\varepsilon) \Big|^2 - |\mathrm{H}_\varepsilon|^2 \right) \mathrm{d}x \\
&+ \int_{\mathbb{R}^d} \frac{1}{2\varepsilon} \left| \sum_{i=1}^{N} (\nabla \cdot \xi_i) \partial_u \psi_i(u_\varepsilon) \right|^2 \mathrm{d}x \\
&- \sum_{i=1}^{N} \int_{\mathbb{R}^d} (\nabla \cdot \xi_i) B \cdot \nabla (\psi_i \circ u_\varepsilon) \,\mathrm{d}x \\
&+ \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{\varepsilon}{2} |B \cdot \xi_{i,j}|^2 |\xi_{i,j}|^2 |\nabla u_\varepsilon|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x \\
&+ \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} (\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) : \mathrm{H}_\varepsilon \otimes B |\nabla u_\varepsilon| \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \,\mathrm{d}x.
\end{aligned}
$$

□

3.4.3 Derivation of the coercivity properties

We next show how our assumption (A4) implies the coercivity properties of our relative entropy.

Proof of Proposition 39. To prove (3.15a)-(3.15c), let $i, j \in \{1, \dots, N\}$, $i \neq j$; suppose that (x,t) is such that $u_\varepsilon(x,t) \in \mathcal{T}_{i,j}$. In particular, we then have $\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) = 1$.

Proof of (3.15a) *and* (3.15b)*:* Starting from (3.12), expanding the second square, and making use of Young's inequality, the fact that for each (x,t) there exists at most three indices $k \in \{1, \dots, N\} \setminus \{i,j\}$ with $\xi_k(x,t) \neq 0$, and (3.13), we obtain

$$
\begin{aligned}
&\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) + \sum_{\ell=1}^{N} \xi_\ell \cdot \nabla (\psi_\ell \circ u_\varepsilon) \\
&\geq \frac{1}{2} \left| \sqrt{\varepsilon} \nabla u_\varepsilon - \frac{1}{\sqrt{\varepsilon}} \Big(\tfrac{1}{2} \partial_u \psi_{i,j}(u_\varepsilon) \otimes \xi_{i,j} - \sum_{\substack{k=1\\ k \notin \{i,j\}}}^{N} \tfrac{1}{2\sqrt{3}} \partial_u \psi_0(u_\varepsilon) \otimes \sqrt{3} \xi_k \Big) \right|^2 \\
&\quad + \frac{1}{2\varepsilon} \Bigg[2W(u_\varepsilon) - \Big(|\xi_{i,j}|^2 + (4 - \delta_{\mathsf{cal}}) \sum_{\substack{k=1\\ k \notin \{i,j\}}}^{N} |\sqrt{3} \xi_{i,j} \cdot \xi_k|^2 \Big) \Big| \tfrac{1}{2} \partial_u \psi_{i,j}(u_\varepsilon) \Big|^2 \\
&\qquad - \sum_{\substack{k=1\\ k \notin \{i,j\}}}^{N} \Big(|\sqrt{3} \xi_k|^2 + \tfrac{1}{4 - \delta_{\mathsf{cal}}} \Big) \Big| \tfrac{1}{2\sqrt{3}} \partial_u \psi_0(u_\varepsilon) \Big|^2 \Bigg].
\end{aligned}
$$

Then, by adding zeros and using $|\sqrt{3}\xi_k| \leq 1$, we obtain

$$
\begin{aligned}
&\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) + \sum_{\ell=1}^N \xi_\ell \cdot \nabla(\psi_\ell \circ u_\varepsilon) \\
&\geq \frac{1}{2}\bigg|\sqrt{\varepsilon}\nabla u_\varepsilon - \frac{1}{\sqrt{\varepsilon}}\bigg(\tfrac{1}{2}\partial_u\psi_{i,j}(u_\varepsilon) \otimes \xi_{i,j} - \sum_{\substack{k=1\\ k\notin\{i,j\}}}^N \tfrac{1}{2\sqrt{3}}\partial_u\psi_0(u_\varepsilon) \otimes \sqrt{3}\xi_k\bigg)\bigg|^2 \\
&\quad + \frac{1}{2\varepsilon}\bigg[2W(u_\varepsilon) - \Big|\tfrac{1}{2}\partial_u\psi_{i,j}(u_\varepsilon)\Big|^2 - (\tfrac{5}{4} + \delta_{\mathsf{cal}} + \delta_{\mathsf{coer},1})\Big|\tfrac{1}{2}\partial_u\psi_0(u_\varepsilon)\Big|^2 \\
&\qquad\qquad - \delta_{\mathsf{coer},2}\,|\partial_u\psi_{i,j}(u_\varepsilon)\cdot\partial_u\psi_0(u_\varepsilon)|\bigg] \\
&\quad + \frac{\delta_{\mathsf{coer},1}}{2\varepsilon}\Big|\tfrac{1}{2}\partial_u\psi_0(u_\varepsilon)\Big|^2 + \frac{\delta_{\mathsf{coer},2}}{2\varepsilon}\,|\partial_u\psi_{i,j}(u_\varepsilon)\cdot\partial_u\psi_0(u_\varepsilon)|\;,
\end{aligned}
\tag{3.29}
$$

where $\delta_{\mathsf{cal}}, \delta_{\mathsf{coer},1}, \delta_{\mathsf{coer},2}, \delta_{\mathsf{coer},3} > 0$ are arbitrarily small constants. Finally, using (A4) and integrating over the set $\{x : u_\varepsilon(x,t) \in \mathcal{T}_{i,j}\}$, we can conclude about the validity of (3.15a) and (3.15b).

Proof of (3.15c)*:* By adding zero, we can write

$$
\begin{aligned}
&\nabla(\psi_0 \circ u_\varepsilon) = \partial_u\psi_0(u_\varepsilon)\cdot\nabla u_\varepsilon \\
&= \frac{1}{\sqrt{\varepsilon}}\partial_u\psi_0(u_\varepsilon)\cdot\bigg[\sqrt{\varepsilon}\nabla u_\varepsilon - \frac{1}{\sqrt{\varepsilon}}\bigg(\tfrac{1}{2}\partial_u\psi_{i,j}\otimes\xi_{i,j} - \sum_{\substack{k=1\\ k\notin\{i,j\}}}^N \tfrac{1}{2\sqrt{3}}\partial_u\psi_0(u_\varepsilon)\otimes\sqrt{3}\xi_k\bigg)\bigg] \\
&\quad + \frac{1}{2\varepsilon}\partial_u\psi_0(u_\varepsilon)\cdot\partial_u\psi_{i,j}(u_\varepsilon)\xi_{i,j} - \frac{1}{\varepsilon}\sum_{\substack{k=1\\ k\notin\{i,j\}}}^N \tfrac{1}{2\sqrt{3}}|\partial_u\psi_0(u_\varepsilon)|^2\sqrt{3}\xi_k\,.
\end{aligned}
$$

Then, Young's inequality yields

$$
\begin{aligned}
&|\nabla(\psi_0\circ u_\varepsilon)| \\
&\leq \frac{1}{2\varepsilon}|\partial_u\psi_0(u_\varepsilon)|^2 \\
&\quad + \frac{1}{2}\bigg|\sqrt{\varepsilon}\nabla u_\varepsilon - \frac{1}{\sqrt{\varepsilon}}\bigg(\tfrac{1}{2}\partial_u\psi_{i,j}(u_\varepsilon)\otimes\xi_{i,j} - \sum_{\substack{k=1\\ k\notin\{i,j\}}}^N \tfrac{1}{2\sqrt{3}}\partial_u\psi_0(u_\varepsilon)\otimes\sqrt{3}\xi_k\bigg)\bigg|^2 \\
&\quad + \frac{1}{2\varepsilon}|\partial_u\psi_0(u_\varepsilon)\cdot\partial_u\psi_{i,j}(u_\varepsilon)||\xi_{i,j}| \\
&\quad + \frac{1}{\varepsilon}\sum_{\substack{k=1\\ k\notin\{i,j\}}}^N \tfrac{1}{2\sqrt{3}}|\partial_u\psi_0(u_\varepsilon)|^2|\sqrt{3}\xi_k|\,,
\end{aligned}
$$

Using Young's inequality and the estimate (3.29), we can conclude about the validity of (3.15c). ☐

Proof of Proposition 38. Proof of (3.14a), (3.14b) *and* (3.14c)*:* Using (3.9) and adding zero, we obtain

$$
E[u_\varepsilon|\xi] = E[u_\varepsilon] - \sum_{\substack{i,j=1\\ i<j}}^N \int_{\mathbb{R}^d} \frac{1}{2}|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x
$$

$$
\begin{aligned}
&+ \sum_{\substack{i,j,k=1\\ i<j, k\notin\{i,j\}}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\xi_k \cdot \nabla(\psi_0 \circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&+ \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\Big(1 - \xi_{i,j}\cdot \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\Big)|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\,.
\end{aligned}
$$

From assumption (A4), we can deduce

$$
\frac{1}{2}|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \le \frac{1}{2}|\partial_u \psi_{i,j}(u_\varepsilon)||\nabla u_\varepsilon|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \le \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,.
$$

Hence, using the definition of $E[u_\varepsilon]$, we have

$$
\begin{aligned}
&E[u_\varepsilon|\xi] + \sum_{\substack{i,j,k=1\\ i<j, k\notin\{i,j\}}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\|\xi_k\|_{L^\infty_x}|\nabla(\psi_0\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\ge \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\bigg(\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{\sqrt{2W(u_\varepsilon)}}{\sqrt{\varepsilon}}\bigg)^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\quad + \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \frac{1}{2}\Big(1 - \xi_{i,j}\cdot \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\Big)|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\,.
\end{aligned}
$$

Then, noting that

$$
\begin{aligned}
&\left|\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} - \xi_{i,j}\right|^2 |\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\le 2\left(1 - \xi_{i,j}\cdot\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\right)|\nabla(\psi_{i,j}\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)
\end{aligned}
$$

together with the fact that $\sum_{i,j=1:i<j}^{N}\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) = 1$, we see that both (3.14a) and (3.14b) follow from the preceding two formulas and (3.15c). Furthermore, using

$$
\min\{\mathrm{dist}^2(x, I_{i,j}), 1\} \le C(1 - |\xi_{i,j}|) \le C\bigg(1 - \xi_{i,j}\cdot\frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}\bigg),
$$

we obtain (3.14c).

Proof of (3.14d)*:* By exploiting (3.9) and by adding zeros, we obtain

$$
\begin{aligned}
&\Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\le \bigg[\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) + \sum_{\ell=1}^{N}\xi_\ell\cdot\nabla(\psi_\ell\circ u_\varepsilon)\bigg]\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\quad + \bigg[\frac{1}{2}|\nabla(\psi_{i,j}\circ u_\varepsilon)| + \frac{1}{2}\,|\nabla(\psi_0\circ u_\varepsilon)|\bigg]\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,.
\end{aligned}
\tag{3.30}
$$

As a consequence, we have

$$
\sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \min\{\mathrm{dist}^2(x, I_{i,j}), 1\}\,\Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x
$$

$$
\begin{aligned}
&\leq E[u_\varepsilon|\xi] + \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \min\{\operatorname{dist}^2(x,I_{i,j}),1\} |\nabla(\psi_{i,j}\circ u_\varepsilon)| \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\quad + \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} |\nabla(\psi_0\circ u_\varepsilon)|\,\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\,,
\end{aligned}
$$

whence we deduce (3.14d) from (3.15c) and (3.14c).

Proof of (3.14e)*:* Expanding the square and using $1 - c\operatorname{dist}^2(\cdot,I_{i,j}) \leq |\xi_{i,j}| \leq \max\{1 - C\operatorname{dist}^2(\cdot,I_{i,j})),0\}$, we obtain

$$
\begin{aligned}
\varepsilon|(\mathrm{Id}-\xi_{i,j}\otimes\xi_{i,j})\nabla u_\varepsilon^{\mathsf{T}}|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \leq{}& \Big[\varepsilon|\nabla u_\varepsilon|^2 - \varepsilon|(\xi_{i,j}\cdot\nabla)u_\varepsilon|^2\Big]\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&+ C\min\{\operatorname{dist}^2(\cdot,I_{i,j}),1\}\varepsilon|\nabla u_\varepsilon|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,.
\end{aligned}
$$

Then, by adding zeros, we obtain

$$
\begin{aligned}
&\varepsilon|(\mathrm{Id}-\xi_{i,j}\otimes\xi_{i,j})\nabla u_\varepsilon^{\mathsf{T}}|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&= \Bigg[\varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}2W(u_\varepsilon) - \xi_{i,j}\cdot\nabla(\psi_{i,j}\circ u_\varepsilon) + \sum_{\substack{k=1\\ k\notin\{i,j\}}}^{N} \xi_k\cdot\nabla(\psi_0\circ u_\varepsilon)\Bigg]\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\quad - \sum_{\substack{k=1\\ k\notin\{i,j\}}}^{N} \xi_k\cdot\nabla(\psi_0\circ u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) + \left[\frac{1}{4\varepsilon}|\partial_u\psi_{i,j}(u_\varepsilon)|^2 - \frac{1}{\varepsilon}2W(u_\varepsilon)\right]\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\quad - \left[\frac{1}{4\varepsilon}|\partial_u\psi_{i,j}(u_\varepsilon)|^2 - \xi_{i,j}\cdot\nabla(\psi_{i,j}\circ u_\varepsilon) + \varepsilon|(\xi_{i,j}\cdot\nabla)u_\varepsilon|^2\right]\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\quad + C\min\{\operatorname{dist}^2(\cdot,I_{i,j}),1\}\varepsilon|\nabla u_\varepsilon|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\leq 2\left[\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) + \sum_{\ell=1}^{N}\xi_\ell\cdot\nabla(\psi_\ell\circ u_\varepsilon)\right]\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\quad + \sum_{\substack{k=1\\ k\notin\{i,j\}}}^{N} \|\xi_k\|_{L^\infty_x}|\nabla(\psi_0\circ u_\varepsilon)|\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\quad + \left[\frac{1}{4\varepsilon}|\partial_u\psi_{i,j}(u_\varepsilon)|^2 - \frac{1}{\varepsilon}2W(u_\varepsilon)\right]\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\quad - \left|\frac{1}{2\sqrt{\varepsilon}}\partial_u\psi_{i,j}(u_\varepsilon) - \sqrt{\varepsilon}(\xi_{i,j}\cdot\nabla)u_\varepsilon\right|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\quad + C\min\{\operatorname{dist}^2(\cdot,I_{i,j}),1\}\varepsilon|\nabla u_\varepsilon|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,,
\end{aligned}
$$

due to (3.9). Noting that assumption (A4) implies

$$
\frac{1}{4}\big|\partial_u\psi_{i,j}(u_\varepsilon)\big|^2\chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \leq 2W(u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,,
$$

the validity of (3.14e) follows from (3.15c) and (3.14d). □

We next prove the additional coercivity properties stated in Lemma 43.

Proof of Lemma 43. Proof of (3.23)*:* Note that (3.30) yields

$$
\begin{aligned}
&\sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \left| \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} - \xi_{i,j} \right|^2 \varepsilon |\nabla u_\varepsilon|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\leq 8E[u_\varepsilon|\xi] + \sum_{\substack{i,j=1\\ i<j}}^{N} \int_{\mathbb{R}^d} \left| \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} - \xi_{i,j} \right|^2 |\nabla(\psi_{i,j}\circ u_\varepsilon)| \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&\quad + \sum_{\substack{i,j=1\\ i<j}}^{N} 4 \int_{\mathbb{R}^d} |\nabla(\psi_0\circ u_\varepsilon)| \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x\,.
\end{aligned}
$$

Hence, using (3.15c) and (3.14b), we obtain (3.23).

Proof of (3.24) *and* (3.25)*:* First, we compute

$$
\begin{aligned}
&\left| \frac{1}{2\sqrt{\varepsilon}} \frac{|\nabla(\psi_{i,j}\circ u_\varepsilon)|}{|\nabla u_\varepsilon|} - \sqrt{\varepsilon}|\nabla u_\varepsilon| \right|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&= \left[\frac{1}{\varepsilon} \frac{|\nabla(\psi_{i,j}\circ u_\varepsilon)|^2}{4|\nabla u_\varepsilon|^2} + \varepsilon|\nabla u_\varepsilon|^2 - |\nabla(\psi_{i,j}\circ u_\varepsilon)| \right] \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| \frac{1}{2\sqrt{\varepsilon}} \partial_u \psi_{i,j}(u_\varepsilon) \otimes \frac{\nabla(\psi_{i,j}\circ u_\varepsilon)}{|\nabla(\psi_{i,j}\circ u_\varepsilon)|} - \sqrt{\varepsilon}\nabla u_\varepsilon \right|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,\mathrm{d}x \\
&= \left[\frac{1}{4\varepsilon} |\partial_u \psi_{i,j}(u_\varepsilon)|^2 + \varepsilon|\nabla u_\varepsilon|^2 - |\nabla(\psi_{i,j}\circ u_\varepsilon)| \right] \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,.
\end{aligned}
$$

Then, from assumption (A4) we deduce

$$
\frac{|\nabla(\psi_{i,j}\circ u_\varepsilon)|^2}{|\nabla u_\varepsilon|^2} \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \leq |\partial_u \psi_{i,j}(u_\varepsilon)|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \leq 8W(u_\varepsilon)\chi_{\mathcal{T}_{i,j}}(u_\varepsilon)\,.
$$

Finally, by exploiting (3.9) and by adding zero, one can conclude about the validity of both (3.25) and (3.24), due to (3.15a).

Proof of (3.26)*:* Since we have

$$
\begin{aligned}
&\left| \xi_{i,j} \otimes \xi_{i,j} - \frac{\nabla u_\varepsilon^{\mathsf{T}} \nabla u_\varepsilon}{|\nabla u_\varepsilon|^2} \right|^2 \varepsilon |\nabla u_\varepsilon|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\leq \left[2 - 2\frac{\xi_{i,j} \cdot \nabla u_\varepsilon^{\mathsf{T}} \nabla u_\varepsilon \cdot \xi_{i,j}}{|\nabla u_\varepsilon|^2} \right] \varepsilon |\nabla u_\varepsilon|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\leq 2 \left[\varepsilon|\nabla u_\varepsilon|^2 - \varepsilon|(\xi_{i,j}\cdot\nabla)u_\varepsilon|^2 \right] \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) \\
&\leq 2\varepsilon |(\mathrm{Id} - \xi_{i,j}\otimes\xi_{i,j}) \cdot \nabla u_\varepsilon^{\mathsf{T}}|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon) + C\min\{\mathrm{dist}^2(\cdot, I_{i,j}), 1\} \varepsilon |\nabla u_\varepsilon|^2 \chi_{\mathcal{T}_{i,j}}(u_\varepsilon)
\end{aligned}
$$

(where in the last step we have used the estimate $1 - C\,\mathrm{dist}^2(\cdot, I_{i,j}) \leq |\xi_{i,j}| \leq \max\{1 - c\,\mathrm{dist}^2(\cdot, I_{i,j})), 0\}$), the bound (3.26) follows from (3.14e) and (3.14d).

□

3.4.4 Convergence of the phase indicator functions

We now show how to obtain the error estimate at the level of the indicator functions.

Proof of Proposition 41. Using (3.1) and the fact that $\operatorname{supp}\partial_t\bar{\chi}_i \subset \partial\operatorname{supp}\bar{\chi}_i$ as well as $\vartheta_i = 0$ on $\partial\operatorname{supp}\bar{\chi}_i$, we compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}(\psi_i(u_\varepsilon)-\bar{\chi}_i)\vartheta_i\,\mathrm{d}x\\
&=\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\partial_u\psi_i(u_\varepsilon)\cdot\Big(\varepsilon\Delta u_\varepsilon-\frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\Big)\vartheta_i\,\mathrm{d}x+\int_{\mathbb{R}^d}(\psi_i(u_\varepsilon)-\bar{\chi}_i)\partial_t\vartheta_i\,\mathrm{d}x\\
&=\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\partial_u\psi_i(u_\varepsilon)\cdot\Big(\varepsilon\Delta u_\varepsilon-\frac{1}{\varepsilon}\partial_u W(u_\varepsilon)\Big)\vartheta_i\,\mathrm{d}x+\int_{\mathbb{R}^d}B\cdot\nabla(\psi_i\circ u_\varepsilon)\vartheta_i\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^d}(\nabla\cdot B)(\psi_i(u_\varepsilon)-\bar{\chi}_i)\vartheta_i\,\mathrm{d}x+\int_{\mathbb{R}^d}(\psi_i(u_\varepsilon)-\bar{\chi}_i)\left(\partial_t\vartheta_i+B\cdot\nabla\vartheta_i\right)\,\mathrm{d}x\,,
\end{aligned}\tag{3.31}$$

where we added a zero and then integrated by parts. Note that we used the fact that $\vartheta_i = 0$ on $\partial\{\bar{\chi}_i = 1\}$.

By (3.6), the last two terms on the right hand side of (3.31) can be bounded by

$$\Big(\|\nabla\cdot B\|_{L^\infty_x}+C\Big)\int_{\mathbb{R}^d}|\psi_i(u_\varepsilon)-\bar{\chi}_i||\vartheta_i|\,\mathrm{d}x.$$

As for the second term on the right hand side of (3.31), we perform the folllowing decomposition

$$\begin{aligned}
\int_{\mathbb{R}^d}B\cdot\nabla(\psi_i\circ u_\varepsilon)\vartheta_i\,\mathrm{d}x=&\sum_{\substack{j,k=1\\ j<k}}^N\int_{\mathbb{R}^d}(B\cdot\xi_{j,k})\xi_{j,k}\cdot\nabla(\psi_i\circ u_\varepsilon)\vartheta_i\chi_{\mathcal{T}_{j,k}}(u_\varepsilon)\,\mathrm{d}x\\
&+\sum_{\substack{j,k=1\\ j<k}}^N\int_{\mathbb{R}^d}[(\mathrm{Id}-\xi_{j,k}\otimes\xi_{j,k})B)]\cdot\nabla(\psi_i\circ u_\varepsilon)\vartheta_i\chi_{\mathcal{T}_{j,k}}(u_\varepsilon)\,\mathrm{d}x\,,
\end{aligned}$$

whence, by adding a zero and using $\sum_{j,k=1:j\neq k}^N\chi_{\mathcal{T}_{j,k}}=1$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}B\cdot\nabla(\psi_i\circ u_\varepsilon)\vartheta_i\,\mathrm{d}x\\
&=\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\,\mathrm{H}_\varepsilon\cdot\frac{\nabla u_\varepsilon^\mathsf{T}}{|\nabla u_\varepsilon|}\cdot\partial_u\psi_i(u_\varepsilon)\,\mathrm{d}x\\
&\quad+\sum_{\substack{j,k=1\\ j<k}}^N\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\left[\varepsilon(B\cdot\xi_{j,k})\xi_{j,k}|\nabla u_\varepsilon|-\mathrm{H}_\varepsilon\right]\cdot\frac{\nabla u_\varepsilon^\mathsf{T}}{|\nabla u_\varepsilon|}\cdot\partial_u\psi_i(u_\varepsilon)\vartheta_i\chi_{\mathcal{T}_{j,k}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad+\sum_{\substack{j,k=1\\ j<k}}^N\int_{\mathbb{R}^d}[(\mathrm{Id}-\xi_{j,k}\otimes\xi_{j,k})B)]\cdot\nabla(\psi_i\circ u_\varepsilon)\vartheta_i\chi_{\mathcal{T}_{j,k}}(u_\varepsilon)\,\mathrm{d}x\,.
\end{aligned}$$

Note that the last two terms are nonzero only if $j = i$ or $k = i$. Hence, using Young's inequality, the second term can be estimated by

$$\frac{1}{2}\sum_{\substack{k=1\\ k\neq i}}^N\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\left|\varepsilon(B\cdot\xi_{i,k})\xi_{i,k}|\nabla u_\varepsilon|-\mathrm{H}_\varepsilon\right|^2\chi_{\mathcal{T}_{i,k}}(u_\varepsilon)\,\mathrm{d}x$$

$$+\frac{1}{2}\sum_{\substack{k=1\\k\neq i}}^{N}\int_{\mathbb{R}^d}\frac{1}{\varepsilon}|\partial_u\psi_i(u_\varepsilon)|^2|\vartheta_i|^2\chi_{\mathcal{T}_{i,k}}(u_\varepsilon)\,\mathrm{d}x\,.$$

As for the third one, we obtain using Young's inequality and exploiting the coercivity property (3.14e)

$$\begin{aligned}
&\sum_{\substack{k=1\\k\neq i}}^{N}\int_{\mathbb{R}^d}[(\mathrm{Id}-\xi_{i,k}\otimes\xi_{i,k})B)]\cdot\nabla(\psi_i\circ u_\varepsilon)\vartheta_i\chi_{\mathcal{T}_{i,k}}(u_\varepsilon)\,\mathrm{d}x\\
&\le\sum_{\substack{k=1\\k\neq i}}^{N}\int_{\mathbb{R}^d}\varepsilon|(\mathrm{Id}-\xi_{i,k}\otimes\xi_{i,k})B)]\nabla u_\varepsilon^{\mathsf{T}}|^2\chi_{\mathcal{T}_{i,k}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad+\sum_{\substack{k=1\\k\neq i}}^{N}\int_{\mathbb{R}^d}\frac{1}{\varepsilon}|\partial_u\psi_i|^2|\vartheta_i|^2\chi_{\mathcal{T}_{i,k}}(u_\varepsilon)\,\mathrm{d}x\\
&\le CE[u_\varepsilon|\xi]+\sum_{\substack{k=1\\k\neq i}}^{N}\int_{\mathbb{R}^d}\frac{1}{\varepsilon}|\partial_u\psi_i|^2|\vartheta_i|^2\chi_{\mathcal{T}_{i,k}}(u_\varepsilon)\,\mathrm{d}x.
\end{aligned}$$

In summary, we have shown

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}(\psi_i(u_\varepsilon)-\bar{\chi}_i)\vartheta_i\,\mathrm{d}x\\
&\le C\int_{\mathbb{R}^d}|\psi_i(u_\varepsilon)-\bar{\chi}_i||\vartheta_i|\,\mathrm{d}x+\sum_{\substack{k=1\\k\neq i}}^{N}\int_{\mathbb{R}^d}\frac{1}{\varepsilon}|\partial_u\psi_i(u_\varepsilon)|^2|\vartheta_i|^2\chi_{\mathcal{T}_{i,k}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad+\frac{1}{2}\sum_{\substack{k=1\\k\neq i}}^{N}\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\left|\varepsilon(B\cdot\xi_{i,k})\xi_{i,k}|\nabla u_\varepsilon|-\mathrm{H}_\varepsilon\right|^2\chi_{\mathcal{T}_{i,k}}(u_\varepsilon)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\partial_u\psi_i(u_\varepsilon)\cdot\Big(\varepsilon\Delta u_\varepsilon-\frac{1}{\varepsilon}\partial_uW(u_\varepsilon)\Big)\vartheta_i\,\mathrm{d}x+\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\,\mathrm{H}_\varepsilon\cdot\frac{\nabla u_\varepsilon^{\mathsf{T}}}{|\nabla u_\varepsilon|}\cdot\partial_u\psi_i(u_\varepsilon)\,\mathrm{d}x.
\end{aligned}$$

We estimate the two terms in the last line

$$\begin{aligned}
&\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\partial_u\psi_i(u_\varepsilon)\cdot\Big(\varepsilon\Delta u_\varepsilon-\frac{1}{\varepsilon}\partial_uW(u_\varepsilon)\Big)\vartheta_i\,\mathrm{d}x+\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\,\mathrm{H}_\varepsilon\cdot\frac{\nabla u_\varepsilon^{\mathsf{T}}}{|\nabla u_\varepsilon|}\cdot\partial_u\psi_i(u_\varepsilon)\vartheta_i\,\mathrm{d}x\\
&=\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\Big[\Big(\varepsilon\Delta u_\varepsilon-\frac{1}{\varepsilon}\partial_uW(u_\varepsilon)\Big)+\mathrm{H}_\varepsilon\cdot\frac{\nabla u_\varepsilon^{\mathsf{T}}}{|\nabla u_\varepsilon|}\Big]\cdot\partial_u\psi_i(u_\varepsilon)\vartheta_i\,\mathrm{d}x\\
&\le\frac{1}{2}\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\Big[\Big(\varepsilon\Delta u_\varepsilon-\frac{1}{\varepsilon}\partial_uW(u_\varepsilon)\Big)+\mathrm{H}_\varepsilon\cdot\frac{\nabla u_\varepsilon^{\mathsf{T}}}{|\nabla u_\varepsilon|}\Big]^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^d}\frac{1}{\varepsilon}|\partial_u\psi_i(u_\varepsilon)|^2|\vartheta_i|^2\,\mathrm{d}x\\
&\le\int_{\mathbb{R}^d}\frac{1}{2\varepsilon}\left(\Big|\varepsilon\Delta u_\varepsilon-\frac{1}{\varepsilon}\partial_uW(u_\varepsilon)\Big|^2-|\,\mathrm{H}_\varepsilon\,|^2\right)\,\mathrm{d}x-\int_{\mathbb{R}^d}\frac{1}{2\varepsilon}\Big[\mathrm{Id}-\frac{\nabla u_\varepsilon^{\mathsf{T}}\nabla u_\varepsilon}{|\nabla u_\varepsilon|^2}\Big]:\mathrm{H}_\varepsilon\otimes\mathrm{H}_\varepsilon\,\,\mathrm{d}x\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^d}\frac{1}{\varepsilon}|\partial_u\psi_i(u_\varepsilon)|^2|\vartheta_i|^2\,\mathrm{d}x\\
&\le\int_{\mathbb{R}^d}\frac{1}{2\varepsilon}\left(\Big|\varepsilon\Delta u_\varepsilon-\frac{1}{\varepsilon}\partial_uW(u_\varepsilon)\Big|^2-|\,\mathrm{H}_\varepsilon\,|^2\right)\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^d}\frac{1}{\varepsilon}|\partial_u\psi_i(u_\varepsilon)|^2|\vartheta_i|^2\,\mathrm{d}x\,,
\end{aligned}$$

where we used the fact that $\left[\mathrm{Id} - \frac{\nabla u_\varepsilon^{\mathsf{T}} \nabla u_\varepsilon}{|\nabla u_\varepsilon|^2}\right]$ is a positive semidefinite matrix. Since $|\partial_u \psi_i(u_\varepsilon)| \leq C\sqrt{2W(u_\varepsilon)}$ and $|\vartheta_i| \leq \min\{\mathrm{dist}^2(\cdot, \partial \operatorname{supp} \bar{\chi}_i), 1\} \leq C \min\{\mathrm{dist}(\cdot, I_{i,k}), 1\}$ (see (3.5)), from (3.14d) it follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{\varepsilon} |\partial_u \psi_i(u_\varepsilon)|^2 |\vartheta_i|^2 \,\mathrm{d}x = \sum_{\substack{k=1\\k\neq i}}^N \int_{\mathbb{R}^d} \frac{1}{\varepsilon} |\partial_u \psi_i(u_\varepsilon)|^2 |\vartheta_i|^2 \chi_{\mathcal{T}_{i,k}}(u_\varepsilon) \,\mathrm{d}x \\
&\leq C \sum_{\substack{k=1\\k\neq i}}^N \int_{\mathbb{R}^d} \frac{1}{\varepsilon} 2W(u_\varepsilon) |\vartheta_i|^2 \chi_{\mathcal{T}_{i,k}}(u_\varepsilon) \,\mathrm{d}x \\
&\leq CE[u_\varepsilon|\xi]\,.
\end{aligned}
$$

Summarizing the previous estimates, we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} (\psi_i(u_\varepsilon) - \bar{\chi}_i)\vartheta_i \,\mathrm{d}x \\
&\quad \leq CE[u_\varepsilon|\xi] + C \int_{\mathbb{R}^d} |\psi_i(u_\varepsilon) - \bar{\chi}_i||\vartheta_i| \,\mathrm{d}x \\
&\quad\quad + \frac{1}{2} \sum_{\substack{k=1\\k\neq i}}^N \int_{\mathbb{R}^d} \frac{1}{\varepsilon} \left|\varepsilon(B\cdot \xi_{i,k})\xi_{i,k}|\nabla u_\varepsilon| - \mathrm{H}_\varepsilon\right|^2 \chi_{\mathcal{T}_{i,k}}(u_\varepsilon) \,\mathrm{d}x \\
&\quad\quad + \int_{\mathbb{R}^d} \frac{1}{2\varepsilon} \left(\left| \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \partial_u W(u_\varepsilon) \right|^2 - |\,\mathrm{H}_\varepsilon\,|^2 \right) \,\mathrm{d}x.
\end{aligned}
$$

An application of the Gronwall inequality to Theorem 40 yields

$$
\begin{aligned}
&\sup_{t\in[0,T]} E[u_\varepsilon|\xi](t) + \sum_{i=1}^N \sum_{\substack{k=1\\k\neq i}}^N \int_{\mathbb{R}^d} \frac{1}{\varepsilon} \left|\varepsilon(B\cdot \xi_{i,k})\xi_{i,k}|\nabla u_\varepsilon| - \mathrm{H}_\varepsilon\right|^2 \chi_{\mathcal{T}_{i,k}}(u_\varepsilon) \,\mathrm{d}x \\
&+ \int_{\mathbb{R}^d} \frac{1}{2\varepsilon} \left(\left| \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \partial_u W(u_\varepsilon) \right|^2 - |\,\mathrm{H}_\varepsilon\,|^2 \right) \mathrm{d}x \leq C(d, T, (\bar{\chi}(t))_{t\in[0,T]}) E[u_\varepsilon|\xi](0).
\end{aligned}
$$

Integrating the previous formula in time and inserting this estimate, we deduce by the condition (3.5) on the weight ϑ_i

$$
\begin{aligned}
&\int_{\mathbb{R}^d} (\psi_i(u_\varepsilon(\cdot,T)) - \bar{\chi}_i(\cdot,T))\vartheta_i(\cdot,T) \,\mathrm{d}x \\
&\leq C(d, T, (\bar{\chi}(t))_{t\in[0,T]}) E[u_\varepsilon|\xi](0) + \int_0^T \int_{\mathbb{R}^d} (\psi_i(u_\varepsilon) - \bar{\chi}_i)\vartheta_i \,\mathrm{d}x\, dt.
\end{aligned}
$$

The Gronwall inequality now implies our result, using again (3.5). □

3.4.5 Proof of the main theorem

Our main theorem (Theorem 29) is a simple consequence of Theorem 40 and Proposition 41.

Proof. Combining Theorem 40 and Proposition 41, we obtain the desired bounds

$$
\sup_{t\in[0,T]} E_\varepsilon[u_\varepsilon|\xi](t) \leq C\varepsilon,
$$

$$\sup_{t\in[0,T]} \max_{i\in\{1,\dots,N\}} \int_{\mathbb{R}^d} |\psi_i(u_\varepsilon(\cdot,t)) - \bar{\chi}_i(\cdot,t)| \operatorname{dist}(x, \partial \operatorname{supp} \bar{\chi}_i(\cdot,t))\, dx \leq C\varepsilon.$$

Finally, proceeding as in [48, Sec. 3], one can conclude about the error estimate of order $\varepsilon^{1/2}$ in terms of the L^1-norm. □

3.5 Construction of well-prepared initial data

In this section we construct an initial datum $u_\varepsilon(\cdot,0)$ complying with the following relative energy estimate:

$$E[u_\varepsilon|\xi](0) \leq C\varepsilon\,, \tag{3.32}$$

where the constant $C > 0$ depends on the initial data $(\bar{\chi}_1(\cdot,0),\dots,\bar{\chi}_N(\cdot,0))$ and the potential W satisfying assumptions (A1)–(A4) (see Sec. 3.2). In particular, we provide an explicit construction of $u_\varepsilon(\cdot,0)$ for a network of interfaces meeting at two-dimensional triple junctions ($d=2$) satisfying the 120 degree angle condition. To this aim, we adopt a geometric setting for the initial network which was already introduced in [46, Sec. 5-6] in the general time-dependent case. A similar construction can be provided for three-dimensional double bubbles ($d=3$) satisfying the correct angle condition along the triple line, this time by exploiting the corresponding geometric setting given by [57, Sec. 3-4].

Note that from our construction and the fast decay of the Modica-Mortola profiles towards the pure phases α_i, $1 \leq i \leq N$, it will also be apparent that our initial data $u_\varepsilon(\cdot,0)$ also satisfies the estimate

$$\max_{i\in\{1,\dots,N\}} \int_{\mathbb{R}^d} |\psi_i(u_\varepsilon(\cdot,0)) - \bar{\chi}_i(\cdot,0)| \operatorname{dist}(x, \partial \operatorname{supp} \bar{\chi}_i(\cdot,0))\, dx \leq C\varepsilon.$$

In fact, for this lower-order quantity one may even show the stronger bound $O(\varepsilon^2)$. In summary, the considerations in the present section will establish Proposition 31.

3.5.1 Rescaled one-dimensional equilibrium profiles

For any distinct $i,j \in \{1,...,N\}$, let $\gamma_{i,j} : [-1,1] \to \mathbb{R}^2$ be the unique constant-speed C^1 path connecting α_i to α_j such that $\gamma_{i,j}(-1) = \alpha_i$, $\gamma_{i,j}(1) = \alpha_j$, and

$$\int_{-1}^{1} \sqrt{2W(\gamma_{i,j}(r))}|\gamma'_{i,j}(r)|\mathrm{d}r = 1\,.$$

Let $\tilde{\theta}_{i,j} : \mathbb{R} \to [-1,+1]$ be the unique solution of the ODE

$$\tilde{\theta}'_{i,j}(s) = |\gamma'_{i,j}(\tilde{\theta}_{i,j}(s))|^{-1}\sqrt{2W(\gamma_{i,j}(\tilde{\theta}_{i,j}(s))}$$

with boundary conditions $\tilde{\theta}_{i,j}(\pm\infty) = \pm 1$. Due to the growth properties of W in the neighborhoods of α_i and α_j (see condition (A1) in Sec. 3.2), the profile $\tilde{\theta}_{i,j}$ approaches its boundary values ± 1 at $\pm\infty$ with a power law of order $\frac{2}{q-2}$ for $q > 2$ and an exponential rate for $q = 2$ [121].

Let $s^0_{i,j} \in \mathbb{R}$ such that $\tilde{\theta}_{i,j}(s^0_{i,j}) = 0$. Let $\rho > 0$ be such that $\tilde{\theta}_{i,j}(\rho + s^0_{i,j}) = \bar{\theta}^+_{i,j}$ and $\tilde{\theta}_{i,j}(-\rho + s^0_{i,j}) = -\bar{\theta}^-_{i,j}$ for $\bar{\theta}^{\pm}_{i,j} \in (0,1)$. We define the rescaled one-dimensional equilibrium profiles $\theta_{i,j} : \mathbb{R} \to \gamma_{i,j}$ as

$$\theta_{i,j}(s) := \begin{cases} \gamma_{i,j}\Big(\frac{1}{\bar\theta^+_{i,j}}\tilde\theta_{i,j}(s^0_{i,j} \vee (s+s^0_{i,j}) \wedge (\rho + s^0_{i,j}))\Big) & \text{for } s \in [0,\infty)\,, \\ \gamma_{i,j}\Big(\frac{1}{\bar\theta^-_{i,j}}\tilde\theta_{i,j}((-\rho + s^0_{i,j}) \vee (s+s^0_{i,j}) \wedge s^0_{i,j})\Big) & \text{for } s \in (-\infty,0)\,, \end{cases} \tag{3.33}$$

so that $\theta_{i,j}(s) = \alpha_i$ for any $s \leq \rho$ and $\theta_{i,j}(s) = \alpha_j$ for any $s \geq \rho$. Furthermore, we have

$$(\theta_{i,j})'(s) \tag{3.34}$$

$$= \begin{cases} \frac{1}{\bar\theta^+_{i,j}}\sqrt{2W(\theta_{i,j}(s))}\frac{\gamma'_{i,j}}{|\gamma'_{i,j}|}\Big(\frac{1}{\bar\theta^+_{i,j}}\tilde\theta_{i,j}(s^0_{i,j} \vee (s+s^0_{i,j}) \wedge (\rho + s^0_{i,j}))\Big) \text{ for } s \in [0,\rho), \\ \frac{1}{\bar\theta^-_{i,j}}\sqrt{2W(\theta_{i,j}(s))}\frac{\gamma'_{i,j}}{|\gamma'_{i,j}|}\Big(\frac{1}{\bar\theta^-_{i,j}}\tilde\theta_{i,j}((-\rho + s^0_{i,j}) \vee (s+s^0_{i,j}) \wedge s^0_{i,j})\Big) \text{ for } s \in (-\rho,0), \\ 0 \text{ for } s \in (-\infty,-\rho] \cup [\rho,+\infty). \end{cases} \tag{3.35}$$

Note that, if W satisfies additional symmetry properties along the path $\gamma_{i,j}$, then $\tilde\theta_{i,j}$ is odd, thus $s^0_{i,j} = 0$ and $\bar\theta^-_{i,j} = \bar\theta^+_{i,j}$. Moreover, if W satisfies additional symmetry properties with respect to all the paths $\gamma_{i,j}$, then all $\bar\theta^{\pm}_{i,j}$ coincide.

3.5.2 Geometry of the initial network

For simplicity of notation, we shall omit the evaluation at initial time throughout this chapter, i. e. we write $\bar\chi$ instead of $\bar\chi(\cdot,0)$. Let $\bar\chi = (\bar\chi_1, ..., \bar\chi_N)$ be an initial partition of $\mathbb{R}^2$ with interfaces $\partial\{\bar\chi_i = 1\} \cap \partial\{\bar\chi_j = 1\} =: I_{i,j}$ for distinct $i,j \in \{1,...,N\}$. We decompose the network of interfaces according to its topological features, i.e., into smooth two-phase interfaces and triple junctions. Suppose that the network has P of such topological features $\mathcal{T}_n$, $n \in \{1,...,P\}$. We then split $\{1,...,P\} =: \mathcal{C} \cup \mathcal{P}$, where $\mathcal{C}$ enumerates the connected components of the two-phase interfaces and $\mathcal{P}$ enumerates the triple junctions. In particular, if $p \in \mathcal{P}$, $\mathcal{T}_p$ is a triple junction, whereas if $c \in \mathcal{C}$, $\mathcal{T}_c$ is a connected component of a two-phase interface $I_{i,j}$ for some distinct $i,j \in \{1,...,N\}$.

In the following we use a suitable notion of neighborhood for a single connected component of the network of interfaces provided by [46, Definition 21]. In particular, we adopt the notion of localization radius, which allows to define the diffeomorphism corresponding to a single connected component of a network as it follows. Let $r_{i,j}$ be a localization radius for the interface $I_{i,j}$ and let $\mathrm{n}_{i,j}$ be the normal vector field to $I_{i,j}$ pointing towards $\{\bar\chi_j = 1\}$ for some distinct $i,j \in \{1,...,N\}$. Then, the map $\Psi_{i,j} : I_{i,j} \times (-r_{i,j}, r_{i,j}) \to \mathbb{R}^2$, $(x,s) \mapsto x + s\mathrm{n}_{i,j}(x)$ defines a diffeomorphism, whose inverse can be splitted as follows $\Psi^{-1}_{i,j} : \mathrm{im}(\Psi_{i,j}) \mapsto I_{i,j} \times (-r_{i,j}, r_{i,j})$, $x \mapsto (P_{I_{i,j}}x, \mathrm{dist}^{\pm}(x, I_{i,j}))$, where $P_{I_{i,j}} : \mathrm{im}\,(\Psi_{i,j}) \to I_{i,j}$ represent the projection onto the nearest point on the interface $I_{i,j}$, whereas $\mathrm{dist}^{\pm}(\cdot, I_{i,j}) : \mathrm{im}\,(\Psi_{i,j}) \to (-r_{i,j}, r_{i,j})$ a signed distance function.

Similarly as in [46, Definition 24], we provide a notion of admissible localization radius for a triple junction.

Definition 44. *Let $d = 2$. Let $\bar\chi = (\bar\chi_1, ..., \bar\chi_N)$ be an initial partition of $\mathbb{R}^d$ with interfaces $\partial\{\bar\chi_i = 1\}$, $i = 1, ..., N$. Let $\mathcal{T}$ be a triple junction present in the network of interfaces of $\bar\chi$, which we assume for simplicity to be formed by the phases 1, 2 and 3. For each $i \in \{1,2,3\}$, denote by $\mathcal{T}_{i,i+1}$ the connected component of $I_{i,i+1}$ with an endpoint at the triple junction $\mathcal{T}$*

and let $r_{i,i+1} \in (0,1]$ be an admissible localization radius for the interface $I_{i,i+1}$ in the sense of [46, Definition 21]. We call a scale $r = r_{\mathcal{T}} \in (0, r_{1,2} \wedge r_{2,3} \wedge r_{3,1}]$ an admissible localization radius for the triple junction $\mathcal{T}$ if there exists a wedge decomposition of the neighborhood $B_r(\mathcal{T})$ of the triple junction in the following sense:

For each $i \in \{1,2,3\}$ there exist sets $W_{i,i+1}$ and W_i with the following properties:

First, the sets $W_{i,i+1}$ and W_i are non-empty subsets of $B_r(\mathcal{T})$ with pairwise disjoint interior such that

$$\bigcup_{i \in \{1,2,3\}} \overline{W_{i,i+1}} \cup \overline{W_i} = \overline{B_r(\mathcal{T})}\,.$$

Second, each of these sets is represented by a cone with apex at the triple junction $\mathcal{T}$ intersected with $B_r(\mathcal{T})$. More precisely, there exist six pairwise distinct unit-length vectors $\left(X^i_{i,i+1}, X^{i+1}_{i,i+1}\right)_{i \in \{1,2,3\}}$ such that for all $i \in \{1,2,3\}$ we have

$$\begin{aligned} W_{i,i+1} &= \left(\mathcal{T} + \left\{a X^i_{i,i+1} + b X^{i+1}_{i,i+1} : a, b \in (0,\infty)\right\}\right) \cap B_r(\mathcal{T}) \\ W_i &= \left(\mathcal{T} + \left\{a X^i_{i,i+1} + b X^i_{i-1,i} : a, b \in (0,\infty)\right\}\right) \cap B_r(\mathcal{T})\,. \end{aligned}$$

The opening angles of these cones are numerically fixed by

$$X^i_{i,i+1} \cdot X^{i+1}_{i,i+1} = \cos(\tfrac{\pi}{2}) = 0\,, \quad X^i_{i,i+1} \cdot X^i_{i-1,i} = \cos(\tfrac{\pi}{6})\,.$$

Third, we require that for all $i \in \{1,2,3\}$ it holds

$$\begin{aligned} B_r(\mathcal{T}) \cap \mathcal{T}_{i,i+1} \subset W_{i,i+1} \cup \mathcal{T} &\subset \mathbb{H}_{i,i+1} \\ W_i &\subset \mathbb{H}_{i,i+1} \cap \mathbb{H}_{i,i-1} \end{aligned}$$

with the domains $\mathbb{H}_{i,i+1} := \{x \in \mathbb{R}^2 : x \in \operatorname{im}(\Psi_{i,i+1})\} \cap B_r(\mathcal{T})$, where $\Psi_{i,i+1}$ is the diffeomorphism defining the neighboorhood of $I_{i,i+1}$ in the sense of [46, Definition 21].

Let $r := \min_{p \in \mathcal{P}} r_p$, where r_p admissible localization radius for the triple junction $\mathcal{T}_p$. Let $\rho \in (0, r)$. Consider a triple junction $\mathcal{T}$, which we assume for simplicity formed by the phases $1, 2$ and 3. Let $\varepsilon < \rho$, then for all $i \in \{1,2,3\}$ we define

(i) the two-dimensional regions

$$\begin{aligned} W^{\rho,+}_{i,i+1} &:= W_{i,i+1} \cap \{x \in \mathbb{R}^2 : 0 \le \operatorname{dist}^{\pm}(x, \mathcal{T}_{i,i+1}) \le \rho\}\,, \\ W^{\rho,-}_{i,i+1} &:= W_{i,i+1} \cap \{x \in \mathbb{R}^2 : -\rho \le \operatorname{dist}^{\pm}(x, \mathcal{T}_{i,i+1}) \le 0\}\,, \\ W^{\rho}_{i,i+1} &:= W^{\rho,+}_{i,i+1} \cup W^{\rho,-}_{i,i+1}\,, \\ W^{\varepsilon,\pm}_{i,i+1} &:= W^{\rho,\pm}_{i,i+1} \cap B_\varepsilon(\mathcal{T})\,, \\ W^{\varepsilon}_{i,i+1} &:= W^{\varepsilon,+}_{i,i+1} \cup W^{\varepsilon,-}_{i,i+1}\,, \end{aligned}$$

satisfying the inclusions

$$\mathcal{T}_{i,i+1} \cap B_r(\mathcal{T}) \subset W^{\rho}_{i,i+1}\,, \quad \mathcal{T}_{i,i+1} \cap B_\varepsilon(\mathcal{T}) \subset W^{\varepsilon}_{i,i+1} \subset W^{\rho}_{i,i+1}\,;$$

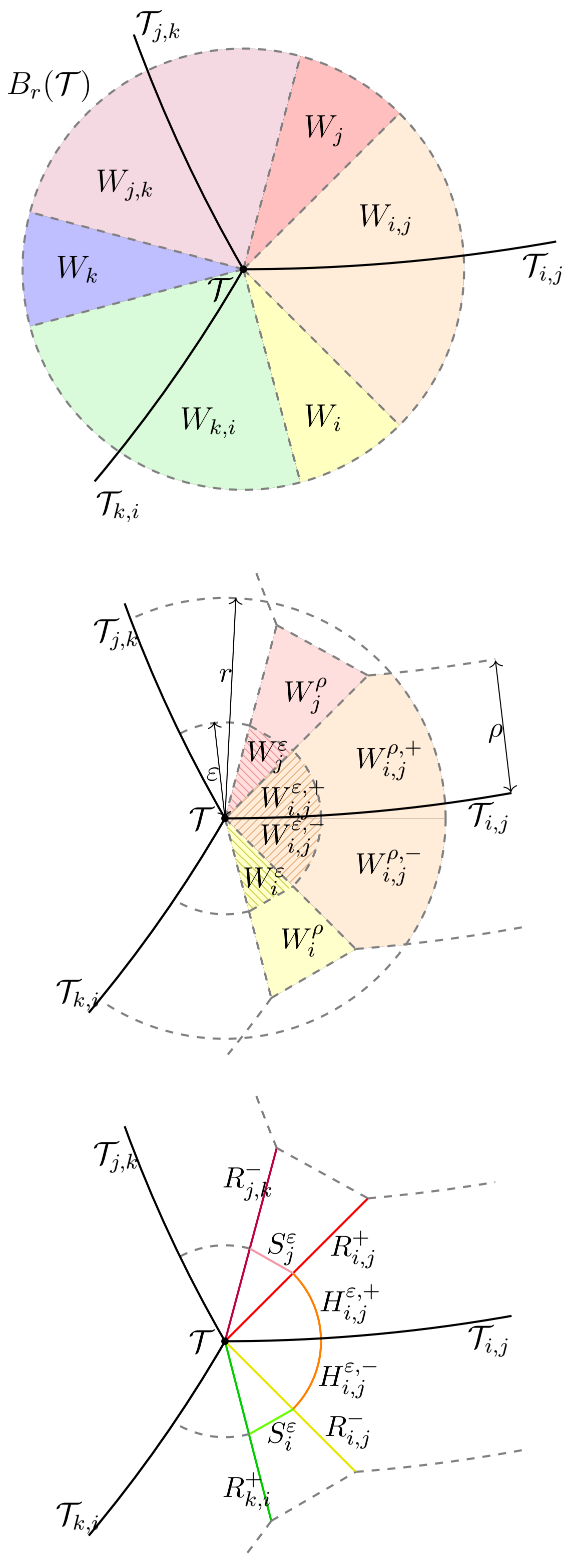


Figure 3.2: Notation and geometry for the construction of well-prepared initial data at a triple junction $\mathcal{T}$ ($d = 2$) formed by the phases i, j and k for mutually distinct $i, j, k \in \{1, ..., N\}$.

(ii) the one-dimensional segments resp. arcs

$$
\begin{aligned}
R^i_{i,i+1} &:= \Big(\mathcal{T} + \Big\{aX^i_{i,i+1} : a \in (0,\infty)\Big\}\Big) \cap \partial W^\rho_{i,i+1}\,,\\
R^+_{i,i+1} &:= R^{i+1}_{i,i+1}, \quad R^-_{i,i+1} := R^i_{i,i+1}\,,\\
H^\varepsilon_{i,i+1} &:= W^\rho_{i,i+1} \cap \partial B_\varepsilon(\mathcal{T})\,,\\
H^{\varepsilon,+}_{i,i+1} &:= H^\varepsilon_{i,i+1} \cap \{x \in \mathbb{R}^2 : 0 \le \operatorname{dist}^\pm(x,\mathcal{T}_{i,i+1}) \le \rho\}\,,\\
H^{\varepsilon,-}_{i,i+1} &:= H^\varepsilon_{i,i+1} \cap \{x \in \mathbb{R}^2 : -\rho \le \operatorname{dist}^\pm(x,\mathcal{T}_{i,i+1}) \le 0\}\,,
\end{aligned}
$$

(iii) S^ε_i as the segment connecting $R^i_{i,i+1} \cap \partial B_\varepsilon(\mathcal{T})$ to $R^i_{i,i-1} \cap \partial B_\varepsilon(\mathcal{T})$;

(iv) the two-dimensional triangular regions W^ρ_i resp. W^ε_i as the one with sides $R^i_{i,i+1}$ and $R^i_{i,i-1}$ resp. the one delimitated by S^ε_i, $R^i_{i,i+1}$ and $R^i_{i,i-1}$, thus satisfying the inclusions

$$
W^\varepsilon_i \subset W^\rho_i \subset W_i\,.
$$

Furthermore, we introduce $P_{R^+_{i,i+1}} : W^\rho_i \cup (W^\rho_{i,i+1} \cap \{x \in \Psi_{i,i+1}(I_{i,i+1} \times [0,\rho))\}) \to R^+_{i,i+1}$, $P_{R^-_{i,i+1}} : W^\rho_i \cup (W^\rho_{i,i+1} \cap \{x \in \Psi_{i,i+1}(I_{i,i+1} \times (-\rho,0])\}) \to R^-_{i,i+1}$, $P_{H^\varepsilon_{i,i+1}} : W^\varepsilon_{i,i+1} \to H^\varepsilon_{i,i+1}$ and $P_{S^\varepsilon_i} : W^\varepsilon_i \to S^\varepsilon_i$ as the orthogonal projections onto the nearest point on $R^+_{i,i+1}$, $R^-_{i,i+1}$, $H^\varepsilon_{i,i+1}$ and S^ε_i, respectively.

3.5.3 Construction of the initial datum

We construct the initial datum $u_{\varepsilon,0} := u_\varepsilon(\cdot,0)$ seperately in each two-dimensional region identified by the geometry of the network as introduced above. Then, we show that in each of these regions the initial relative energy estimate (3.32) holds true.

Neighborhood of a connected component of a two-phase interface. Let $\mathcal{T}_{i,j}$ be a connected component of $I_{i,j}$ with either one or two endpoints at a triple junction for some distinct $i,j \in \{1,...,N\}$. Let $\mathcal{P}_{i,j} \subset \mathcal{P}$ enumerate the numbers of triple junctions as endpoints of $\mathcal{T}_{i,j}$. Then, we can define

$$
\begin{aligned}
M_{i,j} := &\Bigg(\{x \in \mathbb{R}^2 : -\rho \le \operatorname{dist}^\pm(x,\mathcal{T}_{i,j}) \le \rho\} \setminus \bigcup_{p \in \mathcal{P}_{i,j}} B_r(\mathcal{T}_p)\Bigg)\\
&\cup \Bigg(\bigcup_{p \in \mathcal{P}_{i,j}} (W^\rho_{i,j} \setminus W^\varepsilon_{i,j})\Bigg).
\end{aligned}
$$

In the two-dimensional region $M_{i,j}$ we define the initial datum $u_{\varepsilon,0}$ by means of the rescaled one-dimensional equilibrium profile (3.33) as follows

$$
u_{\varepsilon,0}(x) := \theta_{i,j}(\varepsilon^{-1}\operatorname{dist}^\pm(x,\mathcal{T}_{i,j})) \quad \text{for any } x \in M_{i,j}\,, \tag{3.36}
$$

whence we obtain

$$
\begin{aligned}
&E_{M_{i,j}}[u_\varepsilon|\xi](0)\\
&:= \int_{M_{i,j}} \frac{1}{2\varepsilon}|(\theta_{i,j})'(\varepsilon^{-1}\operatorname{dist}^\pm(x,\mathcal{T}_{i,j}))|^2 + \frac{1}{\varepsilon}W(\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}^\pm(x,\mathcal{T}_{i,j})))\\
&\qquad - \frac{1}{2\varepsilon}(\xi_{i,j}\cdot n_{i,j})(\theta_{i,j})'(\varepsilon^{-1}\operatorname{dist}^\pm(x,\mathcal{T}_{i,j}))\cdot\partial_u\psi_{i,j}(\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}^\pm(x,\mathcal{T}_{i,j})))\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{M_{i,j}} \frac{1}{2\varepsilon}|(\theta_{i,j})'(\varepsilon^{-1}\operatorname{dist}^{\pm}(x,\mathcal{T}_{i,j}))|^2 + \frac{1}{\varepsilon}W(\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}^{\pm}(x,\mathcal{T}_{i,j}))) \\
&\qquad - \frac{1}{\varepsilon}|\xi_{i,j}|^2 \frac{1}{\bar{\theta}^{\pm}_{i,j}} 2W(\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}^{\pm}(x,\mathcal{T}_{i,j})))\,\mathrm{d}x \\
&\leq \int_{M_{i,j}} (1-|\xi_{i,j}|^2)\left[\frac{1}{2\varepsilon}|(\theta_{i,j})'(\varepsilon^{-1}\operatorname{dist}^{\pm}(x,\mathcal{T}_{i,j}))|^2 + \frac{1}{\varepsilon}W(\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}^{\pm}(x,\mathcal{T}_{i,j})))\right]\mathrm{d}x.
\end{aligned}
$$

Here, we used that $\psi_k = 0$ along $\gamma_{i,j}$ for any $k \in \{1,...,N\}\setminus\{i,j\}$ and that $\partial_u \psi_{i,j}(\gamma_{i,j}) \cdot \frac{\gamma'_{i,j}}{|\gamma'_{i,j}|} = 2\sqrt{2W(\gamma_{i,j})}$. Indeed, assumption (A4) implies $\partial_u \psi_{i,j}(\gamma_{i,j}) \cdot \gamma'_{i,j} \leq 2\sqrt{2W(\gamma_{i,j})}|\gamma'_{i,j}|$, then (A3) gives the equality sign by contradiction. Then, in the last step we added a zero and we used $|(\theta_{i,j})'(s)|^2 \leq \frac{1}{(\bar{\theta}^{\pm}_{i,j})^2} 2W(\theta_{i,j}(s))$. Note that $1 - |\xi_{i,j}|^2 \leq c\operatorname{dist}^2(\cdot, \mathcal{T}_{i,j})$, and that $W(\theta_{i,j}(s))$ has an exponential resp. a power-law decay of order $\frac{2q}{q-2}$ for $q = 2$ resp. $q > 2$ as s approaches the extrema of $(-\rho, \rho)$, then it vanishes for $s \in (-\infty, -\rho] \cup [\rho, \infty)$. As a consequence, we obtain $E_{M_{i,j}}[u_\varepsilon|\xi](0) \leq C\varepsilon^2$ for some constant $C > 0$.

Pure-phase region. Let $i \in \{1,...,N\}$. Let $\mathcal{P}_i \subset \mathcal{P}$ enumerate the numbers of triple junctions as endpoints of a connected component of an interface between the phase i and any other one. We set $u_{\varepsilon,0} = \alpha_i$ in the pure-phase region $\{\bar{\chi}_i = 1\} \setminus \left(\bigcup_{p\in\mathcal{P}_i} W_i \cup \bigcup_{j:j\neq i}\{x - sn_{i,j}(x), x \in I_{i,j}\,, s \in [0,\rho)\}\right)$. Then $|\nabla u_{\varepsilon,0}| = 0$ and, having $W(\alpha_i) = 0$, the initial relative entropy is equal to zero in the pure-phase region.

Triple junction wedge containing a connected component of a two-phase interface. Given a triple junction $\mathcal{T}$, let $i, j \in \{1,...,N\}$, $i \neq j$, be two of the three phases forming $\mathcal{T}$ and let $k \in \{1,...,N\} \setminus \{i,j\}$ be the third one. The initial datum $u_{\varepsilon,0}$ in the corresponding wedge $W^{\varepsilon,\pm}_{i,j}$ is given by interpolation via orthogonal projections $P_{H^\varepsilon_{i,j}}, P_{I_{i,j}}$ and $P_{R^\pm_{i,j}}$, which reads as

$$
\begin{aligned}
u_{\varepsilon,0}(x) =& \frac{\operatorname{dist}(x, I_{i,j}) + \operatorname{dist}(x, R^{\pm}_{i,j})}{\operatorname{dist}(x, H^\varepsilon_{i,j}) + \operatorname{dist}(x, I_{i,j}) + \operatorname{dist}(x, R^{\pm}_{i,j})} u_{H^\varepsilon_{i,j}}(P_{H^\varepsilon_{i,j}} x) \\
&+ \frac{\operatorname{dist}(x, H^\varepsilon_{i,j}) + \operatorname{dist}(x, R^{\pm}_{i,j})}{\operatorname{dist}(x, H^\varepsilon_{i,j}) + \operatorname{dist}(x, I_{i,j}) + \operatorname{dist}(x, R^{\pm}_{i,j})} u_{I_{i,j}}(P_{I_{i,j}} x) \\
&+ \frac{\operatorname{dist}(x, H^\varepsilon_{i,j}) + \operatorname{dist}(x, I_{i,j})}{\operatorname{dist}(x, H^\varepsilon_{i,j}) + \operatorname{dist}(x, I_{i,j}) + \operatorname{dist}(x, R^{\pm}_{i,j})} u_{R_{i,j}}(P_{R^\pm_{i,j}} x)
\end{aligned}
$$

for any $x \in W^{\varepsilon,\pm}_{i,j}$, where

$$
\begin{aligned}
&u_{H^\varepsilon_{i,j}}(x) = \frac{h^\pm_{i,j}(x,\mathcal{T})}{h^{\varepsilon,\pm}_{i,j}} \theta_{i,j}(\pm\varepsilon^{-1}\bar{h}^{\varepsilon,\pm}_{i,j}) + \frac{h^{\varepsilon,\pm}_{i,j} - h^\pm_{i,j}(x,\mathcal{T})}{h^{\varepsilon,\pm}_{i,j}} \theta_{i,j}(0) \\
&\text{for any } x \text{ along } H^\varepsilon_{i,j}\,, \\
&u_{I_{i,j}}(x) = \frac{l_{i,j}(x,\mathcal{T})}{l^\varepsilon_{i,j}} \theta_{i,j}(0) + \frac{l^\varepsilon_{i,j} - l_{i,j}(x,\mathcal{T})}{l^\varepsilon_{i,j}} \bar{\alpha} \\
&\text{for any } x \text{ along } I_{i,j}\,, \\
&u_{R_{i,j}}(x) = \frac{r^\pm_{i,j}(x,\mathcal{T})}{\varepsilon} \theta_{i,j}(\pm\varepsilon^{-1}\bar{h}^{\varepsilon,\pm}_{i,j}) + \frac{\varepsilon - r^\pm_{i,j}(x,\mathcal{T})}{\varepsilon} \bar{\alpha} \\
&\text{for any } x \text{ along } R^\pm_{i,j}\,,
\end{aligned}
$$

where $\bar{\alpha} = \frac{\alpha_i+\alpha_j+\alpha_k}{3}$, $\bar{h}^{\varepsilon,\pm}_{i,j} := \mathrm{dist}^{\pm}(H^{\varepsilon}_{i,j} \cap R^{\pm}_{i,j}, I_{i,j})$, $l^{\varepsilon}_{i,j}$ resp. $h^{\varepsilon,\pm}_{i,j}$ is the length of $I_{i,j} \cap B_{\varepsilon}(\mathcal{T})$ resp. $\partial B_{\varepsilon}(\mathcal{T}) \cap W^{\varepsilon,\pm}_{i,j}$, whereas $h^{\pm}_{i,j}(\cdot,\mathcal{T})$, $l_{i,j}(\cdot,\mathcal{T})$ resp. $r^{\pm}_{i,j}(\cdot,\mathcal{T})$ is the length of the path along $H^{\varepsilon,\pm}_{i,j}$, $I_{i,j}$ resp. $R^{\pm}_{i,j}$ connecting to $\mathcal{T}$. Since $l^{\varepsilon}_{i,j}$ and $h^{\varepsilon,\pm}_{i,j}$ are of order ε, then this construction gives that $|\nabla u_{\varepsilon,0}(x)|$ is of order $1/\varepsilon$ for any $x \in W^{\varepsilon}_{i,j}$. Hence, being the area of $W^{\varepsilon}_{i,j}$ of order ε^2, we can deduce

$$
\begin{aligned}
E_{W^{\varepsilon}_{i,j}}[u_{\varepsilon}|\xi](0) &:= \int_{W^{\varepsilon}_{i,j}} \frac{\varepsilon}{2}|\nabla u_{\varepsilon,0}|^2 + \frac{1}{\varepsilon}W(u_{\varepsilon,0}) + \sum_{\ell=1}^{N} \xi_{\ell} \cdot \nabla(\psi_{\ell} \circ u_{\varepsilon,0})\,\mathrm{d}x \\
&\leq \int_{W^{\varepsilon}_{i,j}} \frac{\varepsilon}{2}|\nabla u_{\varepsilon,0}|^2 + \frac{C}{\varepsilon} + C|\nabla u_{\varepsilon,0}|\,\mathrm{d}x \leq C\varepsilon\,,
\end{aligned}
$$

for some constant $C > 0$ varying from line to line.

Triple junction wedge not containing any connected component of a two-phase interface. Given a triple junction $\mathcal{T}$, let $i,j,k \in \{1,...,N\}$ be three distinct phases forming $\mathcal{T}$. The initial datum $u_{\varepsilon,0}$ in the region W^{ε}_{j} is given by interpolation via orthogonal projections $P_{S^{\varepsilon}_{j}}, P_{R^{-}_{j,k}}$ and $P_{R^{+}_{i,j}}$, which reads as

$$
\begin{aligned}
u_{\varepsilon,0}(x) =& \frac{\mathrm{dist}(x,R^{-}_{j,k}) + \mathrm{dist}(x,R^{+}_{i,j})}{\mathrm{dist}(x,S^{\varepsilon}_{j}) + \mathrm{dist}(x,R^{-}_{j,k}) + \mathrm{dist}(x,R^{+}_{i,j})} u_{H^{\varepsilon}_{i,j}}(P_{S^{\varepsilon}_{j}}x) \\
&+ \frac{\mathrm{dist}(x,S^{\varepsilon}_{j}) + \mathrm{dist}(x,R^{+}_{i,j})}{\mathrm{dist}(x,S^{\varepsilon}_{j}) + \mathrm{dist}(x,R^{-}_{j,k}) + \mathrm{dist}(x,R^{+}_{i,j})} u_{R^{-}_{j,k}}(P_{R^{-}_{j,k}}x) \\
&+ \frac{\mathrm{dist}(x,S^{\varepsilon}_{j}) + \mathrm{dist}(x,R^{-}_{j,k})}{\mathrm{dist}(x,S^{\varepsilon}_{j}) + \mathrm{dist}(x,R^{-}_{j,k}) + \mathrm{dist}(x,R^{+}_{i,j})} u_{R^{+}_{i,j}}(P_{R^{+}_{i,j}}x)\,,
\end{aligned}
$$

for any $x \in W^{\varepsilon}_{j}$, where

$$
u_{S^{\varepsilon}_{j}}(x) = \frac{s_j(x)}{s^{\varepsilon}_{j}}\theta_{i,j}(\varepsilon^{-1}\bar{h}^{\varepsilon}_{i,j}) + \frac{s^{\varepsilon}_{j} - s_j(x)}{s^{\varepsilon}_{j}}\theta_{j,k}(-\varepsilon^{-1}\bar{h}^{\varepsilon}_{j,k}) \quad \text{for any } x \text{ along } S^{\varepsilon}_{j}\,, \tag{3.37}
$$

$$
u_{R^{-}_{j,k}}(x) = \frac{r_{j,k}(x,\mathcal{T})}{\varepsilon}\theta_{j,k}(-\varepsilon^{-1}\bar{h}^{\varepsilon}_{j,k}) + \frac{\varepsilon - r_{j,k}(x,\mathcal{T})}{\varepsilon}\bar{\alpha} \quad \text{for any } x \text{ along } R^{-}_{j,k}\,, \tag{3.38}
$$

$$
u_{R^{+}_{i,j}}(x) = \frac{r_{i,j}(x,\mathcal{T})}{\varepsilon}\theta_{i,j}(\varepsilon^{-1}\bar{h}^{\varepsilon}_{i,j}) + \frac{\varepsilon - r_{i,j}(x,\mathcal{T})}{\varepsilon}\bar{\alpha} \quad \text{for any } x \text{ along } R^{+}_{i,j}\,, \tag{3.39}
$$

where $\bar{h}^{\varepsilon}_{i,j} := \mathrm{dist}(H^{\varepsilon}_{i,j} \cap R^{+}_{i,j}, I_{i,j})$, $\bar{h}^{\varepsilon}_{j,k} := \mathrm{dist}(H^{\varepsilon}_{j,k} \cap R^{-}_{j,k}, I_{j,k})$, s^{ε}_{j} is the lenght of the segment S^{ε}_{j}, whereas s_j resp. $r_{i,j}(\cdot,\mathcal{T})$ is the lenght of the path along S^{ε}_{j} resp. $R^{+}_{i,j}$ connecting to $R^{-}_{j,k}$ resp. $\mathcal{T}$. Since s^{ε}_{j} is of order ε, we have that $|\nabla u_{\varepsilon,0}(x)|$ is of order $1/\varepsilon$ for any $x \in W^{\varepsilon}_{j}$, whence as above we can deduce that $E_{W^{\varepsilon}_{j}}[u_{\varepsilon}|\xi](0) := \int_{W^{\varepsilon}_{j}} \frac{\varepsilon}{2}|\nabla u_{\varepsilon,0}|^2 + \frac{1}{\varepsilon}W(u_{\varepsilon,0}) + \sum_{\ell=1}^{N} \xi_{\ell} \cdot \nabla(\psi_{\ell} \circ u_{\varepsilon,0})\,\mathrm{d}x \leq C\varepsilon$, for some constant $C > 0$.

Interpolation between two rescaled one-dimensional equilibrium profiles. First, we introduce for any $x \in W^{\rho}_{j} \setminus W^{\varepsilon}_{j}$ the set of coordinates (s,h), where h denotes the distance from S^{ε}_{j} while s is such that $\nabla s \cdot \nabla h = 0$ and $s = 0$ whenever $x \in R^{-}_{j,k}$. Hence, $h \in [0,\tilde{r}_{\varepsilon}]$, where $\tilde{r}_\varepsilon := (r-\varepsilon)\cos(\frac{\pi}{12})$ is fixed, and $s \in [0, 2\tilde{h}_\varepsilon \sin(\frac{\pi}{12})]$, where $\tilde{h}_\varepsilon := (\cos(\frac{\pi}{12}))^{-1}h + \varepsilon$. The initial datum $u_{\varepsilon,0}$ in $W_j^\rho \setminus W_j^\varepsilon$ is given as follows

$$u_{\varepsilon,0}(x) = \frac{s}{2\tilde{h}_\varepsilon \sin(\frac{\pi}{12})}\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})) + \frac{2\tilde{h}_\varepsilon \sin(\frac{\pi}{12}) - s}{2\tilde{h}_\varepsilon \sin(\frac{\pi}{12})}\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j}))\,, \tag{3.40}$$

for any $x \in W_j^\rho \setminus W_j^\varepsilon$, where $P^s_{R^-_{j,k}}$ and $P^s_{R^+_{i,j}}$ projections along the s-axis onto $R^-_{j,k}$ and $R^+_{i,j}$, respectively. Hence, we compute

$$\partial_s u_{\varepsilon,0}(x) = \frac{1}{2\tilde{h}_\varepsilon \sin(\frac{\pi}{12})}\left(\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})) - \theta_{i,j}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j}))\right),$$

and

$$\begin{aligned}
&\partial_h u_{\varepsilon,0}(x) \\
&= \frac{s}{2\tilde{h}_\varepsilon^2 \sin(\frac{\pi}{12})\cos(\frac{\pi}{12})}\left(\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{k,j})) - \theta_{i,j}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j}))\right) \\
&\quad - \frac{s}{2\varepsilon\tilde{h}_\varepsilon \sin(\frac{\pi}{12})}\partial_h \operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})(\theta_{j,k})'(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})) \\
&\quad + \frac{2\tilde{h}_\varepsilon \sin(\frac{\pi}{12}) - s}{2\varepsilon\tilde{h}_\varepsilon \sin(\frac{\pi}{12})}\partial_h \operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j})(\theta_{i,j})'(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j}))\,.
\end{aligned}$$

For the sake of brevity, we introduce the notation $\tilde{\lambda}(x) := \frac{s}{2\tilde{h}_\varepsilon \sin(\frac{\pi}{12})} \in [0,1]$ for any $x \in W_j^\rho \setminus W_j^\varepsilon$. By adding zeros and using the Young inequality together with the fact that $s^2 \leq 4\tilde{h}_\varepsilon^2 \sin^2(\frac{\pi}{12})$, one can obtain

$$\begin{aligned}
\frac{\varepsilon}{2}|\nabla u_{\varepsilon,0}(x)|^2 &= \frac{\varepsilon}{2}|\partial_s u_{\varepsilon,0}(x)|^2 + \frac{\varepsilon}{2}|\partial_h u_{\varepsilon,0}(x)|^2 \\
&\leq C\frac{\varepsilon}{\tilde{h}_\varepsilon^2}|\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})) - \alpha_j|^2 \\
&\quad + C\frac{\varepsilon}{\tilde{h}_\varepsilon^2}|\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j})) - \alpha_j|^2 \\
&\quad + C\tilde{\lambda}^2(x)\frac{1}{\varepsilon}|(\theta_{j,k})'(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k}))|^2 \\
&\quad + C(1-\tilde{\lambda}(x))^2\frac{1}{\varepsilon}|(\theta_{i,j})'(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j})|^2\,,
\end{aligned}$$

for some constant $C > 0$. First, we consider

$$\begin{aligned}
&\int_{W_j^\rho \setminus W_j^\varepsilon} \frac{\varepsilon}{\tilde{h}_\varepsilon^2}|\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})) - \alpha_j|^2\,\mathrm{d}x \\
&= 2\sin(\tfrac{\pi}{12})\int_0^{\tilde{r}_\varepsilon} \frac{\varepsilon}{\tilde{h}_\varepsilon}|\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{k,j})) - \alpha_j|^2\,\mathrm{d}h \\
&\leq C\int_0^{\tilde{r}_\varepsilon} |\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})) - \alpha_j|^2\,\mathrm{d}h
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_0^{\tilde r_\varepsilon} \frac{h}{\varepsilon}|\theta_{j,k}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})) - \alpha_j||(\theta_{j,k})'(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k}))|\,\mathrm{d}h\\
&\leq C\int_0^{\tilde r_\varepsilon} \frac{h}{\varepsilon}|(\theta_{j,k})'(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k}))|\,\mathrm{d}h\,,
\end{aligned}
$$

where we integrated by parts and $C>0$ is a suitable constant varying from line to line. Then, we observe that $\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})$ is a homogeneous and increasing function with respect to h. On the other hand, we recall that $|(\theta_{j,k})'(s)|$ has an exponential resp. a power-law decay of order $\frac{q}{q-2}$ for $q=2$ resp. $q>2$ as s approaches the extrema of $(-\rho,\rho)$, then it vanishes for $s\in(-\infty,-\rho]\cup[\rho,\infty)$. As a consequence, we obtain

$$
\int_{W_j^\rho\setminus W_j^\varepsilon} \frac{\varepsilon}{\tilde h_\varepsilon^2}|\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})) - \alpha_j|^2\,\mathrm{d}x \leq C\varepsilon\,,
$$

and analogously

$$
\int_{W_j^\rho\setminus W_j^\varepsilon} \frac{\varepsilon}{\tilde h_\varepsilon^2}|\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j})) - \alpha_j|^2\,\mathrm{d}x \leq C\varepsilon\,.
$$

Second, we have

$$
\begin{aligned}
&\int_{W_j^\rho\setminus W_j^\varepsilon} \tilde\lambda^2(x)\frac{1}{\varepsilon}|(\theta_{j,k})'(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k}))|^2\,\mathrm{d}x\\
&= 2\sin(\tfrac{\pi}{12})\int_0^{\tilde r_\varepsilon} \tilde\lambda^2(x)\frac{\tilde h_\varepsilon}{\varepsilon}|(\theta_{j,k})'(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k}))|^2\mathrm{d}h\\
&\leq C\int_0^{\tilde r_\varepsilon}\Big(\frac{h}{\varepsilon}+1\Big)|(\theta_{j,k})'(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k}))|^2\mathrm{d}h\\
&\leq C\varepsilon\,,
\end{aligned}
$$

and analogously

$$
\int_{W_j^\rho\setminus W_j^\varepsilon}(1-\tilde\lambda(x))^2\frac{1}{\varepsilon}|(\theta_{i,j})'(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j}))|^2\,\mathrm{d}x \leq C\varepsilon\,.
$$

Observe that by adding zeros we can write

$$
\begin{aligned}
W(u_{\varepsilon,0}(x)) = {}&\tilde\lambda(x)\Big(W(u_{\varepsilon,0}(x)) - W(\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})))\Big)\\
&+(1-\tilde\lambda(x))\Big(W(u_{\varepsilon,0}(x)) - W(\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j})))\Big)\\
&+\tilde\lambda(x)W(\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})))\\
&+(1-\tilde\lambda(x))W(\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j})))\,,
\end{aligned}
$$

where $\tilde\lambda(x), 1-\tilde\lambda(x)\in[0,1]$ for any $x\in W_j^\rho\setminus W_j^\varepsilon$. Since W is Lipschitz (see (A1) in Sec. 3.2), then by adding zeros we obtain

$$
\begin{aligned}
&\Big|\frac{1}{\varepsilon}W(u_{\varepsilon,0}(x)) - \frac{1}{\varepsilon}W(\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})))\Big|\\
&\leq \frac{C}{\varepsilon}|u_{\varepsilon,0}(x) - \theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k}))|
\end{aligned}
$$

$$\le \frac{C}{\varepsilon}(1-\tilde\lambda(x))\left(|\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k}))-\alpha_j| + |\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j}))-\alpha_j|\right)$$

and

$$\begin{aligned}
&\left|\frac{1}{\varepsilon}W(u_{\varepsilon,0}(x)) - \frac{1}{\varepsilon}W(\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j})))\right|\\
&\le \frac{C}{\varepsilon}|u_{\varepsilon,0}(x)-\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j}))|\\
&\le \frac{C}{\varepsilon}\tilde\lambda(x)\left(|\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k}))-\alpha_j| + |\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j}))-\alpha_j|\right).
\end{aligned}$$

In particular, we have

$$\begin{aligned}
&\int_{W^\rho_j\setminus W^\varepsilon_j}\frac{1}{\varepsilon}|\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k}))-\alpha_j|\,\mathrm{d}x\\
&= 2\sin(\tfrac{\pi}{12})\int_0^{\tilde r_\varepsilon}\frac{\tilde h_\varepsilon}{\varepsilon}|\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{k,j}))-\alpha_j|\,\mathrm{d}h\\
&\le C\int_0^{\tilde r_\varepsilon}\frac{(\cos(\frac{\pi}{12}))^{-1}h+\varepsilon}{\varepsilon}|\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k}))-\alpha_j|\,\mathrm{d}h\\
&\le C\int_0^{\tilde r_\varepsilon}\left(\frac{h^2}{\varepsilon^2}+\frac{h}{\varepsilon}\right)|(\theta_{j,k})'(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k}))|\,\mathrm{d}h\,,
\end{aligned}$$

where we integrated by parts and $C>0$ is a suitable constant varying from line to line. Once again since $\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})$ is a homogeneous and increasing function with respect to h, recalling the decay of $|(\theta_{j,k})'(s)|$ in s mentioned above, then we get

$$\int_{W^\rho_j\setminus W^\varepsilon_j}\frac{1}{\varepsilon}|\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k}))-\alpha_j|\,\mathrm{d}x \le C\varepsilon\,,$$

and also

$$\int_{W^\rho_j\setminus W^\varepsilon_j}\frac{1}{\varepsilon}|\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j}))-\alpha_j|\,\mathrm{d}x \le C\varepsilon\,.$$

Similarly, we estimate

$$\begin{aligned}
&\int_{W^\rho_j\setminus W^\varepsilon_j}\frac{1}{\varepsilon}W(\theta_{j,k}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})))\,\mathrm{d}x\\
&\le C\int_0^{\tilde r_\varepsilon}\frac{\tilde h_\varepsilon}{\varepsilon}W(\theta_{j,k}(-\varepsilon^{-1}\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})))\,\mathrm{d}h \le C\varepsilon\,,
\end{aligned}$$

due to the fact that $\operatorname{dist}(P^s_{R^-_{j,k}}x, I_{j,k})$ is a homogeneous and increasing function with respect to h and the decay of $W(\theta_{j,k}(s))$ in s. Analogously, we have

$$\int_{M_j}\frac{1}{\varepsilon}W(\theta_{i,j}(\varepsilon^{-1}\operatorname{dist}(P^s_{R^+_{i,j}}x, I_{i,j})))\,\mathrm{d}x \le C\varepsilon\,.$$

Finally, (3.9) together with (A4) (see Sec. 3.2) and an application of the Young inequality give

$$\left|\sum_{\ell=1}^N\int_{W^\rho_j\setminus W^\varepsilon_j}\xi_\ell\cdot\nabla(\psi_\ell\circ u_{\varepsilon,0})\,\mathrm{d}x\right| \le C\int_{W^\rho_j\setminus W^\varepsilon_j}\sqrt{2W(u_{\varepsilon,0})}|\nabla u_{\varepsilon,0}|\,\mathrm{d}x$$

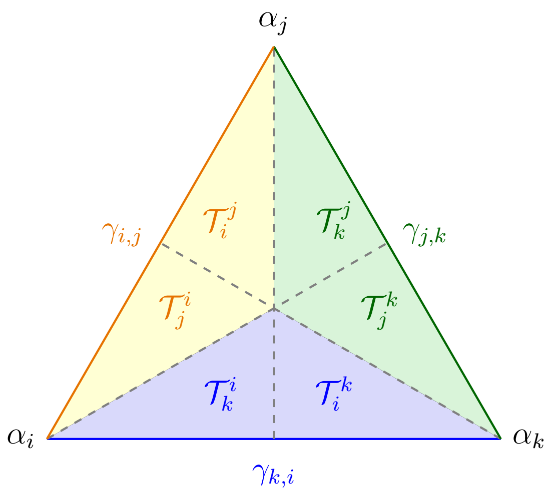


Figure 3.3: Illustration of the disjoint partition $\{\mathcal{T}_j^i\}_{i,j:i\neq j}$ of the simplex $\triangle^2$.

$$\leq C\left[\int_{W_j^\rho\setminus W_j^\varepsilon}\frac{\varepsilon}{2}|\nabla u_{\varepsilon,0}|^2\,\mathrm{d}x+\int_{W_j^\rho\setminus W_j^\varepsilon}\frac{1}{\varepsilon}W(u_{\varepsilon,0})\,\mathrm{d}x\right].$$

Then, from the estimates above we can conclude that

$$E_{W_j^\rho\setminus W_j^\varepsilon}[u_\varepsilon|\xi](0):=\int_{W_j^\rho\setminus W_j^\varepsilon}\frac{\varepsilon}{2}|\nabla u_{\varepsilon,0}|^2+\frac{1}{\varepsilon}W(u_{\varepsilon,0})+\sum_{\ell=1}^N\xi_\ell\cdot\nabla(\psi_\ell\circ u_{\varepsilon,0})\,\mathrm{d}x\leq C\varepsilon$$

for some constant $C>0$.

3.6 Suitable multi-well potentials and a construction for the ψ_i

We next proceed to show that the class of N-well potentials satisfying assumptions (A1)–(A4) is in fact sufficiently broad.

3.6.1 A class of multi-well potentials

Let $\triangle^{N-1}$ be a $(N-1)$-simplex with edges of unit length in $\mathbb{R}^{N-1}$. We denote by $\{\gamma_{i,j}\}_{i,j:i\neq j}$ its edges and by $\{\alpha_i\}_i$ its N vertices, so that $|\alpha_i-\alpha_j|=1$ for any mutually distinct $i,j\in\{1,...,N\}$. We can decompose $\triangle^{N-1}$ (almost) symmetrically into a disjoint partition $\{\mathcal{T}_{i,j}\}_{i,j:i<j}$ such that each point $x\in\triangle^{N-1}$ is assigned to the set $\mathcal{T}_{i,j}$ if $\gamma_{i,j}$ is the edge of $\triangle^{N-1}$ that is closest to x, with x being assigned to the edge with the lowest i and j in case of ties. Each $\mathcal{T}_{i,j}$ can be further split nearly symmetrically into $\mathcal{T}_j^i$ and $\mathcal{T}_i^j$ by defining $\mathcal{T}_j^i$ to consist of the points in $\mathcal{T}_{i,j}$ that are closer to α_i than to α_j. For an illustration of this partition, we refer to Figure 3.3 for $N=3$.

For the purpose of our construction of the ψ_i from condition (A4), we introduce some further notation:

- For $i\in\{1,...,N\}$, we denote by $\mathcal{U}_i:=\overline{B_{r_\mathcal{U}}(\alpha_i)}$ a ball around the vertex α_i with radius $r_\mathcal{U}\in\left(0,\frac{1}{4}\right]$.

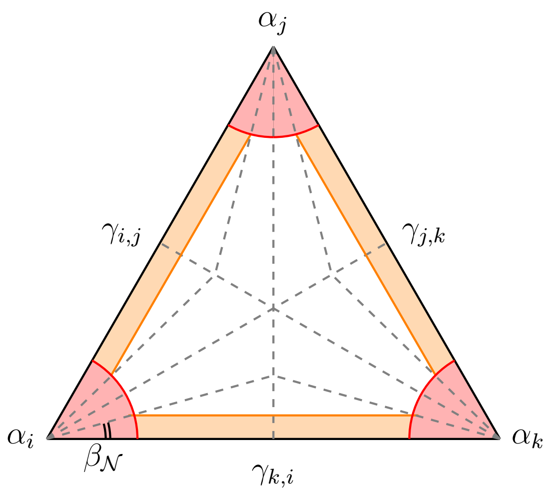


Figure 3.4: Illustration of the partition of the simplex $\triangle^2$ given by $\{\mathcal{U}_i \cap \triangle^2\}_i$, $\{(\mathcal{N}_{i,j} \setminus \mathcal{U}_i) \cap \triangle^2\}_{i<j}$, and $\triangle^2 \setminus \mathcal{N}_{i,j}$.

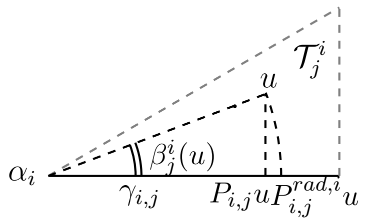


Figure 3.5: Projections of $u \in \mathcal{T}_j^i$ onto $\gamma_{i,j} \subset \triangle^2$.

- For $i < j$, $i, j \in \{1, ..., N\}$, we denote by $\mathcal{N}_{i,j} := \{u \in \mathbb{R}^{N-1} : \operatorname{dist}(u, \gamma_{i,j}) \leq r_{\mathcal{U}} \sin(\beta_{\mathcal{N}})\}$ a neighborhood of the edge $\gamma_{i,j}$. Here, $\beta_{\mathcal{N}} \in (0, \frac{\pi}{12(N-2)}]$ is a fixed positive angle.

For a depiction of the resulting partition in the case $N = 3$, we refer to Figure 3.4.

We furthermore make use of a couple of additional abbreviations.

- For any $i < j$, $i, j \in \{1, ..., N\}$, we denote by $P_{i,j} : \mathcal{T}_{i,j} \to \gamma_{i,j}$ the standard orthogonal projection onto $\gamma_{i,j}$, i. e. the projection onto the nearest point on $\gamma_{i,j}$.
- We denote by $P_{i,j}^{rad,i} : \mathcal{T}_{i,j} \to \gamma_{i,j}$ the radial projection onto $\gamma_{i,j}$ with respect to α_i, i.e. $P_{i,j}^{rad,i} u$ denotes the point on $\gamma_{i,j}$ with $|P_{i,j}^{rad,i} u - \alpha_i| = |u - \alpha_i|$.
- For any $u \in \mathcal{T}_{i,j}$, we denote by $\beta_j^i(u)$ the angle formed by $u - \alpha_i$ and $\gamma_{i,j}$.

For an illustration of these notions, we refer to Figure 3.5 (again in the case $N = 3$).

Definition 45 (Strongly coercive N-well potential on the simplex)**.** *We call a function* $W : \triangle^{N-1} \to [0, \infty)$ a strongly coercive symmetric N-well potential on the simplex *if it satisfies the following list of properties:*

1. *The nonnegative function* $W \in C^{1,1}(\triangle^{N-1};[0,\infty))$ *vanishes exactly in the* N *vertices* $\{\alpha_1,...,\alpha_N\}$ *of the simplex* $\triangle^{N-1}$. *It furthermore has the same symmetry properties as the simplex* $\triangle^{N-1}$.

2. *Given the geodesic distance*

$$\operatorname{dist}_W(v,w) := \inf\Big\{ \int_0^1 \sqrt{2W(\gamma(s))}|\gamma'(s)|\, \mathrm{d}s : \gamma \in C^1([0,1];\mathbb{R}^2) \quad \text{with } \gamma(0)=v, \gamma(1)=w \Big\}, \tag{3.41}$$

the infimum for $\operatorname{dist}_W(\alpha_i,\alpha_j)$ *is achieved by* $\gamma_{i,j}$ *and* $\operatorname{dist}_W(\alpha_i,\alpha_j)=1$ *for any* $i,j \in \{1,...,N\}$, $i \neq j$.

3. *(Growth near the minima* α_i *depending on the angle.) For any distinct* $i,j \in \{1,...,N\}$ *and any* $u \in \mathcal{U}_i \cap \mathcal{T}_{i,j}$, *we have the estimate*

$$(1+\omega(\beta^i_j(u)))W(P^{rad,i}_{i,j}u) \le W(u), \tag{3.42}$$

where $\omega : [0, \frac{\pi}{6(N-2)}] \to [0,\infty)$ *is a* C^1 *increasing function such that*

$$\omega(\beta) = 0 \quad \text{for } \beta = 0\,, \tag{3.43a}$$
$$\omega(\beta) \ge 0 \quad \text{for } \beta \in (0,\beta_{\mathcal{N}}), \tag{3.43b}$$
$$\omega(\beta) > C_\omega \quad \text{for } \beta \in [\beta_{\mathcal{N}}, \tfrac{\pi}{6(N-2)}], \tag{3.43c}$$

where $C_\omega > 0$ *is a suitable large constant depending on* N *and where* $\beta_{\mathcal{N}} \le \frac{\pi}{12(N-2)}$.

4. *(Growth properties of* W *and Lipschitz estimate for* $\sqrt{2W(u)}$ *on the edges* $\gamma_{i,j}$.*) There exist constants* $c_\gamma, C_\gamma > 0$ *such that*

$$c_\gamma(u-\alpha_i)^2(u-\alpha_j)^2 \le W(u) \le C_\gamma(u-\alpha_i)^2(u-\alpha_j)^2, \tag{3.44}$$

holds for all $u \in \gamma_{i,j}$ *and any distinct* $i,j \in \{1,...,N\}$. *Furthermore, there exists a constant* $L_\gamma > 0$ *such that for any* $u_1,u_2 \in \gamma_{i,j}$

$$|\sqrt{2W(u_1)} - \sqrt{2W(u_2)}| \le L_\gamma|u_1-u_2|\,. \tag{3.45}$$

5. *(Growth behavior as one leaves the shortest paths* $\gamma_{i,j}$.*) For any distinct* $i,j \in \{1,...,N\}$ *and any* $u \in \mathcal{T}_{i,j} \cap (\mathcal{N}_{i,j} \setminus (\mathcal{U}_i \cup \mathcal{U}_j))$, *the lower bound*

$$(1+C_{\mathcal{N}}\operatorname{dist}^2(u,\gamma_{i,j}))W(P_{i,j}u) \le W(u) \tag{3.46}$$

holds, where $C_{\mathcal{N}} > 0$ *is a suitable large constant depending on* $r_{\mathcal{U}}, \beta_{\mathcal{N}}, L_\gamma, c_\gamma$.

6. *(Lower bound away from the paths* $\gamma_{i,j}$.*) For any* $u \in \triangle \setminus (\cup_i \mathcal{U}_i \cup_{i<j} \mathcal{N}_{i,j})$

$$\max_{v \in \cup_{i<j}\gamma_{i,j}} W(v) \le \frac{1}{C_{\mathit{int}}} W(u)\,, \tag{3.47}$$

where $C_{\mathit{int}} > 0$ *is a suitable large constant depending on* $N, r_{\mathcal{U}}, \beta_{\mathcal{N}}, c_\gamma, C_\gamma$.

3.6.2 Construction of the approximate phase indicator functions ψ_i

In this subsection we provide an ansatz for the set of functions $\psi_i : \triangle^{N-1} \to [0,1]$, $1 \leq i \leq N$, in the case of a strongly coercive N-well potential on the simplex $W : \triangle^{N-1} \to [0,\infty)$. Recall that the goal is to construct the ψ_i to satisfy condition (A4) (as introduced Section 3.2).

Since $(\psi_1 \circ u_\varepsilon, ..., \psi_N \circ u_\varepsilon)$ represents an approximation of the limit partition $(\bar{\chi}_1, ..., \bar{\chi}_N)$ and since by assumption we have $\mathrm{dist}_W(\alpha_i, \alpha_j) = \delta_{ij}$, our ansatz for ψ_i on the edges $\gamma_{i,j}$ is

$$\psi_i(u_\varepsilon) := \begin{cases} 1 - \mathrm{dist}_W(\alpha_i, u_\varepsilon) \text{ along } \gamma_{i,j} & \text{for } j \in \{1,...,N\} \setminus \{i\}\,, \\ 0 \quad \text{along } \gamma_{j,k} \text{ for } j,k \in \{1,...,N\} \setminus \{i\} : j < k\,. \end{cases} \tag{3.48}$$

In the following we extend this definition of the set of functions ψ_i on the domain $\triangle^{N-1}$. In order to do this, we introduce three interpolation and/or cutoff functions.

Lemma 46 (Interpolation functions)**.** *Let $\beta_{\mathcal{N}} \in (0, \frac{\pi}{12(N-2)}]$ and $r_{\mathcal{U}} \in (0, \frac{1}{4}]$. The following statements hold:*

1. *There exists a function $\lambda : [0, \frac{\pi}{6}] \to [0,1]$ of at least C^1 regularity satisfying the properties $\lambda(\beta) = 0$ and $\partial_\beta \lambda = 0$ for all $\beta \in [0, \beta_{\mathcal{N}}]$, $\lambda(\frac{\pi}{6(N-2)}) = 1$, and*
$$\max_{\beta \in \left[0, \frac{\pi}{6(N-2)}\right]} |\partial_\beta \lambda| \leq 4(N-2)\,. \tag{3.49}$$

2. *There exists a function $\eta : [0,1] \to [0,1]$ of at least C^1 regularity satisfying the properties: $\eta(s) = 1$ for $s \in [0, r_{\mathcal{U}}]$, $\eta(s) = 0$ for $s \in [1 - r_{\mathcal{U}}, 1]$, $\eta(s) + \eta(1-s) = 1$ for all $s \in [0,1]$, and*
$$\max_{s \in [0,1]} |\partial_s \eta| \leq \tfrac{5}{2}\,. \tag{3.50}$$

We omit the proof of the lemma, as it is straightforward. We finally proceed to the construction of the functions ψ_i from (A4) in the case of a strongly coercive N-well potential on the simplex.

Construction 47. *Let $W : \triangle^{N-1} \to [0,\infty)$ be a strongly coercive symmetric N-well potential on the simplex in the sense of Definition 45. We define the associated set of functions $\psi_i : \triangle \to [0,1]$, $1 \leq i \leq N$, as it follows. For $i \in \{1,...,N\}$, we construct ψ_i on the edge between α_i and α_j ($j \in \{1,...,N\} \setminus \{i\}$) by*

$$\psi_i(u) := 1 - \mathrm{dist}_W(\alpha_i, u) \quad \text{for } u \in \gamma_{i,j}. \tag{3.51}$$

Let $j \in \{1,...,N\} \setminus \{i\}$. For any $u \in \mathcal{T}^i_j$, we set

$$\psi_k(u) := 0 \quad \text{for any } k \in \{1,...,N\} \setminus \{i,j\}.$$

Furthermore, we define ψ_i and ψ_j on $\mathcal{T}^i_j \cap (\mathcal{N}_{i,j} \cup \mathcal{U}_i)$ as follows:

- *If $u \in \mathcal{U}_i \cap \mathcal{T}^i_j$, we set*

$$\psi_i(u) := \psi_i(P^{rad,i}_{i,j} u)\,, \tag{3.52a}$$
$$\psi_j(u) := (1 - \lambda(\beta^i_j(u)))\psi_j(P^{rad,i}_{i,j} u)\,. \tag{3.52b}$$

- *If $u \in (\mathcal{N}_{i,j} \setminus \mathcal{U}_i) \cap \mathcal{T}^i_j$, we set*

$$\psi_i(u) := \eta(|P_{i,j}u - \alpha_i|)\psi_i(P^{rad,i}_{i,j}u) + \eta(|P_{i,j}u - \alpha_j|)\psi_i(P^{rad,j}_{i,j}u)\,,$$
$$\psi_j(u) := \eta(|P_{i,j}u - \alpha_i|)\psi_j(P^{rad,i}_{i,j}u) + \eta(|P_{i,j}u - \alpha_j|)\psi_j(P^{rad,j}_{i,j}u). \tag{3.53a}$$

Finally, outside of the domain $\mathcal{M}_i := \bigcup_j \mathcal{U}_j \cup \bigcup_{j:j\neq i} \mathcal{N}_{i,j} \cup \bigcup_{j,k:j,k\neq i,j<k} \mathcal{T}_{j,k}$ on which we have defined ψ_i so far, we define ψ_i as a suitable $C^{1,1}$ extension:

- *If $u \in \triangle^{N-1} \setminus \mathcal{M}_i$, we define*

$$\psi_i(u_\varepsilon) := \psi_i^{int}(u) \tag{3.54a}$$

where $\psi_i^{int} : \triangle^{N-1} \to [0,1]$ is a suitable $C^{1,1}$ extension of $\psi_i : \mathcal{M}_i \cap \triangle^{N-1} \to [0,1]$ that almost preserves the Lipschitz constant of $\psi_i : \mathcal{M}_i \cap \triangle^{N-1} \to [0,1]$.

3.6.3 Existence of a set of suitable approximate phase indicator functions

Proof of Proposition 36. It directly follows from Construction 47 that the set of functions $\psi_i : \triangle \to [0,1]$, $1 \le i \le N$, satisfy $\psi_i = 1$ at α_i and $\psi_i(u) < 1$ for $u \neq \alpha_i$.

We next show the validity of (A4) in a given set $\mathcal{T}^i_j$, which we further decompose into $\mathcal{U}_i \cap \mathcal{T}^i_j$, $(\mathcal{N}_{i,j} \setminus \mathcal{U}_i) \cap \mathcal{T}^i_j$, and $\mathcal{T}^i_j \setminus (\mathcal{U}_i \cup \mathcal{N}_{i,j})$.

Step 1: Proof of (A4) in $\mathcal{U}_i \cap \mathcal{T}^i_j$. Let $u \in \mathcal{U}_i \cap \mathcal{T}^i_j$. Recall $\psi_0 := 1 - \sum_{\ell=1}^N \psi_\ell$. Due to $\mathrm{dist}_W(\alpha_i, \alpha_j) = 1$ and (3.51), it follows that $\psi_i(P^{rad,i}_{i,j}u) = 1 - \psi_j(P^{rad,i}_{i,j}u)$. Thus, we have $\psi_0(u) = \lambda(\beta^i_j(u))\psi_j(P^{rad,i}_{i,j}u)$. We also have $(\frac{\alpha_j-\alpha_i}{|\alpha_j-\alpha_i|} \cdot \nabla)\psi_i(P^{rad,i}_{i,j}u) = \sqrt{2W(P^{rad,i}_{i,j}u)}$. Using (3.52), we can compute

$$\begin{aligned}
\partial_u\psi_{i,j}(u) &= (2 - \lambda(\beta^i_j(u)))\sqrt{2W(P^{rad,i}_{i,j}u)}\mathrm{e}_{rad,i}(u) \\
&\quad - \partial_\beta\lambda(\beta^i_j(u))\frac{1}{|u-\alpha_i|}\psi_j(P^{rad,i}_{i,j}u)\mathrm{e}_{\beta^i_j}(u)\,, \\
\partial_u\psi_0(u) &= \lambda(\beta^i_j(u))\sqrt{2W(P^{rad,i}_{i,j}u)}\mathrm{e}_{rad,i}(u) \\
&\quad + \partial_\beta\lambda(\beta^i_j(u))\frac{1}{|u-\alpha_i|}\psi_j(P^{rad,i}_{i,j}u)\mathrm{e}_{\beta^i_j}(u),
\end{aligned}$$

where $\mathrm{e}_{rad,i}(u), \mathrm{e}_{\beta^i_j}(u)$ are orthogonal vectors associated to the $(N-1)$-dimensional spherical coordinates pointing in the direction of steepest ascent of $|u-\alpha_i|$ respectively $\beta^i_j(u)$; i. e. , in particular we have $\mathrm{e}_{rad,i}(u) := \frac{u-\alpha_i}{|u-\alpha_i|}$. For the sake of brevity, we omit the dependencies on $\beta^i_j(u)$ in the following. Then, it follows that

$$\begin{aligned}
|\partial_u\psi_{i,j}(u)|^2 &\le \Big((2-\lambda)^2 + |\partial_\beta\lambda|^2\Big)\,2W(P^{rad,i}_{i,j}u)\,, \\
|\partial_u\psi_0(u)|^2 &\le \Big(\lambda^2 + |\partial_\beta\lambda|^2\Big)2W(P^{rad,i}_{i,j}u)\,, \\
|\partial_u\psi_{i,j}(u)\cdot\partial_u\psi_0(u)| &\le \Big(\lambda(2-\lambda) + |\partial_\beta\lambda|^2\Big)\,2W(P^{rad,i}_{i,j}u)\,.
\end{aligned}$$

For $\delta > 0$ small enough, we have

$$\left|\tfrac{1}{2}\partial_u\psi_{i,j}(u)\right|^2 + \left(\tfrac{5}{4} + \delta\right)\left|\tfrac{1}{2}\partial_u\psi_0(u)\right|^2 + \delta\,|\partial_u\psi_{i,j}(u)\cdot\partial_u\psi_0(u)|$$

$$
\begin{aligned}
&\leq \Big[\tfrac{1}{4}(2-\lambda)^2 + \tfrac{1}{4}|\partial_\beta \lambda|^2 + (\tfrac{5}{4}+\delta)\tfrac{1}{4}\Big(\lambda^2 + |\partial_\beta\lambda|^2\Big) \\
&\qquad + \delta\left(\lambda(2-\lambda) + |\partial_\beta\lambda|^2\right)\Big] 2W(P^{rad,i}_{i,j}u) \\
&\leq \Big[1 - \lambda + \tfrac{9}{16}\lambda^2 + \tfrac{9}{16}|\partial_\beta\lambda|^2 + C\delta\Big] 2W(P^{rad,i}_{i,j}u) \\
&\leq (1+\omega(\beta^i_j(u)))2W(P^{rad,i}_{i,j}u) \\
&\leq 2W(u)\,,
\end{aligned}
$$

where we used (3.49) and (3.43) together with the fact that δ can be chosen arbitrarily small.

Step 2: Proof of (A4) in $u \in (\mathcal{N}_{i,j}\setminus\mathcal{U}_i)\cap\mathcal{T}^i_j$. Let $u \in (\mathcal{N}_{i,j}\setminus\mathcal{U}_i)\cap\mathcal{T}^i_j$. First, note that $\psi_0 = 1-\psi_i-\psi_j \equiv 0$ on $(\mathcal{N}_{i,j}\setminus\mathcal{U}_i)\cap\mathcal{T}^i_j$. Using (3.53), we compute

$$
\begin{aligned}
&\partial_u\psi_{i,j}(u) \\
&= \eta(|P_{i,j}u-\alpha_i|)2\sqrt{2W(P^{rad,i}_{i,j}u)}\mathrm{e}_{rad,i}(u) - \eta(|P_{i,j}u-\alpha_j|)2\sqrt{2W(P^{rad,j}_{i,j}u)}\mathrm{e}_{rad,j}(u) \\
&\quad + \partial_u\eta(|P_{i,j}u-\alpha_i|)\left[\psi_j(P^{rad,i}_{i,j}u) - \psi_j(P^{rad,j}_{i,j}u) + \psi_i(P^{rad,j}_{i,j}u) - \psi_i(P^{rad,i}_{i,j}u)\right]\,,
\end{aligned}
$$

where $\mathrm{e}_{rad,i}(u) := \frac{u-\alpha_i}{|u-\alpha_i|}$ and $\mathrm{e}_{rad,j}(u) := \frac{u-\alpha_j}{|u-\alpha_j|}$. Note that we have

$$
|P^{rad,i}_{i,j}u-\alpha_i| \leq |P_{i,j}u-\alpha_i| + \frac{\mathrm{dist}^2(u,\gamma_{i,j})}{2|P_{i,j}u-\alpha_i|}\,, \tag{3.55}
$$

$$
|P^{rad,i}_{i,j}u - P^{rad,j}_{i,j}u| \leq \frac{\mathrm{dist}^2(u,\gamma_{i,j})}{2|P_{i,j}u-\alpha_i||P_{i,j}u-\alpha_j|}\,. \tag{3.56}
$$

As a consequence, we obtain

$$
\begin{aligned}
&\max\{\psi_j(P^{rad,i}_{i,j}u) - \psi_j(P^{rad,j}_{i,j}u), \psi_i(P^{rad,j}_{i,j}u) - \psi_i(P^{rad,i}_{i,j}u)\} \\
&\leq \sqrt{2W(v_u)}\frac{\mathrm{dist}^2(u,\gamma_{i,j})}{2|P_{i,j}u-\alpha_i||P_{i,j}u-\alpha_j|}\,,
\end{aligned}
$$

where $v_u \in \gamma_{i,j}$ maximum of $\sqrt{2W}$ on the segment connecting $P^{rad,j}_{i,j}u$ to $P^{rad,i}_{i,j}u$. From (3.50) and $\eta(|P_{i,j}u-\alpha_i|)+\eta(|P_{i,j}u-\alpha_j|)=1$ it follows that

$$
\left|\tfrac{1}{2}\partial_u\psi_{i,j}(u)\right|^2 \leq \left(1 + \frac{5}{4|P_{i,j}u-\alpha_i||P_{i,j}u-\alpha_j|}\,\mathrm{dist}^2(u,\gamma_{i,j})\right)^2 2W(v_u)\,. \tag{3.57}
$$

Using that $1-|P_{i,j}u-\alpha_i| = |P_{i,j}u-\alpha_j|$ and $|P_{i,j}u-\alpha_i|^{-1}(1-|P_{i,j}u-\alpha_i|)^{-1} \leq r_{\mathcal{U}}^{-1}(1-r_{\mathcal{U}})^{-1}$, one can obtain

$$
\begin{aligned}
&\left(1 + \frac{5}{4|P_{i,j}u-\alpha_i||P_{i,j}u_\varepsilon-\alpha_j|}\,\mathrm{dist}^2(u,\gamma_{i,j})\right)^2 \\
&\leq 1 + C_1\,\mathrm{dist}^2(u,\gamma_{i,j})
\end{aligned}
$$

for $C_1 = \frac{5}{2r_{\mathcal{U}}(1-r_{\mathcal{U}})} + \frac{25\sin^2(\beta_{\mathcal{N}})}{16(1-r_{\mathcal{U}})^2}$ since $\mathrm{dist}(u,\gamma_{i,j}) \leq r_{\mathcal{U}}\sin(\beta_{\mathcal{N}})$. On the other hand, first by adding a zero and using (3.45), then noting that $|v_u-\alpha_i| \leq |P^{rad,i}_{i,j}u-\alpha_i|$ and using (3.56), from (3.44) together with the fact that $|P_{i,j}u-\alpha_i| \geq \frac{1}{2}$ we can deduce

$$
2W(v_u) = 2W(P_{i,j}u)\left(1 + \frac{\sqrt{2W(v_u)}-\sqrt{2W(P_{i,j}u)}}{\sqrt{2W(P_{i,j}u)}}\right)^2
$$

$$
\begin{aligned}
&\leq 2W(P_{i,j}u)\left(1+\frac{L_\gamma}{2|P_{i,j}u-\alpha_i|\sqrt{2W(P_{i,j}u)}}\operatorname{dist}^2(u,\gamma_{i,j})\right)^2\\
&\leq 2W(P_{i,j}u)\left(1+\frac{L_\gamma}{\sqrt{2c_\gamma}}\tan^2(\beta^i_j(u))\right)^2\\
&\leq 2W(P_{i,j}u)\left(1+C_2\tan^2(\beta^i_j(u))\right)\,,
\end{aligned}
$$

where $C_2 = 2\frac{L_\gamma}{\sqrt{2c_\gamma}} + \tan^2(\beta_{\mathcal{N}})\frac{L_\gamma^2}{2c_\gamma}$. Moreover, one can compute

$$
\begin{aligned}
&\left(1+C_1\operatorname{dist}^2(u,\gamma_{i,j})\right)\left(1+C_2\tan^2(\beta^i_j(u))\right)\\
&\leq 1+C_1\operatorname{dist}^2(u,\gamma_{i,j})+\frac{C_2}{r_{\mathcal{U}}^2\cos^2\beta_{\mathcal{N}}}\operatorname{dist}^2(u,\gamma_{i,j})+C_1C_2\tan^2(\beta_{\mathcal{N}})\operatorname{dist}^2(u,\gamma_{i,j})\\
&\leq 1+C_{\mathcal{N}}\operatorname{dist}^2(u,\gamma_{i,j})
\end{aligned}
$$

for $C_{\mathcal{N}} = C_1 + \frac{C_2}{r_{\mathcal{U}}^2\cos^2\beta_{\mathcal{N}}} + C_1C_2\tan^2(\beta_{\mathcal{N}})$. Using our assumption (3.46), we can conclude from (3.57) and the preceding three estimates that

$$
\begin{aligned}
\left|\tfrac{1}{2}\partial_u\psi_{i,j}(u)\right|^2 &\leq \left(1+C_{\mathcal{N}}\operatorname{dist}^2(u,\gamma_{i,j})\right)2W(P_{i,j}u)\\
&\leq 2W(u)\,.
\end{aligned}
$$

Step 3: Proof of (A4) in $u\in\mathcal{T}^i_j\setminus\mathcal{M}_i$. Let $u\in\mathcal{T}^i_j\setminus\mathcal{M}_i$. By (3.54) we have

$$
\begin{aligned}
\psi_{i,j}(u_\varepsilon) &= \psi_j^{\mathsf{int}}(u)-\psi_i^{\mathsf{int}}(u)\,,\\
\psi_0(u_\varepsilon) &= 1-\psi_i^{\mathsf{int}}(u)-\psi_j^{\mathsf{int}}(u)\,,
\end{aligned}
$$

where ψ_ℓ^{int}, $\ell\in\{i,j\}$, is a $C^{1,1}$ extension of ψ_ℓ from $\mathcal{M}_\ell\cap\triangle^{N-1}$ to $\triangle^{N-1}$ that approximately preserves the Lipschitz constant $L_{\mathsf{int},\ell}>0$. Thus, we have

$$
|\partial_u\psi_{i,j}|\leq(1+\delta)(L_{\mathsf{int},i}+L_{\mathsf{int},j})\,,\quad\text{and similarly}\quad|\partial_u\psi_0|\leq(1+\delta)(L_{\mathsf{int},i}+L_{\mathsf{int},j})\,,
$$

where $\delta>0$ arbitrary small constant. It is not too difficult to derive an estimate on the Lipschitz constants $L_{\mathsf{int},i}$ and $L_{\mathsf{int},j}$ in terms of $\max_{w\in\cup_{\ell,m:\ell<m}\gamma_{\ell,m}}2W(w)$. To this aim, we first estimate

$$
\begin{aligned}
&|\psi_i(u)|\leq\max_{w\in\gamma_{i,m}}\sqrt{2W(w)}r_{\mathcal{U}}\quad\text{for } u\in\gamma_{i,m}\cap\mathcal{U}_m\text{ and } m\neq i,\\
&|\psi_i(P^{rad,i}_{i,m}u)-\psi_\ell(P^{rad,m}_{i,m}u)|\leq\max_{w\in\gamma_{i,m}}\sqrt{2W(w)}|P^{rad,i}_{i,m}u-P^{rad,m}_{i,m}u|\\
&\qquad\overset{(3.56)}{\leq}\max_{w\in\gamma_{i,m}}\sqrt{2W(w)}\frac{r_{\mathcal{U}}}{2(1-r_{\mathcal{U}})}\sin^2(\beta_{\mathcal{N}})\quad\text{for } u\in\mathcal{N}_{i,m}\text{ and } m\neq i.
\end{aligned}
$$

Using these estimates, the definitions (3.52)-(3.53), and the bounds (3.49) and (3.50), we obtain

$$
\begin{aligned}
&|\partial_u\psi_i(u)|\leq\max_{w\in\gamma_{i,j}}\sqrt{2W(w)}\quad\text{for } u\in\mathcal{U}_i\,,\\
&|\partial_u\psi_i(u)|\leq\left(1+4(N-2)r_{\mathcal{U}}\right)\max_{w\in\gamma_{i,m}}\sqrt{2W(w)}\quad\text{for } u\in\mathcal{U}_m, m\neq i\,,
\end{aligned}
$$

$$|\partial_u \psi_i(u)| \leq \left(1 + \frac{5r_{\mathcal{U}}}{4(1-r_{\mathcal{U}})}\sin^2(\beta_{\mathcal{N}})\right) \max_{w\in\gamma_{i,m}} \sqrt{2W(w)} \text{ for } u \in \mathcal{N}_{i,m},\, m \neq i.$$

Furthermore, we have $|\partial_u \psi_i(u)| = 0$ in $\triangle^{N-1} \cap (\cup_{m<n:m\neq i,n\neq i} \mathcal{T}_{m,n})$. Defining

$$M := \max\left\{1, 1+4(N-2)r_{\mathcal{U}}, 1 + \frac{5r_{\mathcal{U}}}{4(1-r_{\mathcal{U}})}\sin^2(\beta_{\mathcal{N}})\right\} = 1 + 4(N-2)r_{\mathcal{U}}\,,$$

this yields by (3.47) for $u \in \mathcal{M}_i \cap \triangle^{N-1}$

$$|\partial_u \psi_i(u)| \leq M \max_{w\in\bigcup_{\ell,m:\ell<m}\gamma_{\ell,m}} \sqrt{2W(w)} =: M_W$$

for any $u \in \triangle^{N-1} \cap \mathcal{M}_i$. In order to estimate the Lipschitz constant $L_{\text{int,i}}$ of $\psi_i|_{\mathcal{M}_i \cap \triangle^{N-1}}$, one has to address the issue of nonconvexity of $\mathcal{M}_i$. It is not too difficult to see (but rather technical to prove) that for any pair of points $u, v \in \mathcal{M}_i$ there exists a connecting path $\tilde{\gamma}$ in $\mathcal{M}_i$ with len$(\tilde{\gamma}) \leq C_{\mathcal{M}}|u-v|$. This shows $L_{\text{int},i} \leq C_{\mathcal{M}} M_W$. Having an upper bound for $L_{\text{int},i}$, using the fact that our extension of ψ_ℓ to $\triangle^{N-1} \setminus \mathcal{M}_i$ approximately preserves the Lipschitz constant, and choosing $C_{\text{int}} > \frac{9}{4} C_{\mathcal{M}}^2 M^2$ in (3.47), we can compute for $u \in \mathcal{T}_j^i \setminus \mathcal{M}_i$

$$\begin{aligned}&\left|\tfrac{1}{2}\partial_u \psi_{i,j}(u)\right|^2 + \left(\tfrac{5}{4} + \delta\right)\left|\tfrac{1}{2}\partial_u \psi_0(u)\right|^2 + \delta\,|\partial_u \psi_{i,j}(u) \cdot \partial_u \psi_0(u)| \\ &\leq (1+\delta)^2\left(1 + \tfrac{5}{4} + 5\delta\right)\max_m L^2_{\text{int},m} \leq 2W(u).\end{aligned}$$

Here, we have used the fact that $\delta > 0$ can be chosen arbitrarily small. □

3.7 Additional length condition for the gradient flow calibration

In this last section we justify the validity of the condition (3.4i) for the gradient flow calibration constructed in [46] (cf. Definition 34). To this aim, first we show that (3.4i) holds on the network of interfaces, then we motivate the extension of the property (3.4i) to $\mathbb{R}^d$.

The global gradient flow calibration for a network is obtained by gluing together suitable local constructions at each topological feature, i.e., a two phase interface or a triple junction (for more details see [46]). More precisely, a partition of unity is defined in order to localize around each topological feature, and then the global vector fields are defined by gluing together locally constructed vector fields. In the following, we denote by $\xi_{i,j}^{I_{ij}}$ resp. $\xi_\ell^{\mathcal{T}_{ijk}}$ the local construction of the gradient flow calibration in a neighborhood of a single connected component of I_{ij} resp. of the triple junction $\mathcal{T}_{i,j,k}$ (cf. [46, Sec. 5-6]).

As a starting point, we recall some useful properties of the gradient flow calibration. For any distinct $i, j \in \{1, ..., N\}$, we have

$$\xi_{i,j} = \xi_{i,j}^{I_{ij}} = \xi_i^{I_{ij}} - \xi_j^{I_{ij}} = \mathrm{n}_{i,j}, \qquad \xi_i^{I_{ij}} = \tfrac{1}{2}\xi_{i,j}^{I_{ij}},$$

and $\xi_k = \xi_k^{I_{ij}}$ coincides with the null vector for any $k \in \{1, ..., N\} \setminus \{i, j\}$ on $I_{i,j}$ outside of a neighborhood of the triple junctions where $I_{i,j}$ ends. Let $\mathcal{T}_{i,j,k}$ be the triple junction where the phases i, j and k meet, for mutually distinct $i, j, k \in \{1, ..., N\}$. Then, at the triple junction $\mathcal{T}_{i,j,k}$

$$\xi_\ell = \xi_\ell^{\mathcal{T}_{ijk}}, \quad |\xi_\ell| = \tfrac{1}{\sqrt{3}}, \quad \xi_\ell \cdot \xi_m = -\tfrac{1}{6} \quad \text{for any distinct } \ell, m \in \{i, j, k\},$$

and $\xi_\ell = \xi_\ell^{\mathcal{T}_{ijk}}$ coincides with the null vector for any $\ell \in \{1, ..., N\} \setminus \{i, j, k\}$.

Let $i, j \in \{1, ..., N\}$ such that $i \neq j$. If we restrict to the network of interfaces, the vector field $\xi_{i,j}$ is nonzero only on each connected component of $I_{i,j}$, $I_{i,k}$, $I_{j,k}$ for any $k \in \{1, ..., N\} \setminus \{i, j\}$ and at the triple junctions where either the phase i or j meets other phases. Using the properties of the gradient flow calibration listed above, one can easily see that (3.4i) is satisfied at each point belonging to a connected component of $I_{i,j}$, $I_{i,k}$ or $I_{j,k}$ and outside of a neighborhood of the triple junctions where it ends, as well as at the triple junctions where either the phase i or j meets other phases. As a consequence, in order to conclude about (3.4i) on the network of interface, we only need to check its validity in a neighborhood of triple junctions where either the phase i or j meets other phases. In particular, it suffices to consider a neighborhood of $\mathcal{T}_{i,j,k}$ and a neighborhood of $\mathcal{T}_{i,k,k'}$ for some distinct $k, k' \in \{1, ..., N\} \setminus \{i, j\}$.

In a neighborhood of $\mathcal{T}_{i,j,k}$, one can write

$$\begin{aligned} \xi_{i,j} &= \eta \xi_{i,j}^{\mathcal{T}_{ijk}} + (1-\eta)\xi_{i,j}^{I_{ij}}, \quad \xi_k = \eta \xi_k^{\mathcal{T}_{ijk}} \quad \text{along } I_{i,j}, \\ \xi_{i,j} &= \eta \xi_{i,j}^{\mathcal{T}_{ijk}} + (1-\eta)\xi_{i,j}^{I_{ik}}, \quad \xi_k = \eta \xi_k^{\mathcal{T}_{ijk}} + (1-\eta)\xi_k^{I_{ik}} \quad \text{along } I_{i,k}, \end{aligned}$$

where $\eta : \mathbb{R}^d \to [0,1]$ is a cut-off fuction decreasing quadratically with the distance from $\mathcal{T}_{i,j,k}$ and vanishing outside of a neighborhood of $\mathcal{T}_{i,j,k}$. Note that $\xi_{i,j}^{\mathcal{T}_{ijk}} = \xi_{i,j}^{I_{ij}}$ along $I_{i,j}$ in a neighborhood of $\mathcal{T}_{i,j,k}$. It follows that $|\xi_{i,j}|^2 = 1$ and $\xi_{i,j} \cdot \xi_k = 0$ for $k \in \{1, ..., N\} \setminus \{i, j\}$ (cf. [46, Sec. 7.2]), hence (3.4i) holds along $I_{i,j}$ in a neighborhood of $\mathcal{T}_{i,j,k}$. As a next step, we compute $|\xi_{i,j}|^2$ and $|\sqrt{3}\xi_{i,j} \cdot \xi_k|^2$ for $k \in \{1, ..., N\} \setminus \{i, j\}$, and then show that (3.4i) with $\delta_{\mathsf{cal}} = 0$ holds along $I_{i,k}$ in a neighborhood of $\mathcal{T}_{i,j,k}$. Since $\xi_{i,j}^{I_{ik}} = \xi_i^{I_{ik}} = \frac{1}{2}\xi_{i,k}^{I_{ik}} = \frac{1}{2}\xi_{i,k}^{\mathcal{T}_{ijk}}$ and $\xi_k^{I_{ik}} = -\frac{1}{2}\xi_{i,k}^{I_{ik}} = -\frac{1}{2}\xi_{i,k}^{\mathcal{T}_{ijk}}$ along $I_{i,k}$ in a neighborhood of $\mathcal{T}_{i,j,k}$, then one can deduce

$$\begin{aligned} |\xi_{i,j}|^2 &= \eta^2 |\xi_{i,j}^{\mathcal{T}_{ijk}}|^2 + (1-\eta)^2 |\tfrac{1}{2}\xi_{i,k}^{\mathcal{T}_{ijk}}|^2 + \eta(1-\eta)\xi_{i,j}^{\mathcal{T}_{ijk}} \cdot \xi_{i,k}^{\mathcal{T}_{ijk}} \\ &= \eta^2 + (1-\eta)^2 \tfrac{1}{4} + \eta(1-\eta)\tfrac{1}{2}, \\ \xi_{i,j} \cdot \xi_k &= -\tfrac{1}{2}\eta(1-\eta)\xi_{i,j}^{\mathcal{T}_{ijk}} \cdot \xi_{i,k}^{\mathcal{T}_{ijk}} + \tfrac{1}{2}\eta(1-\eta)\xi_{i,k}^{\mathcal{T}_{ijk}} \cdot \xi_k^{\mathcal{T}_{ijk}} - \tfrac{1}{4}(1-\eta)^2 |\xi_{i,k}^{\mathcal{T}_{ijk}}|^2 \\ &= -\tfrac{1}{2}\eta(1-\eta) - \tfrac{1}{4}(1-\eta)^2, \end{aligned}$$

where we used $\xi_{i,j}^{\mathcal{T}_{ijk}} \cdot \xi_k^{\mathcal{T}_{ijk}} = 0$ in a neighborhood of $\mathcal{T}_{i,j,k}$ (for more details see [46, Sec. 7.2]). As a consequence, one can see that

$$|\xi_{i,j}|^2 + 4|\sqrt{3}\xi_{i,j} \cdot \xi_k|^2 \leq 1,$$

thus (3.4i) with $\delta_{\mathsf{cal}} = 0$ is satisfied along $I_{i,k}$ in a neighborhood of $\mathcal{T}_{i,j,k}$.

As a next step, we consider a neighborhood of $\mathcal{T}_{i,k,k'}$, where we can write

$$\begin{aligned} \xi_{i,j} &= \tilde{\eta}\xi_{i,j}^{\mathcal{T}_{ikk'}} + (1-\tilde{\eta})\xi_{i,j}^{I_{ik}}, \quad \xi_k = \tilde{\eta}\xi_k^{\mathcal{T}_{ikk'}} + (1-\tilde{\eta})\xi_k^{I_{ik}}, \quad \xi_{k'} = \tilde{\eta}\xi_{k'}^{\mathcal{T}_{ikk'}} \quad \text{along } I_{i,k}, \\ \xi_{i,j} &= \tilde{\eta}\xi_{i,j}^{\mathcal{T}_{ikk'}}, \quad \xi_k = \tilde{\eta}\xi_k^{\mathcal{T}_{ikk'}} + (1-\tilde{\eta})\xi_k^{I_{kk'}}, \quad \xi_{k'} = \tilde{\eta}\xi_{k'}^{\mathcal{T}_{ikk'}} + (1-\tilde{\eta})\xi_{k'}^{I_{kk'}} \quad \text{along } I_{k,k'}, \end{aligned}$$

where $\tilde{\eta} : \mathbb{R}^d \to [0,1]$ denotes a cut-off fuction decreasing quadratically with the distance from $\mathcal{T}_{i,k,k'}$ and vanishing outside of the neighborhood of $\mathcal{T}_{i,k,k'}$. Moreover, we have $\xi_{i,j}^{I_{ik}} = \xi_i^{I_{ik}} = \frac{1}{2}\xi_{i,k}^{I_{ik}} = \frac{1}{2}\xi_{i,k}^{\mathcal{T}_{ikk'}}$ and $\xi_k^{I_{ik}} = -\frac{1}{2}\xi_{i,k}^{I_{ik}} = -\frac{1}{2}\xi_{i,k}^{\mathcal{T}_{ikk'}}$ along $I_{i,k}$, as well as $\xi_k^{I_{kk'}} = \frac{1}{2}\xi_{k,k'}^{I_{kk'}} = \frac{1}{2}\xi_{k,k'}^{\mathcal{T}_{ikk'}}$ and $\xi_{k'}^{I_{kk'}} = -\frac{1}{2}\xi_{k,k'}^{I_{kk'}} = -\frac{1}{2}\xi_{k,k'}^{\mathcal{T}_{ikk'}}$ along $I_{k,k'}$. Proceeding as above (cf. [46, Sec. 7.2]), one can compute $|\xi_{i,j}|^2$ and $|\sqrt{3}\xi_{i,j} \cdot \xi_k|^2$ for $k \in \{1, ..., N\} \setminus \{i, j\}$, and then show that (3.4i) holds along both $I_{i,k'}$ and $I_{k,k'}$ in a neighborhood of $\mathcal{T}_{i,k,k'}$. In particular, observe that on the network (3.4i) is satisfied with $\delta_{\mathsf{cal}} = 0$.

Finally, observe that the gradient flow calibration satisfies first order compatibility conditions at any triple junction (cf. [46]). As a consequence, the property (3.4i) can be extended to $\mathbb{R}^d$ up to errors of second order in the distance with respect to the network. Hence, if one additionally truncates the gradient flow calibration constructed in [46] by means of an additional cut-off function decreasing quadratically with the distance from the network, then the condition (3.4i) follows for an arbitrarily small $\delta_{\mathsf{cal}} > 0$.

CHAPTER 4

Weak-strong stability for planar multiphase mean curvature flow beyond a circular topology change

This chapter contains a preliminary version of the paper "Stability of planar multiphase mean curvature flow beyond a circular topology change"[47], which is a work in progress together with Julian Fischer, Sebastian Hensel and Maximilian Moser.

Abstract. The evolution of a network of interfaces by mean curvature flow features the occurrence of topology changes and geometric singularities. As a consequence, classical solution concepts for mean curvature flow are in general limited to a finite time horizon. At the same time, the evolution beyond topology changes can be described only in the framework of weak solution concepts (e.g., Brakke solutions), whose uniqueness may fail.

Following the relative energy approach à la Fischer-Hensel-Laux-Simon [46], we prove a a weak-strong stability result beyond the singular time of a circular topology change: Any weak (i.e., BV) solution of planar multiphase mean curvature flow starting sufficiently close to a smooth, closed and simple curve evolving by mean curvature flow has to stay close to it for all times. This implies a weak-strong uniqueness principle for BV solutions to planar multiphase mean curvature flow beyond circular topology changes.

Previous weak-strong stability results of this form are limited to time horizons before the first topology change of the strong solution [46]. The reason is that the time-dependent constant in the associated relative energy inequality is non-integrable. We overcome this issue by developing a weak-strong stability theory for circular topology change up to dynamic space-time shift, which dynamically adapt the strong solution to the weak solution so that the leading-order non-integrable contributions in the relative energy inequality are compensated.

4.1 Main result

Our main result on the weak-strong stability of BV solution to multiphase mean curvature flow (MCF) reads as it follows.

Theorem 48 (Weak-strong stability up to shift for circular topology change)**.** *Let $d = 2$ and $P \geq 2$. Consider a global-in-time BV solution $\chi = (\chi_1, \ldots, \chi_P)$ to multiphase MCF in the* *sense of Definition 49. Consider also a smoothly evolving two-phase strong solution to MCF* $\bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P \equiv 1-\bar{\chi}_1)$ *with extinction time* $T_{ext} =: \frac{1}{2}r_0^2 > 0$*. Fix* $\alpha \in (1, 5)$*.*

There exists $\delta_{\mathrm{asymp}} \ll_\alpha \frac{1}{2}$ *such that if for all* $t \in (0, T_{ext})$ *the interior of the phase* $\{\bar{\chi}_1(\cdot, t){=}1\} \subset \mathbb{R}^2$ *is* δ_{asymp}*-close to a circle with radius* $r(t) := \sqrt{2(T_{ext}-t)}$ *in the sense of Definition 50, the evolution of* $\bar{\chi}$ *is* unique and stable until the extinction time T_{ext} modulo shift *in the following sense:*

There exists $\delta \ll 1$ *as well as an error functional* $E[\chi_0|\bar{\chi}_0] \in [0, \infty)$ *for the initial data of* χ *and* $\bar{\chi}$ *such that if*

$$E[\chi_0|\bar{\chi}_0] < \delta r_0, \tag{4.1}$$

one may then choose

a time horizon $t_\chi > 0$*,*
a path of translations $z \in W^{1,\infty}((0, t_\chi); \mathbb{R}^2)$*, and*
a strictly increasing time reparametrization $T \in W^{1,\infty}((0, t_\chi); (0, T_{ext}))$

with the properties $(z(0), T(0)) = (0, 0)$*,*

$$t_\chi = \sup\{t : T(t) < T_{ext}\}, \tag{4.2}$$

$$\frac{1}{r_0}\|z\|_{L^\infty_t(0,t_\chi)} \leq \sqrt{\frac{1}{r_0}E[\chi_0|\bar{\chi}_0]}, \tag{4.3}$$

$$\frac{1}{T_{ext}}\|T - \mathrm{id}\|_{L^\infty_t(0,t_\chi)} \leq \sqrt{\frac{1}{r_0}E[\chi_0|\bar{\chi}_0]}, \tag{4.4}$$

such that for a.e. $t \in (0, t_\chi)$ *it holds*

$$E[\chi|\bar{\chi}^{z,T}](t) \leq E[\chi_0|\bar{\chi}_0]\Big(\frac{r_T(t)}{r_0}\Big)^\alpha \tag{4.5}$$

where $\bar{\chi}^{z,T}(x, t) := \bar{\chi}(x{-}z(t), T(t))$*,* $(x, t) \in \mathbb{R}^2{\times}[0, t_\chi)$*, denotes the* shifted strong solution, $r_T(t) := r(T(t))$ *for* $t \in [0, t_\chi)$*, and* $E[\chi|\bar{\chi}^{z,T}](t)$ *is an error functional satisfying*

$$E[\chi|\bar{\chi}^{z,T}](t) = 0 \quad \Longleftrightarrow \quad \chi(\cdot, t) = \bar{\chi}^{z,T}(\cdot, t) \text{ a.e. in } \mathbb{R}^2. \tag{4.6}$$

In particular, the BV solution χ *goes extinct and the associated time horizon* t_χ *provides an upper bound for the extinction time.*

Definition 49 (BV solution to multiphase MCF)**.** *Let* $d = 2$ *and* $P \geq 2$*. A measurable map*

$$\chi = (\chi_1, \ldots, \chi_P) \colon \mathbb{R}^2 \times [0, \infty) \to \{0, 1\}^P$$

(or the corresponding tuple of sets $\Omega_i(t) := \{\chi_i(t) = 1\}$ *for* $i = 1, \ldots, P$*) is called a* global-in-time BV solution to multiphase MCF with initial data $\chi_0 = (\chi_{0,1}, \ldots, \chi_{0,P})\colon\ \mathbb{R}^2 \to \{0, 1\}^P$ *if the following conditions are satisfied:*

i) For any $T_{\mathrm{BV}} \in (0, \infty)$*,* χ *is a BV solution to multiphase MCF on* $[0, T_{\mathrm{BV}})$ *with initial data* χ_0 *in the sense of [46, Definition 13] (with trivial surface tension matrix* $\sigma = \mathrm{diag}(1, \ldots, 1) \in \mathbb{R}^{P\times P}$*) such that* *i.a) (Partition with finite interface energy) For almost every $T \in [0, T_{\mathrm{BV}})$, $\chi(T)$ is a partition of $\mathbb{R}^d$ with the interface energy*

$$E[\chi] := \sum_{i,j=1, i\neq j}^{P} \int_{\mathbb{R}^d} \frac{1}{2}\Big(\mathrm{d}|\nabla\chi_i| + \mathrm{d}|\nabla\chi_j| - \mathrm{d}|\nabla(\chi_i{+}\chi_j)|\Big) \tag{4.7}$$

is finite and

$$\operatorname*{ess\,sup}_{T\in[0,T_{\mathrm{BV}})} E[\chi(\cdot,T)] = \operatorname*{ess\,sup}_{T\in[0,T_{\mathrm{BV}})} \sum_{i,j=1,i\neq j}^{P} \int_{I_{i,j}(T)} 1 \,\mathrm{d}\mathcal{H}^{d-1} < \infty, \tag{4.8}$$

where $I_{i,j} = \partial^*\Omega_i \cap \partial^*\Omega_j$ for $i \neq j$ is the interface between Ω_i and Ω_j.

i.b) (Evolution equation) For all $i \in \{1, \dots, P\}$, there exist normal velocities $V_i \in L^2(\mathbb{R}^d \times [0, T_{\mathrm{BV}}), |\nabla\chi_i| \otimes \mathcal{L}^1)$ in the sense that each χ_i satisfies the evolution equation

$$\begin{aligned}&\int_{\mathbb{R}^d} \chi_i(\cdot,T)\varphi(\cdot,T)\,\mathrm{d}x - \int_{\mathbb{R}^d} \chi_{0,i}\varphi(\cdot,0)\,\mathrm{d}x \\ &= \int_0^T \int_{\mathbb{R}^d} V_i\varphi \mathrm{d}|\nabla\chi_i|\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} \chi_i\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{4.9}$$

for almost every $T \in [0, T_{\mathrm{BV}})$ and all $\varphi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d \times [0, T_{\mathrm{BV}}])$. Moreover, the (reflection) symmetry condition $V_i \frac{\nabla\chi_i}{|\nabla\chi_i|} = V_j \frac{\nabla\chi_j}{|\nabla\chi_j|}$ shall hold $\mathcal{H}^{d-1}$-almost everywhere on the interfaces $I_{i,j}$ for $i \neq j$.

i.c) (BV formulation of mean curvature) The normal velocities satisfy the equation

$$\begin{aligned}&\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^{T_{\mathrm{BV}}} \int_{I_{i,j}(t)} V_i \frac{\nabla\chi_i}{|\nabla\chi_i|} \cdot \mathrm{B}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &= \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^{T_{\mathrm{BV}}} \int_{I_{i,j}(t)} \left(\mathrm{Id} - \frac{\nabla\chi_i}{|\nabla\chi_i|} \otimes \frac{\nabla\chi_i}{|\nabla\chi_i|}\right) : \nabla \mathrm{B}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\end{aligned} \tag{4.10}$$

for all $B \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d \times [0, T_{\mathrm{BV}}]; \mathbb{R}^d)$.

ii) For all $[s, \tau] \subset [0, \infty)$, the energy dissipation inequality

$$E[\chi(\cdot,\tau)] + \int_s^\tau \sum_{i,j=1,i\neq j}^{P} \int_{I_{i,j}(t)} \frac{1}{2}|V_{i,j}|^2\, d\mathcal{H}^1 dt \leq E[\chi(\cdot,s)] \tag{4.11}$$

holds true, and the energy $t \mapsto E[\chi(\cdot,t)] := \sum_{i,j=1,i\neq j}^{P} \frac{1}{2}\mathcal{H}^1(I_{i,j}(t))$ is assumed to be locally absolutely continuous. ♢

Definition 50 (Quantitative closeness to circle)**.** *Let $\mathcal{A} \subset \mathbb{R}^2$ be a bounded, open and simply connected set with C^∞ boundary $\partial\mathcal{A}$. Fix two constants $\delta_{\mathrm{asymp}} \in (0, \frac{1}{2})$ and $r > 0$. We refer to $\mathcal{A}$ as* δ_{asymp}-close to a circle with radius r *if there exists an arc-length parametrization $\gamma\colon [0, L) \to \mathbb{R}^2$ of $\partial\mathcal{A}$ such that $\frac{1}{2}r$ is a tubular neighborhood width of $\partial\mathcal{A}$ and*

$$\frac{1}{2\pi r}|L - 2\pi r| \leq \delta_{\mathrm{asymp}}, \tag{4.12}$$

$$\sup_{\theta\in[0,L)} \Big|\mathrm{n}_{\partial\mathcal{A}}\big(\gamma(\theta)\big) - (-e^{2\pi i \frac{\theta}{L}})\Big| \leq \delta_{\mathrm{asymp}}, \tag{4.13}$$

$$\sup_{\theta\in[0,L)} r\Big|H_{\partial\mathcal{A}}\big(\gamma(\theta)\big) - \frac{1}{r}\Big| \leq \delta_{\mathrm{asymp}}, \tag{4.14}$$

$$\sup_{\theta\in[0,L)} r^2\Big|(\nabla^{\tan} H_{\partial\mathcal{A}})\big(\gamma(\theta)\big)\Big| \leq \delta_{\mathrm{asymp}}, \tag{4.15}$$

where $\mathrm{n}_{\partial\mathcal{A}}$ *denotes the unit normal vector field along* $\partial\mathcal{A}$ *pointing inside* $\mathcal{A}$ *and* $H_{\partial\mathcal{A}} := -\nabla^{\tan}\cdot\mathrm{n}_{\partial\mathcal{A}}$ *is the associated scalar mean curvature of* $\partial\mathcal{A}$*.* ◇

Remark 51. *Consider a smoothly evolving two-phase solution to mean curvature flow* $\bar\chi = (\bar\chi_1,\ldots,\bar\chi_P \equiv 1-\bar\chi_1)$ *with initial data* $\bar\chi_{0,1} = \chi_{\mathcal{A}_0}$ *for some smooth, bounded, open and simply connected initial set* $\mathcal{A}_0 \subset \mathbb{R}^2$*. By the Gage–Hamilton–Grayson theorem ([51],[55]), the solution goes extinct at time* $T_{ext} = \frac{\mathrm{vol}(\mathcal{A}_0)}{\pi}$*, and for any given* $\delta_{\mathrm{asymp}} \in (0,1)$*, there exists a time* $t_0 = t_0(\mathcal{A}_0,\delta_{\mathrm{asymp}})$ *such that for all* $t \in [t_0,T_{ext})$ *it holds that the interior of* $\{\bar\chi_1(\cdot,t)=1\}$ *is* δ_{asymp}*-close to a circle with radius* $r(t) := \sqrt{2(T_{ext}-t)}$ *in the sense of Definition 50.*

In particular, from some time onwards one is in the asymptotic regime close to the extinction time for which the conclusions of Theorem 1 apply, at least if at time $t_0 = t_0(\mathcal{A}_0,\delta_{\mathrm{asymp}})$ *the assumption* (4.1) *on the smallness of the initial error is satisfied (i.e., with respect to* $r(t_0)$*). Based on the weak-strong stability estimate prior to topology changes from [46], this requirement can be translated into a condition at the initial time* $t=0$*: there exists a constant* $\mu_0 = \mu_0(t_0,\mathcal{A}_0) > 0$ *such that if* $E[\chi_0|\bar\chi_0] < \frac{1}{\mu_0}\delta r_0$ *then* $E[\chi|\bar\chi](t_0) < \delta r(t_0)$*.*

In summary, for general initial data $\mathcal{A}_0$ *as considered here, one first has, thanks to the main result of [46], at least stability in the sense of* $\frac{d}{dt}E[\chi|\bar\chi](t) \leq C(t)E[\chi|\bar\chi](t)$ *for times* $t\in(0,t_0)$ *where* $C(t) \sim (2(T_{ext}-t))^{-1} = r(t)^{-2}$*. Then, if* $E[\chi_0|\bar\chi_0] < \frac{1}{\mu_0}\delta r_0$ *at time* t_0*, in addition the decay estimate* (4.5) *from Theorem 1 holds true for all times in the asymptotic regime* (t_0,T_{ext})*.* ◇

Notation and some elementary differential geometry For the smoothly evolving $\bar\chi$, we write $\mathrm{n}_{\bar I}(\cdot,t)$ for the unit normal vector field of $\bar I(\cdot,t) := \partial\{\bar\chi_1(\cdot,t)=1\}$ pointing inside $\{\bar\chi_1(\cdot,t)=1\}$, and also define a tangent vector field through $\mathrm{n}_{\bar I}(\cdot,t) = J\tau_{\bar I}(\cdot,t)$ with $J\in\mathbb{R}^{2\times2}$ being counter-clockwise rotation by 90°, $t\in(0,T)$. Curvature is defined by $H_{\bar I}(\cdot,t) := -\nabla^{\tan}\cdot\mathrm{n}_{\bar I}(\cdot,t)$ for $t\in(0,T_{ext})$. In particular, it holds

$$\nabla^{\tan}\mathrm{n}_{\bar I} = -H_{\bar I}\tau_{\bar I}\otimes\tau_{\bar I}, \qquad \nabla^{\tan}\tau_{\bar I} = H_{\bar I}\mathrm{n}_{\bar I}\otimes\tau_{\bar I}. \tag{4.16}$$

Within the tubular neighborhood $\{x\in\mathbb{R}^2\colon \mathrm{dist}(x,\bar I(t)) < r(t)/2\}$, the nearest-point projection onto $\partial\{\bar\chi(\cdot,t)=1\}$ is denoted by $P_{\bar I}(\cdot,t)$, whereas we write $\mathrm{sdist}_{\bar I}(\cdot,t)$ for the signed distance function, with orientation fixed through the requirement $\nabla\,\mathrm{sdist}_{\bar I}(\cdot,t)|_{\bar I} = \mathrm{n}_{\bar I}(\cdot,t)$, $t\in(0,T_{ext})$.

Given a map $f\colon \mathbb{R}^2\times[0,T_{ext})\to\mathbb{R}^m$ (or $f\colon \bigcup_{t\in[0,T_{ext})}\bar I(t)\times\{t\}\to\mathbb{R}^m$), we will use the notation $f^{z,T}$ to refer to the space-time shifted function $\mathbb{R}^2\times(0,t_\chi)\ni(x,t)\mapsto f(x-z(t),T(t))\in\mathbb{R}^m$ (or in the other case $\bigcup_{t\in[0,t_\chi)}(z(t)+\bar I(T(t))\times\{t\}\ni(x,t)\mapsto f(x-z(t),T(t))\in\mathbb{R}^m$) for any data $t_\chi\in(0,\infty)$, $z\colon(0,t_\chi)\to\mathbb{R}^2$ and $T\colon[0,t_\chi)\to[0,T_{ext})$. The shifted geometry itself will be abbreviated by $\bar I^{z,T}(t) := z(t)+\bar I(T(t))$, $t\in(0,t_\chi)$, and analogously for an associated arc-length parametrization $\bar\gamma(\cdot,t)$ of $\bar I(\cdot,t)$: $\bar\gamma^{z,T}(\cdot,t) := z(t)+\bar\gamma(\cdot,T(t))$, $t\in(0,t_\chi)$. Note that

$$\mathrm{sdist}^{z,T}_{\bar I}(\cdot,t) = \mathrm{sdist}_{\bar I^{z,T}}(\cdot,t), \tag{4.17}$$

and thus as a direct consequence

$$\mathrm{n}^{z,T}_{\bar{I}}(\cdot,t) = \mathrm{n}_{\bar{I}^{z,T}}(\cdot,t). \tag{4.18}$$

Indeed, the former simply follows from

$$\mathrm{sdist}_{\bar{I}}(\cdot,t) = \mathrm{dist}\Big(\cdot,\mathbb{R}^2\setminus\{\bar{\chi}_1(\cdot,t){=}1\}\Big) - \mathrm{dist}\Big(\cdot,\{\bar{\chi}_1(\cdot,t){=}1\}\Big).$$

Furthermore, within the tubular neighborhood $\{x\in\mathbb{R}^2\colon\ \mathrm{dist}(x{-}z(t),\bar{I}(T(t))) < r(T(t))/2\} = \{x\in\mathbb{R}^2\colon\ \mathrm{dist}(x,\bar{I}^{z,T}(t)) < r(T(t))/2\}$ it holds

$$P^{z,T}_{\bar{I}}(\cdot,t) = -z(t) + P_{\bar{I}^{z,T}}(\cdot,t). \tag{4.19}$$

Finally, for the sake of shortness, we will denote $\frac{d}{\mathrm{d}t}f$ by $\dot{f}$.

4.2 Overview of the strategy

For the rest of the paper, fix a global-in-time BV solution $\chi = (\chi_1,\ldots,\chi_P)$ to (planar) multiphase MCF in the sense of Definition 49 as well as a smoothly evolving two-phase solution to MCF $\bar{\chi} = (\bar{\chi}_1,\ldots,\bar{\chi}_P\equiv 1{-}\bar{\chi}_1)$ with extinction time $T_{ext} =: \frac{1}{2}r_0^2 > 0$. We also assume that for all $t\in(0,T_{ext})$ the interior of the phase $\{\bar{\chi}_1(\cdot,t){=}1\}\subset\mathbb{R}^2$ is δ_{asymp}-close to a circle with radius $r(t) := \sqrt{2(T_{ext}{-}t)}$ in the sense of Definition 50. Consistent with the claim of Theorem 1, we will choose a suitable value of the constant δ_{asymp} in the course of the proof.

4.2.1 Heuristics: Leading-order behaviour near extinction time

The aim of this subsection is to compute the time evolution of our linearized error functional in the simplified case of a centered self-similarly shrinking circle. As a result, our analysis reveals the instability of our linearized error functional near the extinction time.

Consider a centered circle self-similarly shrinking by mean curvature flow: $t\mapsto\partial B_{r(t)} = \mathrm{im}\,\bar{\gamma}(t)\subset\mathbb{R}^2$, where $\bar{\gamma}(t)\colon[0,2\pi r(t))\to\partial B_{r(t)}$, $\theta\mapsto r(t)e^{i\frac{\theta}{r(t)}}$, is an arc-length parametrization of $\partial B_{r(t)}$. In particular, $\dot{r} = -\frac{1}{r}$ in the interval $(0,\frac{1}{2}r_0^2 = T_{ext})$ for $r_0 := r(0) > 0$, i.e., $r(t) = \sqrt{2(T_{ext}-t)}$.

Apart from the shrinking circle, let us consider a second solution to mean curvature flow, for which we in addition assume that it can be written as a smooth graph over the self-similarly shrinking circle. More precisely, there exists a smooth time-dependent height function $h(\cdot,t)\colon\partial B_{r(t)}\to\mathbb{R}$ with $|h(\cdot,t)|\ll r(t)$ and $|h'(\cdot,t)|\ll 1$ such that this second solution is represented as the image of the curve

$$\gamma_h(\cdot,t) := \Big(\mathrm{id} + h(\cdot,t)\mathrm{n}_{\partial B_{r(t)}}\Big)\circ\bar{\gamma}(\cdot,t) \quad \text{on } [0,2\pi r(t)), \tag{4.20}$$

where $\mathrm{n}_{\partial B_{r(t)}}$ denotes the inward-pointing unit normal along $\partial B_{r(t)}$ and by slight abuse of notation $h'(\cdot,t) := (\tau_{\partial B_{r(t)}}\cdot\nabla^{\tan})h(\cdot,t)$ for the choice of tangent vector field $\tau_{\partial B_{r(t)}}\Big(\bar{\gamma}(\theta,t)\Big) = ie^{i\frac{\theta}{r(t)}}$. As we will show later, our error functional in this perturbative setting corresponds to leading order to

$$E_h(t) := \int_{\partial B_{r(t)}} \frac{1}{2}\frac{h^2(\cdot,t)}{r^2(t)} + \frac{1}{2}(h')^2(\cdot,t)\,d\mathcal{H}^1. \tag{4.21}$$

For the current purposes, we content ourselves with studying the stability of $E_h(t)$ near the extinction time.

To this end, we have to derive the PDE satisfied by the height function h (and its derivative). Dropping from now on for ease of notation the time dependence of all involved quantities, we first note that by definition in case of self-similarly shrinking circle

$$\partial_t\bar{\gamma} = \Big(\frac{1}{r}\mathrm{n}_{\partial B_r} + \lambda\tau_{\partial B_r}\Big) \circ \bar{\gamma} \quad \text{on } [0, 2\pi r), \tag{4.22}$$

where λ denotes the tangential velocity. Second, we may then, on one side, directly compute based on the definition (4.20)

$$\partial_t\gamma_h = \bigg(\Big(\frac{1}{r} + \partial_t h + \lambda h'\Big)\mathrm{n}_{\partial B_r}\bigg) \circ \bar{\gamma} + \bigg(\lambda\Big(1 - \frac{h}{r}\Big)\tau_{\partial B_r}\bigg) \circ \bar{\gamma}. \tag{4.23}$$

On the other side, since γ_h is assumed to evolve by mean curvature flow, it holds

$$H_{\gamma_h} = \partial_t\gamma_h \cdot \mathrm{n}_{\gamma_h} \quad \text{on } [0, 2\pi r), \tag{4.24}$$

where the normal n_{γ_h} and mean curvature H_{γ_h} of the curve γ_h are given by the elementary formulas (with J denoting the counter-clockwise rotation by $90°$)

$$\mathrm{n}_{\gamma_h} = J\frac{\partial_\theta\gamma_h}{|\partial_\theta\gamma_h|} = \left(\frac{\Big(1 - \frac{h}{r}\Big)\mathrm{n}_{\partial B_r} - h'\tau_{\partial B_r}}{\sqrt{\Big(1 - \frac{h}{r}\Big)^2 + (h')^2}}\right) \circ \bar{\gamma} \tag{4.25}$$

and

$$H_{\gamma_h} = \frac{\partial_{\theta\theta}\gamma_h}{|\partial_\theta\gamma_h|^2} \cdot \mathrm{n}_{\gamma_h} = \left(\frac{\Big(1 - \frac{h}{r}\Big)\Big(\frac{1}{r} + h'' - \frac{h}{r^2}\Big) + 2\frac{(h')^2}{r}}{\sqrt{\Big(1 - \frac{h}{r}\Big)^2 + (h')^2}^{\,3}}\right) \circ \bar{\gamma}. \tag{4.26}$$

From (4.23)–(4.26), one may now deduce the non-linear PDE satisfied by the height function h. However, because in what follows we are only interested in identifying the leading-order behavior, we suppose from now on that the height function h instead satisfies the corresponding linearized equation:

$$\partial_t h = h'' + \frac{h}{r^2} \quad \text{on } \partial B_r. \tag{4.27}$$

From this, using $(\partial_t h)' = \partial_t h' + \frac{h}{r^2}$, we in particular deduce

$$\partial_t h' = h''' + 2\frac{h'}{r^2}. \tag{4.28}$$

Indeed, this follows easily from (4.27) and exploiting the change of variables $\widetilde{h}(\theta) := h(re^{i\theta})$ as a useful computational device.

Recalling (4.21), we thus get from the transport theorem as well as (4.27)–(4.28)

$$\begin{aligned}
\frac{d}{dt}E_h &= \int_{\partial B_r} \partial_t\Big(\frac{1}{2}\frac{h^2}{r^2} + \frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1 - \int_{\partial B_r} H^2_{\partial B_r}\Big(\frac{1}{2}\frac{h^2}{r^2} + \frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1 \\
&= \int_{\partial B_r} \frac{h}{r}\Big(2\frac{h}{r^3} + \frac{h''}{r}\Big)\,d\mathcal{H}^1 + \int_{\partial B_r} h'\Big(h''' + 2\frac{h'}{r^2}\Big)\,d\mathcal{H}^1
\end{aligned} \tag{4.29}$$

$$- \int_{\partial B_r} \frac{1}{r^2}\Big(\frac{1}{2}\frac{h^2}{r^2} + \frac{1}{2}(h')^2\Big)\, d\mathcal{H}^1.$$

Integrating by parts and collecting alike terms therefore yields

$$\frac{d}{dt}E_h + \int_{\partial B_r} (h'')^2\, d\mathcal{H}^1 = \int_{\partial B_r} \frac{3}{2}\frac{h^2}{r^4} + \frac{1}{2}\frac{(h')^2}{r^2}\, d\mathcal{H}^1. \tag{4.30}$$

Fourier decomposing

$$[0,2\pi) \ni \theta \mapsto \widetilde{h}(\theta) = h(re^{i\theta}) = a_0 \frac{1}{\sqrt{2\pi}}\chi_{[0,2\pi]} + \sum_{k=1}^{\infty} a_k \frac{\cos(k\theta)}{\sqrt{\pi}} + b_k \frac{\sin(k\theta)}{\sqrt{\pi}}, \tag{4.31}$$

where we also recall the formulas for the associated Fourier coefficients

$$a_0 = \int_0^{2\pi} \frac{1}{\sqrt{2\pi}}\widetilde{h}(\theta)\, d\theta, \quad a_k = \int_0^{2\pi} \widetilde{h}(\theta)\frac{\cos(k\theta)}{\sqrt{\pi}}\, d\theta, \quad b_k = \int_0^{2\pi} \frac{\sin(k\theta)}{\sqrt{\pi}}\, d\theta,$$

then rearranges (4.30) as

$$\begin{aligned} &\frac{d}{dt}E_h + \frac{1}{r^3}\sum_{k=1}^{\infty} k^4\Big(a_k^2 + b_k^2\Big) \\ &\quad = \frac{1}{r^3}\frac{3}{2}a_0^2 + \frac{1}{r^3}\sum_{k=1}^{\infty}\Big(\frac{3}{2} + \frac{1}{2}k^2\Big)\Big(a_k^2 + b_k^2\Big). \end{aligned} \tag{4.32}$$

Since $k^4 - \frac{3}{2} - \frac{1}{2}k^2 > 0$ for $k \geq 2$, we infer that only the modes (a_0, a_1, b_1) are unstable near the extinction time (in the sense that these are precisely those inducing the borderline non-integrable singularity r^{-2} in the Gronwall estimate of E_h).

4.2.2 Heuristics: Decay estimate

Geometrically, the unstable modes correspond to time dilations and spatial translations. The basic idea of the present work is to correct these by dynamically adapting the smoothly evolving strong solution. In the simplified context of a self-similarly shrinking circle, this works as follows.

Consider $t_h > 0$ (to be interpreted as an upper bound for the perturbed solution) as well as a smooth path $z\colon (0,t_h) \to \mathbb{R}^2$ of translations together with a smooth time diffeomorphism $T\colon (0,t_h) \to (0,\frac{1}{2}r_0^2)$, the latter to be thought of as a perturbation of the identity: $T(t) =: t + \mathfrak{T}(t)$ for $t \in (0,t_h)$. Based on this data, we then introduce the dynamically adapted solution

$$\bar{\gamma}^{z,T}(\theta,t) := \bar{\gamma}(\theta, T(t)) + z(t), \quad \theta \in [0, 2\pi r_T(t)),\, t \in (0,t_h), \tag{4.33}$$

where $r_T(t) := r(T(t))$, and assume that the perturbed solution γ_h is given by

$$\gamma_h(\cdot,t) = \Big(\mathrm{id} + h(\cdot,t)\mathrm{n}_{\partial B_{r_T(t)}(z(t))}\Big) \circ \bar{\gamma}^{z,T}(\cdot,t), \quad t \in (0,t_h), \tag{4.34}$$

where $|h(\cdot,t)| \ll r_T(t)$ and $|h'(\cdot,t)| \ll 1$. We are again interested in the stability properties of

$$E_h^{z,T}(t) := \int_{\partial B_{r_T(t)}(z(t))} \frac{1}{2}\frac{h^2(\cdot,t)}{r^2(t)} + \frac{1}{2}(h')^2(\cdot,t)\, d\mathcal{H}^1, \quad t \in (0,t_h). \tag{4.35}$$

In fact, we actually aim to identify ODEs for z and $\mathfrak{T}$ such that $E_h^{z,T}$ satisfies a quantitative decay estimate on $(0,t_h)$. One of course already expects the ODE for $\mathfrak{T}$ to involve the mode a_0, whereas the ODE for z is expected to be encoded in terms of (a_1,b_1). From now on, we again make use of the notational convention of suppressing the time dependence of all involved quantities. To this end, it will be convenient to associate to any map $f(\cdot,t)\colon \partial B_{r(t)} \to \mathbb{R}$ its time-rescaled version $f_T(\cdot,t)\colon \partial B_{r_T(t)} \to \mathbb{R}$ defined by $f_T(\cdot,t) := f(\cdot,T(t))$.

We start by computing the normal speed of $\partial B_{r_T}(z)$. By definition (4.33),

$$\partial_t \bar{\gamma}^{z,T} = (\partial_t \bar{\gamma})_T \Big(1+\dot{\mathfrak{T}}\Big) + \dot{z}. \tag{4.36}$$

Hence, the normal speed of $\partial B_{r_T}(z)$ in the direction of $\mathrm{n}_{\partial B_{r_T}(z)}$ is given by

$$V_{\partial B_{r_T}(z)} = \frac{1}{r_T}\Big(1+\dot{\mathfrak{T}}\Big) + \mathrm{n}_{\partial B_{r_T}(z)} \cdot \dot{z}. \tag{4.37}$$

The tangential speed in the direction of $\tau_{\partial B_{r_T}(z)}$ is furthermore given by

$$\lambda_{\partial B_{r_T}(z)} = \lambda_T\Big(1+\dot{\mathfrak{T}}\Big) + \tau_{\partial B_{r_T}(z)} \cdot \dot{z}. \tag{4.38}$$

In particular, we may now compute

$$\begin{aligned}\partial_t \gamma_h = {}&\bigg(\Big(V_{\partial B_{r_T}(z)} + \partial_t h + \lambda_{\partial B_{r_T}(z)} h'\Big)\mathrm{n}_{\partial B_{r_T}(z)}\bigg) \circ \bar{\gamma}^{z,T} \\ &+ \bigg(\Big(\lambda_{\partial B_{r_T}(z)} - \lambda_T\Big(1+\dot{\mathfrak{T}}\Big)\frac{h}{r_T}\Big)\tau_{\partial B_{r_T}(z)}\bigg) \circ \bar{\gamma}^{z,T}.\end{aligned} \tag{4.39}$$

Furthermore, the analogous versions of the formulas (4.25)–(4.26) hold true:

$$\mathrm{n}_{\gamma_h} = \left(\frac{\Big(1-\frac{h}{r_T}\Big)\mathrm{n}_{\partial B_{r_T}(z)} - h'\tau_{\partial B_{r_T}(z)}}{\sqrt{\Big(1-\frac{h}{r_T}\Big)^2 + (h')^2}}\right) \circ \bar{\gamma}^{z,T} \tag{4.40}$$

and

$$H_{\gamma_h} = \left(\frac{\Big(1-\frac{h}{r_T}\Big)\Big(\frac{1}{r_T} + h'' - \frac{h}{r_T^2}\Big) + 2\frac{(h')^2}{r_T}}{\sqrt{\Big(1-\frac{h}{r_T}\Big)^2 + (h')^2}^{\,3}}\right) \circ \bar{\gamma}^{z,T}. \tag{4.41}$$

Combining the information provided by (4.37)–(4.40), we deduce

$$\begin{aligned}\partial_t \gamma_h \cdot \mathrm{n}_{\gamma_h} &= \Big(1-\frac{h}{r_T}\Big)\Big(V_{\partial B_{r_T}(z)} + \partial_t h + \lambda_{\partial B_{r_T}(z)} h'\Big) \\ &\quad - h'\bigg(\Big(1-\frac{h}{r_T}\Big)\lambda_T\Big(1+\dot{\mathfrak{T}}\Big) + \tau_{\partial B_{r_T}(z)} \cdot \dot{z}\bigg) \\ &= \Big(1-\frac{h}{r_T}\Big)\Big(\frac{1}{r_T}\Big(1+\dot{\mathfrak{T}}\Big) + \mathrm{n}_{\partial B_{r_T}(z)} \cdot \dot{z} + \partial_t h\Big) - \frac{h}{r_T} h'\tau_{\partial B_{r_T}(z)} \cdot \dot{z}.\end{aligned}$$

Turning as above to the linearized PDE satisfied by the height function, we therefore obtain

$$\partial_t h = h'' + \frac{h}{r_T^2} - \frac{\dot{\mathfrak{T}}}{r_T} - \mathrm{n}_{\partial B_{r_T}(z)} \cdot \dot{z} \tag{4.42}$$

as well as

$$(\partial_t h') = h''' + (2+\dot{\mathfrak{T}})\frac{h'}{r_T^2} + \frac{1}{r_T}\tau_{\partial B_{r_T}(z)}\cdot \dot z. \tag{4.43}$$

We may now finally compute based on the transport theorem, the definition (4.35), and the formulas (4.37) as well as (4.42)–(4.43)

$$\begin{aligned}
\frac{d}{dt}E_h^{z,T} &= \int_{\partial B_{r_T}(z)} \partial_t\Big(\frac{1}{2}\frac{h^2}{r_T^2} + \frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1 \\
&\quad - \int_{\partial B_{r_T}(z)} H_{\partial B_{r_T}(z)} V_{\partial B_{r_T}(z)}\Big(\frac{1}{2}\frac{h^2}{r_T^2} + \frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1 \\
&= \int_{\partial B_{r_T}(z)} \frac{h}{r_T}\Big(\frac{1+\dot{\mathfrak{T}}}{r_T^3}h + \frac{1}{r_T}\Big(h'' + \frac{h}{r_T^2} - \frac{\dot{\mathfrak{T}}}{r_T} - \mathrm{n}_{\partial B_{r_T}(z)}\cdot \dot z\Big)\Big)\,d\mathcal{H}^1 \\
&\quad + \int_{\partial B_{r_T}(z)} h'\Big(h''' + (2+\dot{\mathfrak{T}})\frac{h'}{r_T^2} + \frac{1}{r_T}\tau_{\partial B_{r_T}(z)}\cdot \dot z\Big)\,d\mathcal{H}^1 \\
&\quad - \int_{\partial B_{r_T}(z)} \frac{1}{r_T^2}\Big(\frac{1}{2}\frac{h^2}{r_T^2} + \frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1 \\
&\quad - \int_{\partial B_{r_T}(z)} H_{\partial B_{r_T}(z)}\Big(V_{\partial B_{r_T}(z)} - H_{\partial B_{r_T}(z)}\Big)\Big(\frac{1}{2}\frac{h^2}{r_T^2} + \frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1.
\end{aligned}$$

Hence, integrating by parts and collecting again alike terms yields

$$\begin{aligned}
\frac{d}{dt}E_h^{z,T} &= \int_{\partial B_{r_T}(z)} \frac{3}{2}\frac{h^2}{r_T^4}\,d\mathcal{H}^1 - \int_{\partial B_{r_T}(z)} \frac{h}{r_T^3}\dot{\mathfrak{T}}\,d\mathcal{H}^1 \\
&\quad + \int_{\partial B_{r_T}(z)} \frac{1}{2}\frac{(h')^2}{r_T^2}\,d\mathcal{H}^1 - \int_{\partial B_{r_T}(z)} 2\frac{h}{r_T^2}\mathrm{n}_{\partial B_{r_T}(z)}\cdot \dot z\,d\mathcal{H}^1 \\
&\quad - \int_{\partial B_{r_T}(z)} (h'')^2\,d\mathcal{H}^1 \\
&\quad + R_{h.o.t.},
\end{aligned} \tag{4.44}$$

where

$$R_{h.o.t.} := \int_{\partial B_{r_T}(z)} \frac{1}{r_T}\Big(\frac{\dot{\mathfrak{T}}}{r_T} - \mathrm{n}_{\partial B_{r_T}(z)}\cdot \dot z\Big)\Big(\frac{1}{2}\frac{h^2}{r_T^2} + \frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1. \tag{4.45}$$

Based on the Fourier decomposition (4.31), the identity (4.44) now motivates to define

$$\dot{\mathfrak{T}} = \frac{c_T}{r_T}\fint_0^{2\pi} \widetilde{h}\,d\theta, \quad \dot z = \frac{c_z}{r_T^2}\fint_0^{2\pi} \widetilde{h}(-e^{i\theta})\,d\theta, \tag{4.46}$$

where the constants (c_T, c_z) are yet to be chosen. Indeed, with these choices we get

$$\begin{aligned}
&\frac{d}{dt}E_h^{z,T} + \frac{(c_T - 3/2)}{r_T^2}\frac{a_0^2}{r_T} + \frac{(c_z - 1)}{r_T^2}\frac{a_1^2 + b_1^2}{r_T} \\
&\qquad + \frac{1}{r_T^2}\sum_{k=2}^{\infty}\Big(k^4 - \frac{3}{2} - \frac{1}{2}k^2\Big)\frac{a_k^2 + b_k^2}{r_T} = R_{h.o.t.},
\end{aligned} \tag{4.47}$$

where, due to $|\dot{\mathfrak{T}}| \leq c_T\frac{1}{r_T}\|h\|_{L^\infty(\partial B_{r_T}(z))}$ and $|\dot{z}| \leq c_z\frac{1}{r_T^2}\|h\|_{L^\infty(\partial B_{r_T}(z))}$, one has an estimate for the remainder term in the form of

$$\Big|R_{h.o.t.}\Big| \leq \Big(c_T{+}c_z\Big)\frac{\|h\|_{L^\infty(\partial B_{r_T}(z))}}{r_T}\frac{1}{r_T^3}\Big(\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}\frac{1}{2}(1{+}k^2)(a_k^2+b_k^2)\Big). \tag{4.48}$$

Hence, for given $\delta \in (0,1)$, if $|h| \ll_{\delta,c_T,c_z} r_T$, one gets an upgrade of (4.44) in the form of

$$\begin{aligned}\frac{d}{dt}E_h^{z,T} + \frac{c_T{-}3/2(1{+}\delta)}{r_T^2}\frac{a_0^2}{r_T} + \frac{c_z{-}1(1{+}\delta)}{r_T^2}\frac{a_1^2+b_1^2}{r_T}& \\ + \frac{1}{r_T^2}\sum_{k=2}^{\infty}\Big(k^4{-}(1{+}\delta)\Big(\frac{3}{2}{+}\frac{1}{2}k^2\Big)\Big)\frac{a_k^2+b_k^2}{r_T} &\leq 0.\end{aligned} \tag{4.49}$$

Because of

$$E_h^{z,T} = \frac{1}{2}\frac{a_0^2}{r_T} + \sum_{k=1}^{\infty}\frac{1}{2}(1{+}k^2)\frac{a_k^2+b_k^2}{r_T}, \tag{4.50}$$

we deduce that for any constant $\alpha > 1$ satisfying

$$\alpha \leq \min\{2c_T - 3(1{+}\delta), c_z - 1(1{+}\delta)\}, \tag{4.51}$$

$$\alpha\frac{1}{2}(1{+}k^2) \leq k^4{-}(1{+}\delta)\Big(\frac{3}{2}{+}\frac{1}{2}k^2\Big), \qquad k \geq 2, \tag{4.52}$$

it holds

$$\frac{d}{dt}E_h^{z,T} + \frac{\alpha}{r_T^2}E_h^{z,T} \leq 0. \tag{4.53}$$

Choosing $c_T := 4$ and $c_z := 6$, optimizing shows that for any desired exponent $\alpha \in (1,5)$ there exists a choice of the constant δ such that (4.53) holds true (in the perturbative regime $|h| \ll_{c_T,c_z,\delta} 1$ with linearized evolution law (4.42)). Indeed, the function $f\colon [2,\infty) \to [0,\infty)$, $x \mapsto \big(\frac{1}{2}(1{+}x^2)\big)^{-1}\big(x^4 - (\frac{3}{2}{+}\frac{1}{2}x^2)\big)$ is monotonically increasing and satisfies $f(2) = 5$. Since $\frac{d}{dt}r_T^\alpha = -\alpha r_T^{\alpha-1}\frac{1}{r_T} = -(1{+}\dot{\mathfrak{T}})\frac{\alpha}{r_T^2}r_T^\alpha$ and $|\dot{\mathfrak{T}}| \leq c_T\frac{1}{r_T}\|h\|_{L^\infty(\partial B_{r_T}(z))}$, one may choose for any $\alpha \in (1,5)$ the constant δ such that

$$E_h^{z,T}(t) \leq E_h^{z,T}(0)\Big(\frac{r_T(t)}{r_0}\Big)^\alpha, \quad t \in (0,t_h). \tag{4.54}$$

This is precisely the type of decay estimate (or, weak-strong stability estimate up to shift) claimed in our main result, Theorem 1.

Before we turn in the upcoming subsections to a description of the key ingredients and steps for our proof of Theorem 48 (with the above considerations, of course, being their main motivation), let us provide some final remarks on the main assumptions behind the derivation of the decay estimate (4.54).

First, one may derive a version of (4.44) also in the case where the time-evolving curve $\bar{\gamma}$ is not parametrizing a perfect circle. The main difference in this case is that the coefficients are not anymore simply constant along $\bar{\gamma}$ (i.e., not proportional to inverse powers of r_T). It is precisely at this stage where we exploit our notion of quantitative closeness of the strong solution to a circular solution, cf. Definition 50, allowing us to effectively reduce the situation to the constant-coefficient computation (4.44) (i.e., in PDE jargon, we perform nothing else than a global freezing of coefficients).

A second simplifying assumption was the usage of the linearized evolution law (4.42) for the height function h as well as that we only considered the stability of the leading order contribution E_h to our actual error functional. It will turn out that these linearization errors are harmless and only impact the final stability estimate qualitatively in the same manner as the term $R_{h.o.t.}$ from (4.44).

Needless to say, in the general setting of Theorem 48 where we aim for quantitative stability beyond circular topology change even for the broader class of weak (i.e., here BV) solutions, we can not rely on the above considerations (e.g., transport theorem, derivation of the (linearized) evolution law (4.42)) in order to rigorously derive the evolution of the error functional. In order to still unravel the structure of the right hand side of (4.44), we instead make use of the recently introduced notions of gradient flow calibrations and relative entropies for multiphase mean curvature flow from [46], serving as a robust replacement of the above considerations to the weak setting.

Last but not least, one of course also needs an independent argument ensuring that one can reduce the whole estimation strategy to a perturbative graph setting as above. This, however, is precisely one of the key points of the upcoming subsections.

4.2.3 A general stability estimate for multiphase MCF

Starting point of our strategy is a stability estimate which one may essentially directly infer from the combination of [46, Proposition 17] and [46, Lemma 20] (or more precisely, their proofs).

Lemma 52 (Preliminary stability estimate)**.** *Let $((\xi_i)_{i=1,\dots,P},(\vartheta_i)_{i=1,\dots,P-1},B)$—to be thought of as being constructed from $\bar\chi$—such that*

$$\begin{aligned}
&\xi_i\in W^{1,\infty}_{loc}\Big([0,T_{ext});W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)\Big)\cap L^\infty_{loc}\Big([0,T_{ext});W^{2,\infty}(\mathbb{R}^2;\mathbb{R}^2)\Big),\\
&\vartheta_i\in W^{1,1}_{loc}\Big([0,T_{ext});L^1(\mathbb{R}^2)\Big)\cap L^1_{loc}\Big([0,T_{ext});(W^{1,1}\cap W^{1,\infty})(\mathbb{R}^2)\Big),\\
&B\in L^\infty_{loc}\Big([0,T_{ext});W^{2,\infty}(\mathbb{R}^2;\mathbb{R}^2)\Big),
\end{aligned}$$

where $(\vartheta_i)_{i=1,\dots,P-1}$ is supposed to satisfy, for all $t\in(0,T_{ext})$,

$$\begin{aligned}
&\vartheta_1(\cdot,t)<0 && \text{in the interior of } \{\bar\chi_1(\cdot,t)=1\},\\
&\vartheta_1(\cdot,t)>0 && \text{in the exterior of } \overline{\{\bar\chi_1(\cdot,t)=1\}},\\
&\vartheta_i(\cdot,t)=1 && \text{throughout } \mathbb{R}^2 \text{ for } i\neq 1.
\end{aligned}$$

Consider in addition data (t_χ,z,T) such that $t_\chi\in(0,\infty)$, $z\in W^{1,\infty}_{loc}((0,t_\chi);\mathbb{R}^2)$ and $T\in W^{1,\infty}_{loc}((0,t_\chi);(0,T_{ext}))$, and define for all $t\in(0,t_\chi)$

$$E_{\mathrm{int}}[\chi|\bar\chi^{z,T}](t):=\sum_{i,j=1,i\neq j}^{P}\frac12\int_{I_{i,j}(t)}1-\mathrm{n}_{i,j}(\cdot,t)\cdot\xi^{z,T}_{i,j}(\cdot,t)\,d\mathcal{H}^1, \tag{4.55}$$

$$E_{\mathrm{bulk}}[\chi|\bar\chi^{z,T}](t):=\sum_{i=1}^{P-1}\int_{\mathbb{R}^2}|\chi_i(\cdot,t)-\bar\chi^{z,T}_i(\cdot,t)||\vartheta^{z,T}_i(\cdot,t)|\,dx, \tag{4.56}$$

$$E[\chi|\bar\chi^{z,T}](t):=E_{\mathrm{int}}[\chi|\bar\chi^{z,T}](t)+E_{\mathrm{bulk}}[\chi|\bar\chi^{z,T}](t). \tag{4.57}$$

where we also defined $\xi_{i,j} := \xi_i - \xi_j$ for all distinct $i, j \in \{1, \dots, P\}$.

Then, for all $[s,\tau] \subset [0, t_\chi)$, it holds

$$\begin{aligned} &E_{\mathrm{int}}[\chi|\bar{\chi}^{z,T}](\tau) + \int_s^\tau \sum_{i,j=1, i\neq j}^P \frac{1}{2}\mathcal{D}_{i,j}[\chi|\bar{\chi}^{z,T}](t)\,dt \\ &\quad \leq E_{\mathrm{int}}[\chi|\bar{\chi}^{z,T}](s) + \int_s^\tau \sum_{i,j=1, i\neq j}^P \frac{1}{2} RHS_{i,j}^{\mathrm{int}}[\chi|\bar{\chi}^{z,T}](t)\,dt, \end{aligned} \tag{4.58}$$

as well as

$$E_{\mathrm{bulk}}[\chi|\bar{\chi}^{z,T}](\tau) = E_{\mathrm{bulk}}[\chi|\bar{\chi}^{z,T}](s) + \int_s^\tau \sum_{i=1}^{P-1} RHS_i^{\mathrm{bulk}}[\chi|\bar{\chi}^{z,T}](t)\,dt, \tag{4.59}$$

where the individual terms are given by

$$\begin{aligned} \mathcal{D}_{i,j}[\chi|\bar{\chi}^{z,T}](t) := &\int_{I_{i,j}(t)} \frac{1}{2}\Big|V_{i,j} + \nabla\cdot\xi_{i,j}^{z,T}\Big|^2(\cdot,t)\,d\mathcal{H}^1 \\ &+ \int_{I_{i,j}(t)} \frac{1}{2}\Big|V_{i,j}\mathrm{n}_{i,j} - (B^{z,T}\cdot\xi_{i,j}^{z,T})\xi_{i,j}^{z,T}\Big|^2(\cdot,t)\,d\mathcal{H}^1, \end{aligned}$$

and

$$\begin{aligned} &RHS_{i,j}^{\mathrm{int}}[\chi|\bar{\chi}^{z,T}](t) \\ &:= -\int_{I_{i,j}(t)} \Big(\partial_t\xi_{i,j}^{z,T} + (B^{z,T}\cdot\nabla)\xi_{i,j}^{z,T} + (\nabla B^{z,T})^{\mathsf{T}}\xi_{i,j}^{z,T}\Big)(\cdot,t)\cdot(\mathrm{n}_{i,j} - \xi_{i,j}^{z,T})(\cdot,t)\,d\mathcal{H}^1 \\ &\quad - \int_{I_{i,j}(t)} \Big(\partial_t\xi_{i,j}^{z,T} + (B^{z,T}\cdot\nabla)\xi_{i,j}^{z,T}\Big)(\cdot,t)\cdot\xi_{i,j}^{z,T}(\cdot,t)\,d\mathcal{H}^1 \\ &\quad + \int_{I_{i,j}(t)} \frac{1}{2}\Big|\nabla\cdot\xi_{i,j}^{z,T} + B^{z,T}\cdot\xi_{i,j}^{z,T}\Big|^2(\cdot,t)\,d\mathcal{H}^1 \\ &\quad - \int_{I_{i,j}(t)} \frac{1}{2}\Big|B^{z,T}\cdot\xi_{i,j}^{z,T}\Big|(\cdot,t)\Big(1 - |\xi_{i,j}^{z,T}|^2\Big)(\cdot,t)\,d\mathcal{H}^1 \\ &\quad - \int_{I_{i,j}(t)} (1 - \mathrm{n}_{i,j}\cdot\xi_{i,j}^{z,T})(\cdot,t)\nabla\cdot\xi_{i,j}^{z,T}(\cdot,t)(B^{z,T}\cdot\xi_{i,j}^{z,T})(\cdot,t)\,d\mathcal{H}^1 \\ &\quad + \int_{I_{i,j}(t)} \Big((\mathrm{Id} - \xi_{i,j}^{z,T}\otimes\xi_{i,j}^{z,T})B^{z,T}\Big)(\cdot,t)\cdot\Big((V_{i,j} + \nabla\cdot\xi_{i,j}^{z,T})\mathrm{n}_{i,j}\Big)(\cdot,t)\,d\mathcal{H}^1 \\ &\quad + \int_{I_{i,j}(t)} (1 - \mathrm{n}_{i,j}\cdot\xi_{i,j}^{z,T})(\cdot,t)\nabla\cdot B^{z,T}(\cdot,t)\,d\mathcal{H}^1 \\ &\quad - \int_{I_{i,j}(t)} (\mathrm{n}_{i,j} - \xi_{i,j}^{z,T})(\cdot,t)\otimes(\mathrm{n}_{i,j} - \xi_{i,j}^{z,T})(\cdot,t) : \nabla B^{z,T}(\cdot,t)\,d\mathcal{H}^1, \end{aligned}$$

as well as

$$\begin{aligned} &RHS_i^{\mathrm{bulk}}[\chi|\bar{\chi}^{z,T}](t) \\ &:= -\sum_{j=1, j\neq i}^P \int_{I_{i,j}(t)} \vartheta_i^{z,T}(\cdot,t)(B^{z,T}\cdot\xi_{i,j}^{z,T} - V_{i,j})(\cdot,t)\,d\mathcal{H}^1 \\ &\quad - \sum_{j=1, j\neq i}^P \int_{I_{i,j}(t)} \vartheta_i^{z,T}(\cdot,t)B^{z,T}(\cdot,t)\cdot(\mathrm{n}_{i,j} - \xi_{i,j}^{z,T})(\cdot,t)\,d\mathcal{H}^1 \end{aligned}$$

$$+\int_{\mathbb{R}^2}(\chi_i-\bar{\chi}_i^{z,T})(\cdot,t)\vartheta_i^{z,T}(\cdot,t)\nabla\cdot B^{z,T}(\cdot,t)\,dx$$
$$+\int_{\mathbb{R}^2}(\chi_i-\bar{\chi}_i^{z,T})(\cdot,t)\Big(\partial_t\vartheta_i^{z,T}+(B^{z,T}\cdot\nabla)\vartheta_i^{z,T}\Big)(\cdot,t)\,dx.$$

The next two steps of our strategy are concerned with the construction of the input data for Lemma 52: first (t_χ, z, T) and second $((\xi_i)_{i=1,\ldots,P}, (\vartheta_i)_{i=1,\ldots,P-1}, B)$.

4.2.4 Construction of dynamic shifts

In case of a single closed curve, a characteristic length scale associated with the evolution of $\bar{\chi}$ is given by $r(t) := \sqrt{\frac{\mathrm{vol}(\{\bar{\chi}_1(\cdot,t)=1\})}{\pi}}$, $t\in[0,T_{ext})$. Since

$$\frac{d}{dt}\mathrm{vol}(\{\bar{\chi}(\cdot,t)=1\}) = -2\pi$$

we infer that

$$\begin{cases}\dot{r}(t) = -\frac{1}{r(t)}, & t\in(0,T_{ext}),\\ r(0)=r_0 := \sqrt{\frac{\mathrm{vol}(\{\bar{\chi}_1(\cdot,0)=1\})}{\pi}}, & \end{cases} \tag{4.60}$$

Hence, $r(t)=\sqrt{2(T_{ext}-t)}$ and $T_{ext}=\frac{1}{2}r_0^2$.

In Subsection 4.2.2, we already derived the defining ODEs for $(z, T=\mathrm{id}+\mathfrak{T})$, at least in a regime where the weak solution is represented as a sufficiently regular graph over the smooth solution, cf. (4.46). Of course, there is no guarantee to be in that regime for all times, so that the general construction needs a robust version of (4.46). To this end, it is convenient to work with the notion of interface error heights.

Construction 53 (Interface error heights)**.** *Let (t_χ, z, T) be as in Lemma 52. Let $\zeta\colon\mathbb{R}\to[0,1]$ be a smooth cutoff function such that $\zeta(s)=1$ for $|s|\le 1/(16C_\zeta)$ and $\zeta(s)=0$ for $|s|>1/(8C_\zeta)$, where $C_\zeta\in[1,\infty)$ is a given constant. We then define* interface error heights

$$\rho(\cdot,\cdot;z,T),\rho_\pm(\cdot,\cdot;z,T)\colon \bigcup_{t\in[0,t_\chi)}\bar{I}^{z,T}(t)\times\{t\}\to\mathbb{R}$$

through a slicing construction:

$$\rho_+(x,t;z,T) := \int_0^{\frac{1}{2}r_T(t)}(\bar{\chi}_1^{z,T}-\chi_1)\Big(x+\ell\mathrm{n}_{\bar{I}^{z,T}}(\cdot,t),t\Big)\zeta\Big(\frac{\ell}{r_T(t)}\Big)\,d\ell, \tag{4.61}$$
$$\rho_-(x,t;z,T) := \int_{-\frac{1}{2}r_T(t)}^0(\chi_1-\bar{\chi}_1^{z,T})\Big(x+\ell\mathrm{n}_{\bar{I}^{z,T}}(\cdot,t),t\Big)\zeta\Big(\frac{\ell}{r_T(t)}\Big)\,d\ell, \tag{4.62}$$
$$\rho(x,t;z,T) := \rho_+(x,t;z,T)-\rho_-(x,t;z,T). \tag{4.63}$$

We have everything in place to construct (t_χ, z, T).

Lemma 54 (Existence of space-time shifts)**.** *There exists a unique choice of*

- *a time horizon $t_\chi>0$,*
- *a path of translations $z\in W^{1,\infty}_{loc}((0,t_\chi);\mathbb{R}^2)$, and*

- *a map* $\mathfrak{T} \in W^{1,\infty}((0,t_\chi);(0,T_{ext}))$, $T := \mathrm{id} + \mathfrak{T}$ *strictly increasing,*

which in addition satisfy $(z(0), \mathfrak{T}(0)) = (0,0)$,

$$t_\chi := \sup\{t : T(t) < \tfrac{1}{2}r_0^2\}, \tag{4.64}$$

as well as

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\mathfrak{T}}(t) \end{bmatrix} = \begin{bmatrix} \frac{6}{r_T^2(t)} \fint_{\bar{I}^{z,T}(t)} \rho(\cdot,t;z,T)\mathrm{n}_{\bar{I}^{z,T}}(\cdot,t)\,d\mathcal{H}^1 \\ \frac{4}{r_T(t)} \fint_{\bar{I}^{z,T}(t)} \rho(\cdot,t;z,T)\,d\mathcal{H}^1 \end{bmatrix}, \quad t \in (0,t_\chi). \tag{4.65}$$

In particular, for given $\delta_{\mathrm{err}} \in (0,\frac{1}{2})$ *one may choose the constant* $C_\zeta \gg_{\delta_{\mathrm{err}}} 1$ *from Construction 53 such that*

$$|\dot{z}(t)| \leq \delta_{\mathrm{err}}\frac{1}{r_T(t)}, \quad |\dot{\mathfrak{T}}(t)| \leq \delta_{\mathrm{err}}, \quad t \in (0,t_\chi). \tag{4.66}$$

The proof of Lemma 54 is given in Section 4.5.1.

4.2.5 Construction of gradient flow calibrations

In contrast to [46], in the present work the smoothly evolving solution $\bar{\chi}$ stems from a simple two-phase geometry instead of a more complicated multiphase geometry with branching interfaces. As a consequence, the construction of a gradient flow calibration (cf. [46, Definition 2 and Definition 4]) is particularly simple and can be given directly as follows.

Construction 55 (Gradient flow calibration up to extinction time)**.** *Consider a smooth cutoff function* $\eta\colon \mathbb{R} \to [0,1]$ *such that* $\eta(s) = 1$ *for* $|s| \leq 1/8$, $\eta(s) = 0$ *for* $|s| \geq 1/4$ *and* $\|\eta'\|_{L^\infty(\mathbb{R})} \leq 16$. *We then define an extension* $\xi\colon \mathbb{R}^2\times[0,T_{ext}) \to \mathbb{R}^2$ *of the unit vector field* $\bar{\mathrm{n}}$ *by means of*

$$\xi(x,t) = \eta\Big(\frac{\mathrm{sdist}_{\bar{I}}(x,t)}{r(t)}\Big)\mathrm{n}_{\bar{I}}\Big(P_{\bar{I}}(x,t),t\Big), \quad (x,t) \in \mathbb{R}^2\times[0,T_{ext}). \tag{4.67}$$

Based on this auxiliary construction, we may now introduce families of vector fields $(\xi_i)_{i=1,\dots,P}$ *and* $(\xi_{i,j})_{i,j\in\{1,\dots,P\},i\neq j}$ *(defined as maps* $\mathbb{R}^2\times[0,T_{ext}) \to \mathbb{R}^2$*) by the following simple procedure.*

- $\xi_{i,j} := \xi_i - \xi_j$ *for any* $i,j \in \{1,\dots,P\}$, $i \neq j$.
- $\xi_i \equiv 0$ *for all* $i \notin \{1,P\}$.
- $\xi_1 := -\frac{1}{2}\xi$ *and* $\xi_P := \frac{1}{2}\xi$.

Furthermore, we define an extension $B\colon \mathbb{R}^2\times[0,T_{ext}) \to \mathbb{R}^2$ *of the normal velocity field* $H_{\bar{I}}\mathrm{n}_{\bar{I}}$ *of* $\bar{\chi}$ *through*

$$B(x,t) := \eta\Big(\frac{\mathrm{sdist}_{\bar{I}}(x,t)}{r(t)}\Big)(H_{\bar{I}}\mathrm{n}_{\bar{I}})\Big(P_{\bar{I}}(x,t),t\Big), \quad (x,t) \in \mathbb{R}^2\times[0,T_{ext}). \tag{4.68}$$

Finally, for the construction of the family $(\vartheta_i)_{i=1,\dots,P-1}$ *(defined as functions mapping* $\mathbb{R}^2\times[0,T_{ext}) \to [-1,1]$*), we proceed as follows. Let* $\bar{\vartheta}\colon \mathbb{R} \to [-1,1]$ *be a smooth function such that* $\bar{\vartheta}(s) = -s$ *for* $|s| \leq 1/4$, $\bar{\vartheta}(s) = -1$ *for* $s \geq 1/2$, $\bar{\vartheta}(s) = 1$ *for* $s \leq -1/2$, *and* $\|\bar{\vartheta}'\|_{L^\infty(\mathbb{R})} \leq 4$. *We then define*

$$\vartheta(x,t) := \frac{1}{r(t)}\bar{\vartheta}\Big(\frac{\mathrm{sdist}_{\bar{I}}(x,t)}{r(t)}\Big), \quad (x,t) \in \mathbb{R}^2\times[0,T_{ext}), \tag{4.69}$$

and, at last,

- $\vartheta_1 := \vartheta$,
- $\vartheta_i := 1$ *for all* $i \notin \{1,P\}$.

Note that the gradient flow calibration $((\xi_i)_{i=1,\dots,P}, (\vartheta_i)_{i=1,\dots,P-1}, B)$ from Construction 55 is an admissible input for Lemma 52. From now on, whenever we refer to an admissible element from either (t_χ, z, T) or $((\xi_i)_{i=1,\dots,P}, (\vartheta_i)_{i=1,\dots,P-1}, B)$, we always mean their specific realizations provided by Lemma 54 or Construction 55, respectively.

4.2.6 Time splitting: Regular vs. non-regular times

With the input for Lemma 52 being constructed, the main remaining major task is to upgrade the preliminary stability estimates (4.58) and (4.59) to the decay estimate (4.5) for the overall error. The main idea here is to reduce the whole estimation strategy to a regime where the weak solution χ is only a small perturbation of $\bar{\chi}$, for which we in turn already formally identified the leading-order contributions to the stability estimates in Subsections 4.2.1–4.2.2.

Definition 56 (Regular and non-regular times)**.** *Fix* $\Lambda > 0$. *We then define a disjoint decomposition*

$$(0,t_\chi) = \mathcal{T}_{\text{non-reg}}(\Lambda) \cup \mathcal{T}_{\text{reg}}(\Lambda)$$

such that

$$\mathcal{T}_{\text{non-reg}}(\Lambda) := \bigg\{ t \in (0,t_\chi) : \sum_{i,j=1,i\neq j}^{P} \int_{I_{i,j}(t)} \frac{1}{2}|V_{i,j}(\cdot,t)|^2 \, d\mathcal{H}^1 \geq \Lambda\frac{2\pi}{r_T(t)} \bigg\}. \tag{4.70}$$

The motivation behind the previous definition is as follows. On one side, for non-regular times, the right hand sides of the preliminary stability estimates (4.58) and (4.59) turn out to be easily estimated thanks to the defining condition of disproportionally large dissipation of the weak solution, cf. (4.70). On the other side, the opposite of (4.70) together with a smallness assumption on the overall error (consistent with the decay (4.5)) imply for regular times the desired perturbative setting. The latter is formalized in the following result.

Proposition 57 (Perturbative regime at regular times)**.** *Fix* $\Lambda > 0$ *and let* $t \in \mathcal{T}_{\text{reg}}(\Lambda)$, *i.e.,* $t \in (0,t_\chi)$ *such that*

$$\sum_{i,j=1,i\neq j}^{P} \int_{I_{i,j}(t)} \frac{1}{2}|V_{i,j}(\cdot,t)|^2 \, d\mathcal{H}^1 < \Lambda\frac{2\pi}{r_T(t)}. \tag{4.71}$$

Given $C_\zeta \geq 1$ *from Construction 53 and given any* $C, C' \geq 1$, *there exists a constant* $\delta \ll_{\Lambda,C,C',C_\zeta} \frac{1}{2}$ *such that*

$$E[\chi|\bar{\chi}^{z,T}](t) \leq \delta r_T(t) \tag{4.72}$$

implies:

- $\chi_i(\cdot,t) \equiv 0$ *for all* $i \notin \{1,P\}$.
- *There exists a* height function

$$h(\cdot,t) \in H^2(\bar{I}^{z,T}(t)) \tag{4.73}$$

such that the only remaining interface is given by

$$I_{1,P}(t) = \Big\{x \in \bar{I}^{z,T}(t) \colon x + h(x,t)\mathrm{n}_{\bar{I}^{z,T}}(x,t)\Big\}. \tag{4.74}$$

- *Finally, it holds*

$$\|h(\cdot,t)\|_{L^\infty(\bar{I}^{z,T}(t))} \leq \frac{r_T(t)}{16\max\{C, C_\zeta\}}, \tag{4.75}$$

$$\|h'(\cdot,t)\|_{L^\infty(\bar{I}^{z,T}(t))} \leq \frac{1}{C'}. \tag{4.76}$$

In particular, the height function $h(\cdot,t)$ *coincides with the interface error height* $\rho(\cdot,t;z,T)$ *from Construction 53 and* (4.65) *simply reads*

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\mathfrak{T}}(t) \end{bmatrix} = \begin{bmatrix} \frac{6}{r_T^2(t)} \fint_{\bar{I}^{z,T}(t)} h(\cdot,t)\mathrm{n}_{\bar{I}^{z,T}}(\cdot,t)\,d\mathcal{H}^1 \\ \frac{4}{r_T(t)} \fint_{\bar{I}^{z,T}(t)} h(\cdot,t)\,d\mathcal{H}^1 \end{bmatrix}. \tag{4.77}$$

In the perturbative regime of Proposition 57, our error functionals take the following form.

Lemma 58 (Error functionals in perturbative regime)**.** *Fix* $t \in (0,t_\chi)$ *and assume that the conclusions of Proposition 57 hold true. Given* $\delta_{\mathrm{err}} \in (0,1)$*, one may select* $C, C' \gg_{\delta_{\mathrm{err}}} 1$ *from* (4.75)–(4.76) *such that*

$$\begin{aligned} &(1-\delta_{\mathrm{err}}) \int_{\bar{I}^{z,T}(t)} \frac{1}{2}\Big(\frac{h(\cdot,t)}{r_T(t)}\Big)^2 d\mathcal{H}^1 \\ &\leq E_{\mathrm{bulk}}[\chi|\bar{\chi}^{z,T}](t) \leq (1+\delta_{\mathrm{err}}) \int_{\bar{I}^{z,T}(t)} \frac{1}{2}\Big(\frac{h(\cdot,t)}{r_T(t)}\Big)^2 d\mathcal{H}^1 \end{aligned} \tag{4.78}$$

as well as

$$\begin{aligned} &(1-\delta_{\mathrm{err}}) \int_{\bar{I}^{z,T}(t)} \frac{1}{2}|h'(\cdot,t)|^2\,d\mathcal{H}^1 \\ &\leq E_{\mathrm{rel}}[\chi|\bar{\chi}^{z,T}](t) \leq (1+\delta_{\mathrm{err}}) \int_{\bar{I}^{z,T}(t)} \frac{1}{2}|h'(\cdot,t)|^2\,d\mathcal{H}^1. \end{aligned} \tag{4.79}$$

The proofs of Proposition 57 and of Lemma 58 are given in Section 4.6.1 and in Section 4.6.2, respectively.

4.2.7 Stability estimates at non-regular times

In a next step, we take care of the estimation of the right hand sides of (4.58) and (4.59) in the case of disproportionally large dissipation.

Lemma 59. *There exist* $\Lambda \gg_{\delta,\delta_{\mathrm{asymp}}} 1$ *as well as* $\delta, \delta_{\mathrm{asymp}} \ll \frac{1}{2}$ *such that for every* $t \in \mathcal{T}_{\text{non-reg}}(\Lambda)$ *satisfying* (4.12) *and* $E[\chi|\bar{\chi}^{z,T}](t) \leq \delta r_T(t)$ *it holds*

$$\begin{aligned}\sum_{i,j=1,i\neq j}^{P} \frac{1}{2}\Big(- \mathcal{D}_{i,j}[\chi|\bar{\chi}^{z,T}](t) + RHS^{\mathrm{int}}_{i,j}[\chi|\bar{\chi}^{z,T}](t)\Big) + \sum_{i=1}^{P-1} RHS^{\mathrm{bulk}}_{i}[\chi|\bar{\chi}^{z,T}](t) \\ \leq -\frac{1}{2}\sum_{i,j=1,i\neq j}^{P}\int_{I_{i,j}(t)} \frac{1}{2}|V_{i,j}(\cdot,t)|^2\,d\mathcal{H}^1.\end{aligned} \tag{4.80}$$

We may easily post-process the estimate (4.80) to an estimate in terms of our error functional consistent with the final decay estimate (4.5).

Corollary 60. *There exist* $\Lambda \gg_{\delta,\delta_{\mathrm{asymp}}} 1$ *as well as* $\delta, \delta_{\mathrm{asymp}} \ll \frac{1}{2}$ *such that for every* $t \in \mathcal{T}_{\text{non-reg}}(\Lambda)$ *satisfying* (4.12) *and* $E[\chi|\bar{\chi}^{z,T}](t) \leq \delta r_T(t)$ *it holds*

$$\begin{aligned}\sum_{i,j=1,i\neq j}^{P} \frac{1}{2}\Big(- \mathcal{D}_{i,j}[\chi|\bar{\chi}^{z,T}](t) + RHS^{\mathrm{int}}_{i,j}[\chi|\bar{\chi}^{z,T}](t)\Big) + \sum_{i=1}^{P-1} RHS^{\mathrm{bulk}}_{i}[\chi|\bar{\chi}^{z,T}](t) \\ \leq -\frac{5}{r_T^2(t)}E[\chi|\bar{\chi}^{z,T}](t).\end{aligned} \tag{4.81}$$

The proofs of Lemma 59 and of Corollary 60 can be found in Section 4.4.2.

4.2.8 Stability estimates for perturbative regime

We proceed with the estimation of the right hand sides of (4.58) and (4.59) in the perturbative regime described by Proposition 57. We first derive the version of the stability estimate (4.44) without making use of the assumption on $\bar{\chi}$ being quantitatively close to a shrinking circle. The derivation of these estimates, namely the proofs of the following lemmas, are contained in Sections 4.4.3-4.4.4-4.4.5 .

Lemma 61 (Stability estimate in perturbative setting: variable coefficients)**.** *Fix* $t \in (0, t_\chi)$ *and assume that the conclusions of Proposition 57 hold true. Given* $\delta_{\mathrm{err}} \in (0,1)$*, one may choose the constants* $C, C' \gg_{\delta_{\mathrm{err}}} 1$ *from* (4.75)–(4.76) *such that*

$$\begin{aligned}\sum_{i,j=1,i\neq j}^{P} \frac{1}{2}\Big(- \mathcal{D}_{i,j}[\chi|\bar{\chi}^{z,T}](t) + \frac{1}{2}RHS^{\mathrm{int}}_{i,j}[\chi|\bar{\chi}^{z,T}](t)\Big) \\ + \sum_{i=1}^{P-1} RHS^{\mathrm{bulk}}_{i}[\chi|\bar{\chi}^{z,T}](t) \leq R_{l.o.t.} + R_{h.o.t.},\end{aligned} \tag{4.82}$$

where the leading order terms are given by

$$\begin{aligned}R_{l.o.t.} := &-\int_{\bar{I}^{z,T}(t)} (h'')^2(\cdot,t)\,d\mathcal{H}^1 \\ &+ \int_{\bar{I}^{z,T}(t)} \Big(\frac{3}{2}H^2_{\bar{I}^{z,T}}(\cdot,t) - \frac{1}{r_T^2(t)}\Big)(h')^2(\cdot,t)\,d\mathcal{H}^1 \\ &+ \int_{\bar{I}^{z,T}(t)} \frac{1}{r_T^2(t)}\Big(\frac{1}{2}H^2_{\bar{I}^{z,T}}(\cdot,t) + \frac{1}{r_T^2(t)}\Big)h^2(\cdot,t)\,d\mathcal{H}^1 \\ &- \int_{\bar{I}^{z,T}(t)} \Big(\frac{1}{r_T^2(t)} + H^2_{\bar{I}^{z,T}}(\cdot,t)\Big)h(\cdot,t)\mathrm{n}_{\bar{I}^{z,T}}(\cdot,t)\cdot\dot{z}(t)\,d\mathcal{H}^1\end{aligned}$$

$$
\begin{aligned}
&- \int_{\bar{I}^{z,T}(t)} \frac{1}{r_T^2(t)} H_{\bar{I}^{z,T}}(\cdot,t) h(\cdot,t) \dot{\mathfrak{T}}(t) \, d\mathcal{H}^1 \\
&- \int_{\bar{I}^{z,T}(t)} H'_{\bar{I}^{z,T}}(\cdot,t) \Big(\tau_{\bar{I}^{z,T}}(\cdot,t) \cdot \dot{z}(t) \Big) h(\cdot,t) \, d\mathcal{H}^1 \\
&- \int_{\bar{I}^{z,T}(t)} H'_{\bar{I}^{z,T}}(\cdot,t) \dot{\mathfrak{T}}(t) h'(\cdot,t) \, d\mathcal{H}^1 \\
&+ \int_{\bar{I}^{z,T}(t)} 2 H_{\bar{I}^{z,T}}(\cdot,t) H'_{\bar{I}^{z,T}}(\cdot,t) h(\cdot,t) h'(\cdot,t) \, d\mathcal{H}^1
\end{aligned}
$$

and the higher order terms are given by

$$
\begin{aligned}
R_{h.o.t.} := {}& \delta_{\mathrm{err}} \int_{\bar{I}^{z,T}(t)} (h'')^2(\cdot,t) \, d\mathcal{H}^1 \\
&+ \delta_{\mathrm{err}} \int_{\bar{I}^{z,T}(t)} \Big(\frac{1}{r_T^2(t)} + \Big| H'_{\bar{I}^{z,T}} \Big| (\cdot,t) \Big) (h')^2(\cdot,t) \, d\mathcal{H}^1 \\
&+ \delta_{\mathrm{err}} \int_{\bar{I}^{z,T}(t)} \Big(\frac{1}{r_T^4(t)} + \Big(H'_{\bar{I}^{z,T}} \Big)^2 (\cdot,t) \Big) h^2(\cdot,t) \, d\mathcal{H}^1 \\
&+ \delta_{\mathrm{err}} \int_{\bar{I}^{z,T}(t)} \frac{1}{r_T(t)} \Big| h'(\cdot,t) \tau_{\bar{I}^{z,T}} \cdot \dot{z} \Big| + \frac{1}{r_T^2(t)} \Big| h(\cdot,t) \mathrm{n}_{\bar{I}^{z,T}} \cdot \dot{z} \Big| \, d\mathcal{H}^1 \\
&+ \delta_{\mathrm{err}} \int_{\bar{I}^{z,T}(t)} \frac{1}{r_T^3(t)} \Big| h(\cdot,t) \dot{\mathfrak{T}} \Big| + \Big| H'_{\bar{I}^{z,T}}(\cdot,t) h'(\cdot,t) \dot{\mathfrak{T}} \Big| \, d\mathcal{H}^1.
\end{aligned}
$$

In a second step, we post-process the previous estimate (4.82) to the constant-coefficient estimate (4.44). In PDE jargon, this amounts to nothing else than a freezing of coefficients, only exploiting the estimates from Definition 50.

Lemma 62 (Stability estimate in perturbative setting: frozen coefficients)**.** *Fix $t \in (0, t_\chi)$, assume that the conclusions of Proposition 57 hold true, and define $\widetilde{h}(\cdot,t) \colon [0,2\pi) \to \mathbb{R}$, $\theta \mapsto h\Big(\bar{\gamma}^{z,T}\Big(\frac{L_{\bar{\gamma}^{z,T}}}{2\pi}\theta, t\Big), t\Big)$. Given $\delta_{\mathrm{err}} \in (0,1)$, one may choose the constants $C, C' \gg_{\delta_{\mathrm{err}}} 1$ from* (4.75)–(4.76) *as well as the constant $\delta_{\mathrm{asymp}} \ll_{\delta_{\mathrm{err}}} \frac{1}{2}$ from Definition 50 such that*

$$
\begin{aligned}
\sum_{i,j=1, i \neq j}^{P} \frac{1}{2} \Big(- \mathcal{D}_{i,j}[\chi|\bar{\chi}^{z,T}](t) + \frac{1}{2} RHS_{i,j}^{\mathrm{int}}[\chi|\bar{\chi}^{z,T}](t) \Big)& \\
+ \sum_{i=1}^{P-1} RHS_i^{\mathrm{bulk}}[\chi|\bar{\chi}^{z,T}](t) \leq \widetilde{R}_{l.o.t.} + \widetilde{R}_{h.o.t.},&
\end{aligned}
\tag{4.83}
$$

where the leading order terms are given by

$$
\begin{aligned}
\widetilde{R}_{l.o.t.} := {}& -\frac{1}{r_T^3(t)} \int_0^{2\pi} (\partial_\theta^2 \widetilde{h})^2(\cdot,t) - \frac{1}{2} (\partial_\theta \widetilde{h})^2(\cdot,t) - \frac{3}{2} \widetilde{h}^2(\cdot,t) \, d\theta \\
&- 4 \frac{1}{r_T^3(t)} \Big(\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \widetilde{h}(\cdot,t) \, d\theta \Big)^2 - 6 \frac{1}{r_T^3(t)} \Big| \frac{1}{\sqrt{\pi}} \int_0^{2\pi} \widetilde{h}(\cdot,t) e^{i\theta} \, d\theta \Big|^2
\end{aligned}
$$

and the higher order term is simply given by

$$
\widetilde{R}_{h.o.t.} := \delta_{\mathrm{err}} \frac{1}{r_T^3(t)} \int_0^{2\pi} (\partial_\theta^2 \widetilde{h})^2(\cdot,t) + \frac{1}{2} (\partial_\theta \widetilde{h})^2(\cdot,t) + \frac{3}{2} \widetilde{h}^2(\cdot,t) \, d\theta.
$$

Since we reduced matters to the constant coefficient case, we may now in a third step employ Fourier methods to obtain in the perturbative regime a stability estimate consistent with the decay estimate (4.5).

Lemma 63 (Final stability estimate in perturbative setting)**.** *Fix $t \in (0, t_\chi)$ and assume that the conclusions of Proposition 57 hold true. Given $\alpha \in (1, 5)$, one may choose the constants $C, C' \gg_\alpha 1$ from* (4.75)–(4.76)*, the constant $\delta_{\mathrm{asymp}} \ll_\alpha \frac{1}{2}$ from Definition 50, and the constant $C_\zeta \gg_\alpha 1$ from Construction 53 such that*

$$\begin{aligned}\sum_{i,j=1, i\neq j}^{P} \frac{1}{2}\Big(- \mathcal{D}_{i,j}[\chi|\bar{\chi}^{z,T}](t) + \frac{1}{2} RHS^{\mathrm{int}}_{i,j}[\chi|\bar{\chi}^{z,T}](t)\Big) + \sum_{i=1}^{P-1} RHS^{\mathrm{bulk}}_{i}[\chi|\bar{\chi}^{z,T}](t) \\ \leq -\frac{\alpha}{r_T^2(t)}(1+\dot{\mathfrak{T}})E[\chi|\bar{\chi}^{z,T}](t).\end{aligned} \tag{4.84}$$

4.2.9 A priori stability estimate up to extinction time

The penultimate step of our strategy is simply a summary of our estimates from Subsections 4.2.6–4.2.8.

Theorem 64. *Fix a decay exponent $\alpha \in (1, 5)$ and a time $\widetilde{t}_\chi \in (0, t_\chi)$. One may choose the constant $C_\zeta \gg_\alpha 1$ from Construction 53 as well as constants $\delta_{\mathrm{asymp}} \ll_\alpha \frac{1}{2}$ and $\delta \ll_{\alpha, C_\zeta} \frac{1}{2}$ (all independent of $\widetilde{t}_\chi$) such that if for all $t \in (0, T_{ext})$ the interior of $\{\bar{\chi}_1(\cdot, t){=}1\} \subset \mathbb{R}^2$ is δ_{asymp}-close to a circle with radius $r(t) := \sqrt{2(T_{ext}{-}t)}$ in the sense of Definition 50 and*

$$E[\chi|\bar{\chi}^{z,T}](t) \leq \delta r_T(t) \text{ for all } t \in [0, \widetilde{t}_\chi), \tag{4.85}$$

then it holds for all $[s, \tau] \subset [0, \widetilde{t}_\chi)$

$$E[\chi|\bar{\chi}^{z,T}](\tau) + \int_s^\tau \frac{\alpha}{r_T^2(t)}\big(1+\dot{\mathfrak{T}}(t)\big)E[\chi|\bar{\chi}^{z,T}](t)\,dt \leq E[\chi|\bar{\chi}^{z,T}](s). \tag{4.86}$$

The unconditional decay estimate (4.5) from Theorem 48 now follows by means of a simple ODE argument (cf. Section 4.3). The asserted estimates (4.3)–(4.4) on the space-time shifts are in turn the content of the following result.

Lemma 65. *In the setting of Theorem 64, one may choose the constants such that assumption* (4.85) *implies*

$$\frac{1}{r_0}\|z\|_{L^\infty_t(0,t_\chi)} \leq \sqrt{\frac{1}{r_0}E[\chi_0|\bar{\chi}_0]}, \tag{4.87}$$

$$\frac{1}{T_{ext}}\|T - \mathrm{id}\|_{L^\infty_t(0,t_\chi)} \leq \sqrt{\frac{1}{r_0}E[\chi_0|\bar{\chi}_0]}. \tag{4.88}$$

4.3 Proof of Theorem 48

We proceed in two steps. For the whole proof, fix $\alpha \in (1, 5)$, and choose $C_\zeta \gg_\alpha 1$, $\delta_{\mathrm{asymp}} \ll_\alpha \frac{1}{2}$ and $\delta \ll_{\alpha, C_\zeta} \frac{1}{2}$ such that Theorem 64 applies. We then also fix an auxiliary constant $\kappa \in (0, \delta r_0)$.

Step 1: Post-processed a priori stability estimate. As the conclusion of Theorem 64 holds true, we may deduce that

$$E[\chi|\bar{\chi}^{z,T}](t) \leq \Big(E[\chi_0|\bar{\chi}_0]+\kappa\Big)\Big(\frac{r_T(t)}{r_0}\Big)^{\alpha} =: e(t) \text{ for any } \kappa > 0, \text{ for all } t \in (0,t_\chi). \quad (4.89)$$

Indeed, it is not hard to show that $(0,t_\chi) \ni t \mapsto E[\chi|\bar{\chi}^{z,T}](t) \in [0,\infty)$ is absolutely continuous due to the conditions satisfied by χ being a BV solution for multiphase mean curvature flow in the sense of Definition 49, and the identity

$$\begin{aligned} E[\chi|\bar{\chi}^{z,T}](t) = E[\chi(\cdot,t)] &- \sum_{i=1}^{P}\int_{\mathbb{R}^2}\chi_i(\cdot,t)(\nabla\cdot\xi^{z,T})(\cdot,t)\,dx \\ &+ \sum_{i=1}^{P-1}\int_{\mathbb{R}^2}(\chi_i-\bar{\chi}_i^{z,T})(\cdot,t)\vartheta_i^{z,T}(\cdot,t)\,dx. \end{aligned}$$

As a consequence, we infer from (4.86) that $\frac{d}{dt}E[\chi|\bar{\chi}^{z,T}] \leq -\frac{\alpha}{r_T^2}(1+\dot{\mathfrak{T}})E[\chi|\bar{\chi}^{z,T}]$ a.e. in $(0,t_\chi)$. Since also $\frac{d}{dt}e = -\frac{\alpha}{r_T^2}(1+\dot{\mathfrak{T}})e$, we obtain $\frac{d}{dt}\frac{E[\chi|\bar{\chi}^{z,T}]}{e} \leq 0$ a.e. in $(0,t_\chi)$. Hence, by absolute continuity of the ratio $E[\chi|\bar{\chi}^{z,T}]/e$, the claim (4.89) holds true.

Step 2: Proof of (4.5) *under assumption* (4.1). Define

$$\mathcal{T} := \Big\{t \in (0,t_\chi)\colon E[\chi|\bar{\chi}^{z,T}](t) > e(t)\Big\}, \quad (4.90)$$

and we argue in favor of (4.5) by contradiction. Hence, we assume $\mathcal{T} \neq \emptyset$ and define $\widetilde{t}_\chi := \inf\mathcal{T} \in [0,t_\chi)$. Since $E[\chi_0|\bar{\chi}_0^{z,T}] < e(0)$, it is not hard to show that $\widetilde{t}_\chi \neq 0$. Then, by construction and hypothesis (4.1), we observe that assumption (4.85) is in place for all $t \in [0,\widetilde{t}_\chi)$. In other words, the estimate (4.89) applies on $(0,t_\chi)$ so that continuity of the ratio $E[\chi|\bar{\chi}^{z,T}]/e$ contradicts our assumption $\mathcal{T} \neq \emptyset$. Hence, $\mathcal{T} = \emptyset$ and taking the limit $\kappa \downarrow 0$ implies the decay estimate (4.5). Finally, the bounds (4.3) and (4.4) follow from Lemma 65. □

4.4 Weak-strong stability estimates

4.4.1 Proof of Lemma 52: Preliminary stability estimate

For a proof of (4.58) and (4.59), we refer the interested reader to the proofs of [46, Proposition 17] and [46, Lemma 20]. □

4.4.2 Proof of Lemma 59 and Corollary 60: Stability at non-regular times

Before we turn to the proofs of Lemma 59 and Corollary 60, respectively, we start with a useful auxiliary result.

Lemma 66. *Consider gradient flow calibration $((\xi_i)_{i=1,\dots,P},(\vartheta_i)_{i=1,\dots,P-1},B)$ from Construction 55 and recall that $\xi_{i,j} := \xi_i - \xi_j$ for all distinct $i,j \in \{1,\dots,P\}$. There exists a universal constant $\widetilde{C} \in [1,\infty)$ such that for all $t \in [0,t_\chi)$ and all $i,j \in \{1,\dots,P\}$ with $i \neq j$, it holds*

$$\Big\|\Big(\partial_t\xi_{i,j}^{z,T}\Big)(\cdot,t)\Big\|_{L^\infty(\mathbb{R}^2)} \leq \frac{\widetilde{C}}{r_T^2(t)}, \quad (4.91)$$

$$\left\|\left(\nabla\cdot\xi^{z,T}_{i,j}\right)(\cdot,t)\right\|_{L^\infty(\mathbb{R}^2)}\leq\frac{\widetilde{C}}{r_T(t)},\tag{4.92}$$

$$\left\|\left(\partial_t\vartheta^{z,T}_{i}\right)(\cdot,t)\right\|_{L^\infty(\mathbb{R}^2)}\leq\frac{\widetilde{C}}{r_T^3(t)}.\tag{4.93}$$

Proof of Lemma 66. Fix $t\in[0,t_\chi)$ and $i,j\in\{1,\dots,P\}$ with $i\neq j$. Recalling Construction 55, we observe that $\xi_{i,j}\in\{\pm\xi,\pm\frac{1}{2}\xi,0\}$, so that it suffices to estimate in terms of the vector field ξ. Recalling its definition (4.67) in the form of

$$\begin{aligned}\xi^{z,T}(\cdot,t)&=\eta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\mathrm{n}_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big)\\&=\eta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\Big(\nabla\,\mathrm{sdist}_{\bar I^{z,T}}\Big)(\cdot,t),\end{aligned}\tag{4.94}$$

we directly compute

$$\begin{aligned}\Big(\nabla\cdot\xi^{z,T}\Big)(\cdot,t)&=\frac{1}{r_T(t)}\eta'\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\\&\quad-\eta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{H_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big)}{1-H_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big)\,\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)},\end{aligned}$$

so that (4.92) follows from $|\eta'|\leq 16$, $|H_{\bar I}(\cdot,t)|\leq 2/r(t)$ and $|\,\mathrm{sdist}_{\bar I}(\cdot,t)|\leq r(t)/4$ on $\operatorname{supp}\xi(\cdot,t)$, $t\in[0,T_{ext})$.

Furthermore, since

$$\partial_t s_{\bar I^{z,T}}(\cdot,t)=-H_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big)(1+\dot{\mathfrak{T}})-\mathrm{n}_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big)\cdot\dot z,\tag{4.95}$$

which itself one may either directly read off from the obvious generalization of (4.37) or alternatively from (4.17), we also get

$$\begin{aligned}\Big(\partial_t\xi^{z,T}\Big)(\cdot,t)&=\eta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\Big(\nabla\partial_t\,\mathrm{sdist}_{\bar I^{z,T}}\Big)(\cdot,t)\\&\quad+\frac{1}{r_T(t)}\eta'\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\Big(\partial_t\,\mathrm{sdist}_{\bar I^{z,T}}\Big)(\cdot,t)\mathrm{n}_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big)\\&\quad+\eta'\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T^3(t)}(1+\dot{\mathfrak{T}})\mathrm{n}_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big)\\&=-\eta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{H'_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big)(1+\dot{\mathfrak{T}})}{1-H_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big)\,\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}\\&\quad+\eta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{H_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big)\tau_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big)\cdot\dot z}{1-H_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big)\,\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}\\&\quad+\frac{1}{r_T(t)}\eta'\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\Big(\partial_t\,\mathrm{sdist}_{\bar I^{z,T}}\Big)(\cdot,t)\mathrm{n}_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big)\\&\quad+\eta'\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T^3(t)}(1+\dot{\mathfrak{T}})\mathrm{n}_{\bar I^{z,T}}\Big(P_{\bar I^{z,T}}(\cdot,t),t\Big).\end{aligned}\tag{4.96}$$

Hence, (4.91) follows from (4.96), (4.95), (4.15), (4.66), and the estimates used for the derivation of (4.92).

Recalling Construction 55, we observe that $\vartheta_i \in \{\vartheta, 1\}$, so that it suffices to estimate in terms of the function ϑ. Recalling its definition (4.67) in the form of

$$\vartheta^{z,T}(\cdot,t) = \frac{1}{r_T(t)}\bar{\vartheta}\Big(\frac{\mathrm{sdist}_{\bar{I}^{z,T}}(\cdot,t)}{r_T(t)}\Big), \tag{4.97}$$

we obtain

$$\begin{aligned}\Big(\partial_t\vartheta^{z,T}\Big)(\cdot,t) &= \frac{(1+\dot{\mathfrak{T}})}{r_T^3(t)}\bar{\vartheta}\Big(\frac{\mathrm{sdist}_{\bar{I}^{z,T}}(\cdot,t)}{r_T(t)}\Big)\\ &\quad+\frac{1}{r_T(t)}\bar{\vartheta}'\Big(\frac{\mathrm{sdist}_{\bar{I}^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{\Big(\partial_t\,\mathrm{sdist}_{\bar{I}^{z,T}}\Big)(\cdot,t)}{r_T(t)}\\ &\quad+\frac{1}{r_T(t)}\bar{\vartheta}'\Big(\frac{\mathrm{sdist}_{\bar{I}^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{\mathrm{sdist}_{\bar{I}^{z,T}}(\cdot,t)}{r_T^3(t)}(1+\dot{\mathfrak{T}}),\end{aligned} \tag{4.98}$$

so that, based on the previous ingredients, we have $|\bar{\vartheta}'| \leq 2$ and $|\,\mathrm{sdist}_{\bar{I}}(\cdot,t)| \leq r(t)/4$ on $\mathrm{supp}\,\bar{\vartheta}'\Big(\mathrm{sdist}_{\bar{I}}(\cdot,t)/(r(t)/4)\Big)$, $t\in[0,T_{ext})$, and we may deduce (4.93). ☐

Lemma 67. *Fix $t \in (0,t_\chi)$. For every $\delta_{\mathrm{err}} \in (0,1)$ there exist $\delta, \delta_{\mathrm{asymp}} \ll_{\delta_{\mathrm{err}}} \frac{1}{2}$ such that* (4.12)-(4.15) *and the condition $E[\chi|\bar{\chi}^{z,T}](t) \leq \delta r_T(t)$ imply*

$$\sum_{i,j=1,i\neq j}^{P}\frac{1}{2}\int_{I_{i,j}(t)} 1\,\mathrm{d}\mathcal{H}^{d-1} \leq (1+\delta_{\mathrm{err}})\tilde{C}\pi r_T(t), \tag{4.99}$$

where $\widetilde{C} \in [0,\infty)$ is the constant from Lemma 66.

Proof. One can compute

$$\begin{aligned}&\sum_{i,j=1,i\neq j}^{P}\frac{1}{2}\int_{I_{i,j}(t)} 1\,\mathrm{d}\mathcal{H}^{d-1} = E_{\mathrm{int}}[\chi|\bar{\xi}^{z,T}](t) + \sum_{i=1}^{P}\int_{\mathbb{R}^d}\chi_i\nabla\cdot\xi_i^{z,T}\,\mathrm{d}x\\ &\leq \delta r_T(t) + \sum_{i=1}^{P}\int_{\mathbb{R}^d\cap\mathrm{supp}\,\eta}|\nabla\cdot\xi_i^{z,T}|\,\mathrm{d}x,\end{aligned}$$

whence we can deduce (4.99) using (4.92) together with (4.14). ☐

Proof of Lemma 59. Fix $t \in \mathcal{T}_{\text{non-reg}}(\Lambda)$ for yet to be chosen $\Lambda \gg 1$. For notational simplicity, let us in the sequel drop the dependence on t of all quantities. Since the definition of the error functionals is independent of the precise choice of the vector field B, we may interpret the right hand sides of (4.58) and (4.59) with $B\equiv 0$ and therefore obtain for all $i,j \in \{1,\ldots,P\}$ with $i\neq j$:

$$\begin{aligned}&-\mathcal{D}_{i,j}[\chi|\bar{\chi}^{z,T}] + RHS^{\mathrm{int}}_{i,j}[\chi|\bar{\chi}^{z,T}]\\ &= -\int_{I_{i,j}}|V_{i,j}|^2\,d\mathcal{H}^1 - \int_{I_{i,j}}V_{i,j}\nabla\cdot\xi_{i,j}^{z,T}\,d\mathcal{H}^1 - \int_{I_{i,j}}\mathrm{n}_{i,j}\cdot\partial_t\xi_{i,j}^{z,T}\,d\mathcal{H}^1\end{aligned} \tag{4.100}$$

and for all $i \in \{1,\ldots,P-1\}$

$$\begin{aligned}&RHS^{\mathrm{bulk}}_i[\chi|\bar{\chi}^{z,T}]\\ &= \int_{\mathbb{R}^2}(\chi_i - \bar{\chi}_i^{z,T})\partial_t\vartheta_i^{z,T}\,dx + \sum_{j=1,\,i\neq j}^{P}\int_{I_{i,j}}V_{i,j}\vartheta_i^{z,T}\,d\mathcal{H}^1.\end{aligned} \tag{4.101}$$

Before we start estimating the right hand sides of (4.100) and (4.101), we fix $\delta, \delta_{\mathrm{asymp}} \ll 1$ such that the conclusion of Lemma 67 applies for the choice $\delta_{\mathrm{err}} = \frac{1}{2}$.

From Hölder's inequality, (4.99) and (4.92), we then directly infer

$$\bigg| \int_{I_{i,j}} V_{i,j} \nabla \cdot \xi_{i,j}^{z,T} \, d\mathcal{H}^1 \bigg| \lesssim \frac{1}{\sqrt{r_T}} \bigg(\int_{I_{i,j}} |V_{i,j}|^2 \, d\mathcal{H}^1 \bigg)^{\frac{1}{2}}.$$

Similarly, we may estimate due to (4.91)

$$\bigg| \int_{I_{i,j}} \mathrm{n}_{i,j} \cdot \partial_t \xi_{i,j}^{z,T} \, d\mathcal{H}^1 \bigg| \lesssim \frac{1}{r_T}$$

and, since $|\vartheta_i^{z,T}| \leq 1/r_T$, also

$$\bigg| \int_{I_{i,j}} V_{i,j} \vartheta_i^{z,T} \, d\mathcal{H}^1 \bigg| \lesssim \frac{1}{\sqrt{r_T}} \bigg(\int_{I_{i,j}} |V_{i,j}|^2 \, d\mathcal{H}^1 \bigg)^{\frac{1}{2}}.$$

Finally, the estimates (4.99), (4.12) and (4.93) together with the isoperimetric inequality imply

$$\bigg| \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i^{z,T}) \partial_t \vartheta_i^{z,T} \, dx \bigg| \lesssim \frac{1}{r_T}.$$

Plugging these estimates back into (4.100) and (4.101), we may infer the claim (4.80) from employing the defining condition (4.70) of non-regular times for $\Lambda \gg 1$. □

Proof of Corollary 60. Denote by $\widetilde{\Lambda}$ and $(\widetilde{\delta}, \widetilde{\delta}_{\mathrm{asymp}})$ the constants from Lemma 59. The choices $\Lambda := \max\{\widetilde{\Lambda}, 10\}$ and $(\delta, \delta_{\mathrm{asymp}}) := (\widetilde{\delta}, \widetilde{\delta}_{\mathrm{asymp}})$ then imply the claim. Indeed, for $t \in \mathcal{T}_{\text{non-reg}}(\Lambda)$ satisfying the assumption $E[\chi|\bar{\chi}^{z,T}](t) \leq \delta r_T(t)$, it follows from the defining condition (4.70)

$$\frac{5}{r_T^2(t)} E[\chi|\bar{\chi}^{z,T}](t) \leq \frac{5}{r_T(t)} \leq \frac{1}{2} \frac{\Lambda}{r_T(t)} \leq \frac{1}{2} \sum_{i,j=1, i \neq j}^{P} \int_{I_{i,j}(t)} \frac{1}{2} |V_{i,j}(\cdot, t)|^2 \, d\mathcal{H}^1,$$

so that the validity of (4.80) implies (4.81). □

4.4.3 Proof of Lemma 61: Stability estimate in perturbative setting I

The asserted bound (4.82) follows directly from the estimates (4.126)–(4.139) established in Section 4.7. □

4.4.4 Proof of Lemma 62: Stability estimate in perturbative setting II

For notational simplicity, we again neglect the dependence on t of all quantities. Our proof of the estimate (4.83) proceeds in several steps.

Step 1: Leading order terms involving $H'_{\bar{I}^{z,T}}$. We start by providing a preliminary estimate for the last three right hand side terms of $R_{l.o.t.}$ from Lemma 61. To this end, for each of the three terms we make use of Definition 50 in the form of $|H'_{\bar{I}^{z,T}}| \leq \delta_{\mathrm{asymp}}/r_T^2$. Hence, by Young's inequality and $|H_{\bar{I}^{z,T}}| \leq 2/r_T$

$$\int_{\bar{I}^{z,T}} 2H_{\bar{I}^{z,T}} H'_{\bar{I}^{z,T}} hh' \, d\mathcal{H}^1 \lesssim \delta_{\mathrm{asymp}} \int_{\bar{I}^{z,T}} \frac{1}{r_T^2}(h')^2 + \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1. \tag{4.102}$$

Furthermore, by the defining ODE for the space-time shift in the form of (4.77), Jensen's inequality and (4.12), we obtain

$$-\int_{\bar{I}^{z,T}} H'_{\bar{I}^{z,T}} \Big(\tau_{\bar{I}^{z,T}} \cdot \dot{z} \Big) h \, d\mathcal{H}^1 \lesssim \delta_{\mathrm{asymp}} \int_{\bar{I}^{z,T}} \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1 \tag{4.103}$$

as well as

$$-\int_{\bar{I}^{z,T}} H'_{\bar{I}^{z,T}} \dot{\mathfrak{T}} h' \, d\mathcal{H}^1 \lesssim \delta_{\mathrm{asymp}} \int_{\bar{I}^{z,T}} \frac{1}{r_T^2}(h')^2 + \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1, \tag{4.104}$$

where for the latter we also used Young's inequality.

Step 2: Freezing of coefficients in leading order quadratic terms. As a simple consequence of (4.14), $|H_{\bar{I}^{z,T}}| \leq 2/r_T$ and $a^2 - b^2 = (a-b)(a+b)$, it holds

$$\begin{aligned} &\int_{\bar{I}^{z,T}} \Big(\frac{3}{2} H^2_{\bar{I}^{z,T}} - \frac{1}{r_T^2} \Big) (h')^2 \, d\mathcal{H}^1 + \int_{\bar{I}^{z,T}} \frac{1}{r_T^2} \Big(\frac{1}{2} H^2_{\bar{I}^{z,T}} + \frac{1}{r_T^2} \Big) h^2 \, d\mathcal{H}^1 \\ &\quad \leq \int_{\bar{I}^{z,T}} \frac{1}{2} \frac{1}{r_T^2} (h')^2 + \frac{3}{2} \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1 + \frac{9}{2} \delta_{\mathrm{asymp}} \int_{\bar{I}^{z,T}} \frac{1}{r_T^2}(h')^2 + \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1. \end{aligned} \tag{4.105}$$

Step 3: Freezing of coefficients in leading order correction terms. By the arguments from the previous two steps, we may estimate

$$\begin{aligned} &-\int_{\bar{I}^{z,T}} \Big(\frac{1}{r_T^2} + H^2_{\bar{I}^{z,T}} \Big) h \mathrm{n}_{\bar{I}^{z,T}} \cdot \dot{z} \, d\mathcal{H}^1 - \int_{\bar{I}^{z,T}} \frac{1}{r_T^2} H_{\bar{I}^{z,T}} h \dot{\mathfrak{T}} \, d\mathcal{H}^1 \\ &\quad \leq -\int_{\bar{I}^{z,T}} \frac{2}{r_T^2} h \mathrm{n}_{\bar{I}^{z,T}} \cdot \dot{z} \, d\mathcal{H}^1 - \int_{\bar{I}^{z,T}} \frac{1}{r_T^3} h \dot{\mathfrak{T}} \, d\mathcal{H}^1 \\ &\qquad + \widetilde{C} \delta_{\mathrm{asymp}} \int_{\bar{I}^{z,T}} \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1, \end{aligned} \tag{4.106}$$

where $\widetilde{C} > 0$ is some universal constant.

Step 4: Change of variables in quadratic terms. Recalling the definition $[0, 2\pi) \ni \theta \mapsto h\Big(\bar{\gamma}^{z,T}\big(\frac{L_{\bar{\gamma}^{z,T}}}{2\pi}\theta\big)\Big)$, a simple change of variables together with condition (4.12) trivially entails

$$\frac{1}{(1+\delta_{\mathrm{asymp}})^3} \frac{1}{r_T^3} \int_0^{2\pi} (\partial_\theta^2 \widetilde{h})^2 \, d\theta \leq \int_{\bar{I}^{z,T}} (h'')^2 \, d\mathcal{H}^1, \tag{4.107}$$

$$\int_{\bar{I}^{z,T}} (h'')^2 \, d\mathcal{H}^1 \leq \frac{1}{(1-\delta_{\mathrm{asymp}})^3} \frac{1}{r_T^3} \int_0^{2\pi} (\partial_\theta^2 \widetilde{h})^2 \, d\theta, \tag{4.108}$$

$$\int_{\bar{I}^{z,T}} \frac{1}{r_T^2} (h')^2 \, d\mathcal{H}^1 \leq \frac{1}{(1-\delta_{\mathrm{asymp}})} \frac{1}{r_T^3} \int_0^{2\pi} (\partial_\theta \widetilde{h})^2 \, d\theta, \tag{4.109}$$

$$\int_{\bar{I}^{z,T}} \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1 \leq (1+\delta_{\mathrm{asymp}}) \frac{1}{r_T^3} \int_0^{2\pi} \widetilde{h}^2 \, d\theta. \tag{4.110}$$

Step 5: Change of variables in correction terms. We claim that

$$\begin{aligned}
&\bigg| -\int_{\bar{I}^{z,T}} \frac{2}{r_T^2} h \mathrm{n}_{\bar{I}^{z,T}} \cdot \dot{z}\, d\mathcal{H}^1 - \bigg(-6\frac{1}{r_T^3(t)}\bigg|\frac{1}{\sqrt{\pi}}\int_0^{2\pi} \widetilde{h}(\cdot,t)e^{i\theta}\,d\theta\bigg|^2\bigg)\bigg| \\
&\quad + \bigg| -\int_{\bar{I}^{z,T}} \frac{1}{r_T^2} H_{\bar{I}^{z,T}} h \dot{\mathfrak{T}}\, d\mathcal{H}^1 - \bigg(-4\frac{1}{r_T^3(t)}\Big(\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \widetilde{h}(\cdot,t)\,d\theta\Big)^2\bigg)\bigg| \\
&\qquad \leq \widetilde{C}\delta_{\mathrm{asymp}}\frac{1}{r_T^3}\int_0^{2\pi} \widetilde{h}^2\,d\theta,
\end{aligned} \tag{4.111}$$

where $\widetilde{C} > 0$ is some universal constant. Indeed, this follows similarly to the previous steps, exploiting in the process the two conditions (4.12) and (4.13) as well as the defining ODE of the space-time shift in the form of (4.77).

Step 6: Conclusion. Based on the previous steps, we infer that, for given $\delta_{\mathrm{err}} \in (0,1)$, one may choose $\delta_{\mathrm{asymp}} \ll_{\delta_{\mathrm{err}}} 1$ such that the leading order contribution $R_{l.o.t.}$ from Lemma 61 is estimated by $\widetilde{R}_{l.o.t.} + \frac{1}{2}\widetilde{R}_{h.o.t.}$. Since the higher order contribution $R_{h.o.t.}$ from Lemma 61 can be easily estimated in terms of $\frac{1}{2}\widetilde{R}_{h.o.t.}$ for a suitable choice of $\delta_{\mathrm{asymp}} \ll_{\delta_{\mathrm{err}}} 1$ by means of the previous arguments, this concludes the proof of (4.83). □

4.4.5 Proof of Lemma 63: Final stability estimate in perturbative setting

First, we observe that by Lemma 58 and the estimates (4.109)–(4.110) from the previous proof that, for given $\delta_{\mathrm{err}} \in (0,1)$, one may choose $C, C' \gg_{\delta_{\mathrm{err}}} 1$ and $\delta_{\mathrm{asymp}} \ll_{\delta_{\mathrm{err}}} 1$ such that

$$E[\chi|\bar{\chi}^{z,T}] \leq (1+\delta_{\mathrm{err}})\frac{1}{r_T}\frac{1}{2}\big\|\widetilde{h}\big\|^2_{H^1(0,2\pi)} =: I. \tag{4.112}$$

Second, thanks to Lemma 62, for given $\delta_{\mathrm{err}} \in (0,1)$, one may choose $C, C' \gg_{\delta_{\mathrm{err}}} 1$ and $\delta_{\mathrm{asymp}} \ll_{\delta_{\mathrm{err}}} 1$ such that

$$\begin{aligned}
&\sum_{i,j=1, i\neq j}^{P} \frac{1}{2}\Big(-\mathcal{D}_{i,j}[\chi|\bar{\chi}^{z,T}] + \frac{1}{2}RHS^{\mathrm{int}}_{i,j}[\chi|\bar{\chi}^{z,T}]\Big) + \sum_{i=1}^{P-1} RHS^{\mathrm{bulk}}_i[\chi|\bar{\chi}^{z,T}] \\
&\quad \leq -\frac{1}{r_T^3}\int_0^{2\pi} (1-\delta_{\mathrm{err}})(\partial_\theta^2\widetilde{h})^2 - (1+\delta_{\mathrm{err}})\frac{1}{2}(\partial_\theta\widetilde{h})^2 - (1+\delta_{\mathrm{err}})\frac{3}{2}\widetilde{h}^2\,d\theta \\
&\qquad - 4\frac{1}{r_T^3}\bigg(\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\widetilde{h}\,d\theta\bigg)^2 - 6\frac{1}{r_T^3}\bigg|\frac{1}{\sqrt{\pi}}\int_0^{2\pi}\widetilde{h}e^{i\theta}\,d\theta\bigg|^2 \\
&\quad =: II.
\end{aligned} \tag{4.113}$$

Now, fix $\alpha \in (1,5)$. We claim that there exist $\delta_{\mathrm{err}} \ll_\alpha 1$ as well as a choice of the constant $C_\zeta \gg_\alpha 1$ from Construction 53 such that

$$II \leq -\frac{\alpha}{r_T^2}(1+\dot{\mathfrak{T}})I, \tag{4.114}$$

so that the claim (4.84) follows from (4.112)–(4.114). Fourier decomposing both sides of the asserted inequality (4.114), we may indeed derive the validity of (4.114) for suitably chosen $\delta_{\mathrm{err}} \ll_\alpha 1$ and $C_\zeta \gg_\alpha 1$ analogously to our analysis towards the end of Subsection 4.2.2 (cf. (4.49)–(4.52)), exploiting in the process also the bound (4.66). □

4.4.6 Proof of Theorem 64: Overall a priori stability estimate

The stability estimate (4.86) follows directly from combining all results from Subsections 4.2.3–4.2.8, in particular Lemma 52, Corollary 60, and Lemma 63. Note in this context that assumption (4.85) imply $E[\chi|\bar{\chi}^{z,T}](t) \leq \delta r_T(t)$ for all $t \in (0, t_\chi)$. $\square$

4.5 Construction and properties of space-time shifts

4.5.1 Proof of Lemma 54: Existence of space-time shifts

Our aim is to prove the existence of a time horizon $t_\chi \in (0,\infty)$, a locally Lipschitz map $z\colon [0,t_\chi) \to \mathbb{R}^d$ and a strictly increasing Lipschitz map $T =: \mathrm{id}+\mathfrak{T}\colon [0,t_\chi) \to [0,\infty)$ such that $(z(0),T(0)) = (0,0)$ and

$$t_\chi = \sup\Big\{t : T(t) < \frac{1}{2}r_0^2 = T_{ext}\Big\}, \tag{4.115}$$

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\mathfrak{T}}(t) \end{bmatrix} = F(z,T,t), \quad t \in (0,t_\chi), \tag{4.116}$$

where

$$F(z,T,t) := \begin{bmatrix} \frac{6}{r_T^2(t)} \fint_{\bar{I}^{z,T}(t)} \rho(\cdot,t;z,T)\mathrm{n}_{\bar{I}^{z,T}}(\cdot,t)\,d\mathcal{H}^1 \\ \frac{4}{r_T(t)} \fint_{\bar{I}^{z,T}(t)} \rho(\cdot,t;z,T)\,d\mathcal{H}^1 \end{bmatrix}. \tag{4.117}$$

Note that the asserted Lipschitz bounds (4.66) are then immediate consequences of integrating (4.116) and $|\rho(x,t;z,T)| \leq r_T(t)/(8C_\zeta)$, cf. Construction 53 .

The proof of existence of the solution is obtained by successive approximations and an application of the Picard–Lindelöf argument. To this end, we introduce an auxiliary version of our problem labeled by integers $k \geq 1$, which reads as

$$\begin{bmatrix} \dot{z}_k(t) \\ \dot{\mathfrak{T}}_k(t) \end{bmatrix} = F_k(z_k,T_k,t), \quad (z_k(0),T_k(0)) = (0,0), \tag{4.118}$$

where the right hand side $F_k\colon C_b([0,\infty);\mathbb{R}^2)\times C_b([0,\infty);[0,\infty))\times[0,\infty) \to \mathbb{R}^3$ is defined by truncation:

$$F_k(z,T,t) = F\Big(z, \min\Big\{T, \tfrac{1}{2}r_0^2(1-\tfrac{1}{k})\Big\}, t\Big), \quad t \in [0,\infty). \tag{4.119}$$

Here, $C_b([0,\infty);X)$ denotes the space of bounded and continuous functions taking values in a Banach space X.

We will show below that the fixed point equation obtained from integrating (4.118) admits a unique solution $z_k \in C_b([0,\infty);\mathbb{R}^2)$ and $T_k \in C_b([0,\infty);[0,\infty))$, where $t \mapsto T_k(t)$ is strictly increasing such that

$$\frac{1}{2}t \leq T_k(t) \leq \frac{3}{2}t. \tag{4.120}$$

(The latter two properties are again consequences of $|\dot{\mathfrak{T}}_k| \leq \frac{1}{2}$ due to the estimates $|\rho(x,t;z_k,\min\{T_k,\frac{1}{2}r_0^2(1-\frac{1}{k})\})| \leq 1/(8C_\zeta)$ and $C_\zeta \geq 1$.)

Taking the existence of such a sequence of solutions $(z_k, T_k)_{k\geq 1}$ for granted for the moment, we then define $t_0 := 0$ and for $k \geq 1$

$$t_k = \sup\Big\{T_k(t) < \tfrac{1}{2}r_0^2(1-\tfrac{1}{k})\Big\}. \tag{4.121}$$

By the properties of T_k, uniqueness of solutions to (4.118), as well as the definitions (4.119) and (4.121), the sequence $(t_k)_{k\geq 1}$ is strictly increasing and bounded. The solution to (4.116) is then constructed by

$$(z(t), T(t)) := (z_k(t), T_k(t)), \quad t \in [0, t_k), \tag{4.122}$$

$$t_\chi := \sup_{k\geq 1} t_k = \lim_{k\to\infty} t_k < \infty. \tag{4.123}$$

Note that (4.122) is indeed well-defined by uniqueness of solutions to (4.118), and that the identities (4.115)–(4.116) hold true by construction. Hence, it remains to verify the existence of solutions to (4.118) with the asserted properties.

Fix an integer $k \geq 1$. In order to apply the Picard–Lindelöf argument, we have to show that for given $t \in (0, \infty)$, the function $(z, T) \to F_k(z, T, t)$ is globally Lipschitz with Lipschitz constant independent of t. For notational convenience, we abbreviate the truncation by $\widehat{T} := \big\{T, \frac{1}{2}r_0^2(1-\frac{1}{k})\big\}$. First, we compute

$$\frac{1}{r_{\widehat{T}}^2(t)} \frac{1}{\mathcal{H}^1(\bar{I}^{z,\widehat{T}}(t))} = \frac{1}{2\pi} \frac{1}{r_{\widehat{T}}^3(t)} \frac{2\pi r_{\widehat{T}}(t)}{\mathcal{H}^1(\bar{I}^{0,\widehat{T}}(t))},$$

so that the normalization factor has the required regularity due to the action of the truncation and the smoothness of the evolution of $\bar{\chi}$. Second, since the Jacobian of the tubular neighborhood diffeomorphism

$$x \mapsto (P_{\bar{I}^{z,\widehat{T}}}(x,t), s_{\bar{I}^{z,\widehat{T}}}(x,t))$$

is given by $x \mapsto 1/(1-(H_{\bar{I}^{z,\widehat{T}}} \circ P_{\bar{I}^{z,\widehat{T}}})(x,t)s_{\bar{I}^{z,\widehat{T}}}(x,t))$, plugging in the definition (4.61) together with a change of variables yields

$$\begin{aligned}
&\int_{\bar{I}^{z,\widehat{T}}(t)} \rho_+(\cdot, t; z, \widehat{T})\mathrm{n}_{\bar{I}^{z,\widehat{T}}}(\cdot, t)\, d\mathcal{H}^1 \\
&= \int_{\{0\leq \mathrm{sdist}_{\bar{I}^{z,\widehat{T}}}(\cdot,t)\leq r_{\widehat{T}}(t)/8\}} \Big(1-(H_{\bar{I}^{z,\widehat{T}}} \circ P_{\bar{I}^{z,\widehat{T}}})(\cdot, t)s_{\bar{I}^{z,\widehat{T}}}(\cdot, t)\Big) \\
&\qquad\qquad \times \Big(\bar{\chi}_1^{z,\widehat{T}} - \chi_1\Big)(\cdot, t)\zeta\Big(\frac{s_{\bar{I}^{z,\widehat{T}}}(\cdot, t)}{r_{\widehat{T}}(t)}\Big)\nabla s_{\bar{I}^{z,\widehat{T}}}(\cdot, t)\, dx.
\end{aligned}$$

Shifting variables, we obtain from the relations (4.17)–(4.19)

$$\begin{aligned}
&\int_{\bar{I}^{z,\widehat{T}}(t)} \rho_+(\cdot, t; z, \widehat{T})\mathrm{n}_{\bar{I}^{z,\widehat{T}}}(\cdot, t)\, d\mathcal{H}^1 \\
&= \int_{\{0\leq \mathrm{sdist}_{\bar{I}^{0,\widehat{T}}}(\cdot,t)\leq r_{\widehat{T}}(t)/8\}} \Big(1-(H_{\bar{I}^{0,\widehat{T}}} \circ P_{\bar{I}^{0,\widehat{T}}})(\cdot, t)s_{\bar{I}^{0,\widehat{T}}}(\cdot, t)\Big) \\
&\qquad\qquad \times \Big(\bar{\chi}_1^{0,\widehat{T}} - \chi_1^{-z,\mathrm{id}}\Big)(\cdot, t)\zeta\Big(\frac{s_{\bar{I}^{0,\widehat{T}}}(\cdot, t)}{r_{\widehat{T}}(t)}\Big)\nabla s_{\bar{I}^{0,\widehat{T}}}(\cdot, t)\, dx,
\end{aligned}$$

and the required regularity estimate follows from this representation by smoothness of the evolution of $\bar{\chi}$, the action of the truncation, and Lipschitz continuity of translations of volumes. Since an analogous formula also holds for ρ_+ replaced by ρ_-, this concludes the proof. □

4.5.2 Proof of Lemma 65: Bounds for space-time shifts

Our goal is to prove (4.87)–(4.88). Fix $t \in (0, t_\chi)$. Note that from (4.56), Construction 55, a change to tubular neighborhood coordinates, $|H_{\bar{I}^{z,T}}(\cdot,t)| \leq 2/r_T(t)$, and (4.63) it follows that

$$\begin{aligned}
&E_{\mathrm{bulk}}[\chi|\bar{\chi}^{z,T}](t)\\
&\geq \int_{\{\mathrm{dist}(\cdot,\bar{I}^{z,T}(t))<r_T(t)/8\}} \Big|\chi_1(\cdot,t)-\bar{\chi}_1^{z,T}(\cdot,t)\Big|\Big|\vartheta_1(\cdot,t)\Big|\,dx\\
&= \int_{\bar{I}^{z,T}(t)}\int_{-\frac{r_T(t)}{8}}^{\frac{r_T(t)}{8}} \frac{(\chi_1-\bar{\chi}_1^{z,T})\Big(\cdot+s\mathrm{n}_{\bar{I}^{z,T}}(\cdot,t),t\Big)}{1-H_{\bar{I}^{z,T}}(\cdot,t)s}\frac{-s}{r_T^2(t)}\,dsd\mathcal{H}^1\\
&\geq \frac{4}{5}\int_{\bar{I}^{z,T}(t)}\frac{1}{2}\frac{1}{r_T^2(t)}\Big(\rho_+^2(\cdot,t;z,T)+\rho_-^2(\cdot,t;z,T)\Big)\,d\mathcal{H}^1\\
&\geq \frac{1}{10}\int_{\bar{I}^{z,T}(t)}\Big(\frac{\rho(\cdot,t;z,T)}{r_T(t)}\Big)^2\,d\mathcal{H}^1.
\end{aligned} \tag{4.124}$$

Hence, plugging in (4.65) and (4.66), recalling (4.12), and using Jensen's inequality

$$\begin{aligned}
\frac{1}{r_0}|z(t)| &\leq \int_0^t \frac{1}{r_0}|\dot{z}(s)|\,ds\\
&\lesssim (1+\delta_{\mathrm{asymp}})\frac{1}{r_0}\int_0^t \frac{1}{r_T^2(s)}\Big(\fint_{\bar{I}^{z,T}(s)}\Big|\rho(\cdot,s;z,T)\Big|^2\,d\mathcal{H}^1\Big)^{\frac{1}{2}}\,ds\\
&\lesssim (1+\delta_{\mathrm{asymp}})\frac{1}{r_0}\int_0^t \frac{1}{r_T^{3/2}(s)}\Big(\int_{\bar{I}^{z,T}(s)}\Big(\frac{\rho(\cdot,s;z,T)}{r_T(s)}\Big)^2\,d\mathcal{H}^1\Big)^{\frac{1}{2}}\,ds.
\end{aligned} \tag{4.125}$$

Inserting the estimate (4.125) into (4.124) and afterward exploiting the assumption (4.85) further entails

$$\frac{1}{r_0}|z(t)| \lesssim \delta_{\mathrm{err}}\sqrt{\frac{1}{r_0}E[\chi_0|\bar{\chi}_0]}\int_0^t \frac{1}{r_T^2(s)}\Big(\frac{r_T(s)}{r_0}\Big)^{\frac{1}{2}}\,ds,$$

where $\delta_{\mathrm{err}} \simeq \delta(1+\delta_{\mathrm{asymp}})$, which in turn by (4.66) upgrades to

$$\begin{aligned}
\frac{1}{r_0}|z(t)| &\lesssim \delta_{\mathrm{err}}\sqrt{\frac{1}{r_0}E[\chi_0|\bar{\chi}_0]}\int_0^t \frac{1}{r_T^2(s)}\Big(\frac{r_T(s)}{r_0}\Big)^{\frac{1}{2}}\big(1+\dot{\mathfrak{T}}(s)\big)\,ds\\
&\lesssim -\delta_{\mathrm{err}}\sqrt{\frac{1}{r_0}E[\chi_0|\bar{\chi}_0]}\int_0^t \frac{d}{ds}\Big(\frac{r_T(s)}{r_0}\Big)^{\frac{1}{2}}\,ds\\
&\lesssim \delta_{\mathrm{err}}\sqrt{\frac{1}{r_0}E[\chi_0|\bar{\chi}_0]}\Big(1-\Big(\frac{r_T(t)}{r_0}\Big)^{\frac{1}{2}}\Big).
\end{aligned}$$

Now, choosing $\delta_{\mathrm{err}} \ll 1$ (hence $\delta, \delta_{\mathrm{asymp}} \ll 1$) such that the implicit constant in the last estimate gets canceled, we obtain the claim for the path of translations z. Analogously, one derives a bound of same type for $\frac{1}{T_{ext}}|\mathfrak{T}(t)|$. □

4.6 Reduction to perturbative graph setting

4.6.1 Proof of Proposition 57: Reduction to graph setting

The proof of Proposition 57 is still work in progress and its details will be included in a future version of the work [47]. We expect the statement of Proposition 57 to follow from an application of the Allard Regularity Theorem [117, Theorem 5.2]. Then, one can deduce the H^2 regularity (4.73) from the hypothesis (4.71), the weak definition of $V_{i,j}$ (cf. [46, Definition 13, (17c)]), in particular (4.153), and the bounds (4.75)-(4.76).

4.6.2 Proof of Lemma 58: Error functionals in perturbative regime

The estimates (4.78) follow directly from (4.156), (4.56), (4.142) as well as (4.75), whereas the estimates (4.79) are immediate consequences of (4.155), (4.55), (4.140), (4.150) as well as (4.75)–(4.76). □

4.7 Auxiliary computations in perturbative regime

Let (ξ, ϑ, B) be the maps from Construction 55, and let (t_χ, z, T) be the space-time shifts from Lemma 54. Fix $t \in (0, t_\chi)$ and assume the existence of an height function $h(\cdot, t)$ satisfying the properties as in the conclusions of Proposition 57. For ease of notation, we will drop in the following any dependence on the time t. Furthermore, we will abbreviate in the tubular neighborhood $\{\mathrm{dist}(\cdot, \bar{I}^{z,T}) < r_T/2\}$

$$\begin{aligned} s_{\bar{I}^{z,T}} &:= \mathrm{sdist}_{\bar{I}^{z,T}}, \\ \bar{\mathrm{n}}_{\bar{I}^{z,T}} &:= \mathrm{n}_{\bar{I}^{z,T}} \circ P_{\bar{I}^{z,T}}, \quad \bar{\tau}_{\bar{I}^{z,T}} := \tau_{\bar{I}^{z,T}} \circ P_{\bar{I}^{z,T}}, \\ \bar{H}_{\bar{I}^{z,T}} &:= H_{\bar{I}^{z,T}} \circ P_{\bar{I}^{z,T}}. \end{aligned}$$

Finally, define $I := I_{1,P}$, which is by assumption subject to the graph representation (4.73)–(4.76), and denote $V := V_1$, $\mathrm{n} := \mathrm{n}_{P,1}$ as well as, by slight abuse of notation, $\chi := \chi_1$ and $\bar{\chi} := \bar{\chi}_1$.

We claim that for given $\delta_{\mathrm{err}} \in (0,1)$, one may choose the constants $C, C' \gg_{\delta_{\mathrm{err}}} 1$ from (4.75)–(4.76) such that the individual elements of the stability estimates (4.58) and (4.59) are estimated as follows:

$$\begin{aligned} &- \int_I \Big(\partial_t \xi^{z,T} + (B^{z,T} \cdot \nabla)\xi^{z,T} + (\nabla B^{z,T})^\mathsf{T} \xi^{z,T}\Big) \cdot (\mathrm{n} - \xi^{z,T}) \, d\mathcal{H}^1 \\ &\qquad \leq - \int_{\bar{I}^{z,T}} H^2_{\bar{I}^{z,T}} h (\mathrm{n}_{\bar{I}^{z,T}} \cdot \dot{z}) \, d\mathcal{H}^1 \\ &\qquad\quad - \int_{\bar{I}^{z,T}} H'_{\bar{I}^{z,T}} (\tau_{\bar{I}^{z,T}} \cdot \dot{z}) h \, d\mathcal{H}^1 - \int_{\bar{I}^{z,T}} H'_{\bar{I}^{z,T}} \dot{\mathfrak{T}} h' \, d\mathcal{H}^1 \\ &\qquad\quad + \int_{\bar{I}^{z,T}} \delta_{\mathrm{err}} \Big(\frac{1}{r_T} \Big| (\tau_{\bar{I}^{z,T}} \cdot \dot{z}) h' \Big| + \Big| H'_{\bar{I}^{z,T}} \dot{\mathfrak{T}} h' \Big| \Big) \, d\mathcal{H}^1, \end{aligned} \tag{4.126}$$

$$- \int_I \Big(\partial_t \xi^{z,T} + (B^{z,T} \cdot \nabla)\xi^{z,T}\Big) \cdot \xi^{z,T} \, d\mathcal{H}^1 = 0, \tag{4.127}$$

$$\begin{aligned} &\int_I \frac{1}{2} \Big| \nabla \cdot \xi^{z,T} + B^{z,T} \cdot \xi^{z,T} \Big|^2 \, d\mathcal{H}^1 \\ &\qquad \leq \int_{\bar{I}^{z,T}} \frac{1}{2} H^4_{\bar{I}^{z,T}} h^2 + \delta_{\mathrm{err}} \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1, \end{aligned} \tag{4.128}$$

$$- \int_I \frac{1}{2} \Big| B^{z,T} \cdot \xi^{z,T} \Big| \Big(1 - |\xi^{z,T}|^2 \Big) \, d\mathcal{H}^1 = 0, \tag{4.129}$$

$$- \int_I (1 - \mathrm{n} \cdot \xi^{z,T}) \nabla \cdot \xi^{z,T} (B^{z,T} \cdot \xi^{z,T}) \, d\mathcal{H}^1$$

$$\leq \int_{\bar{I}^{z,T}} \frac{1}{2} H^2_{\bar{I}^{z,T}} (h')^2 + \delta_{\mathrm{err}} \frac{1}{r_T^2} (h')^2 \, d\mathcal{H}^1, \tag{4.130}$$

$$\int_I \Big((\mathrm{Id} - \xi^{z,T} \otimes \xi^{z,T}) B^{z,T} \Big) \cdot (V + \nabla \cdot \xi^{z,T}) \mathrm{n} \, d\mathcal{H}^1 = 0, \tag{4.131}$$

$$\int_I (1 - \mathrm{n} \cdot \xi^{z,T}) \nabla \cdot B^{z,T} \, d\mathcal{H}^1$$
$$\leq - \int_{\bar{I}^{z,T}} \frac{1}{2} H^2_{\bar{I}^{z,T}} (h')^2 \, d\mathcal{H}^1 + \int_{\bar{I}^{z,T}} \delta_{\mathrm{err}} \frac{1}{r_T^2} (h')^2 \, d\mathcal{H}^1, \tag{4.132}$$

$$- \int_I (\mathrm{n} - \xi^{z,T}) \otimes (\mathrm{n} - \xi^{z,T}) : \nabla B^{z,T} \, d\mathcal{H}^1$$
$$\leq \int_{\bar{I}^{z,T}} H^2_{\bar{I}^{z,T}} (h')^2 \, d\mathcal{H}^1 + \int_{\bar{I}^{z,T}} \delta_{\mathrm{err}} \Big(\frac{1}{r_T^2} + \Big| H'_{\bar{I}^{z,T}} \Big| \Big) (h')^2 \, d\mathcal{H}^1, \tag{4.133}$$

$$- \int_I \frac{1}{2} \Big| V + \nabla \cdot \xi^{z,T} \Big|^2 \, d\mathcal{H}^1$$
$$\leq - \int_{\bar{I}^{z,T}} \frac{1}{2} (h'')^2 \, d\mathcal{H}^1 \tag{4.134}$$
$$+ \int_{\bar{I}^{z,T}} \delta_{\mathrm{err}} \Big((h'')^2 + \frac{1}{r_T^2} (h')^2 + \Big((H'_{\bar{I}^{z,T}})^2 + \frac{1}{r_T^4} \Big) h^2 \Big) \, d\mathcal{H}^1$$

$$- \int_I \frac{1}{2} \Big| V \mathrm{n} - (B^{z,T} \cdot \xi^{z,T}) \xi^{z,T} \Big|^2 \, d\mathcal{H}^1$$
$$\leq - \int_{\bar{I}^{z,T}} \frac{1}{2} \Big((h'')^2 + H^4_{\bar{I}^{z,T}} h^2 - H^2_{\bar{I}^{z,T}} (h')^2 \Big) \, d\mathcal{H}^1 \tag{4.135}$$
$$+ \int_{\bar{I}^{z,T}} 2 H_{\bar{I}^{z,T}} H'_{\bar{I}^{z,T}} h h' \, d\mathcal{H}^1$$
$$+ \int_{\bar{I}^{z,T}} \delta_{\mathrm{err}} \Big((h'')^2 + \frac{1}{r_T^2} (h')^2 + \Big((H'_{\bar{I}^{z,T}})^2 + \frac{1}{r_T^4} \Big) h^2 \Big) \, d\mathcal{H}^1,$$

$$\int_I \vartheta^{z,T} (B^{z,T} \cdot \xi^{z,T} - V) \, d\mathcal{H}^1$$
$$\leq \int_{\bar{I}^{z,T}} \frac{H^2_{\bar{I}^{z,T}}}{r_T^2} h^2 \, d\mathcal{H}^1 - \int_{\bar{I}^{z,T}} \frac{1}{r_T^2} (h')^2 \, d\mathcal{H}^1 \tag{4.136}$$
$$+ \int_{\bar{I}^{z,T}} \delta_{\mathrm{err}} \Big((h'')^2 + \frac{1}{r_T^2} (h')^2 + \Big((H'_{\bar{I}^{z,T}})^2 + \frac{1}{r_T^4} \Big) h^2 \Big) \, d\mathcal{H}^1,$$

$$\int_I \vartheta^{z,T} B^{z,T} \cdot (\mathrm{n} - \xi^{z,T}) \, d\mathcal{H}^1$$
$$\leq \int_{\bar{I}^{z,T}} \delta_{\mathrm{err}} \Big(\frac{1}{r_T^4} h^2 + \frac{1}{r_T^2} (h')^2 \Big) \, d\mathcal{H}^1, \tag{4.137}$$

$$\int_{\mathbb{R}^2} (\chi - \bar{\chi}^{z,T}) \vartheta^{z,T} \nabla \cdot B^{z,T} \, dx$$
$$\leq - \int_{\bar{I}^{z,T}} \frac{1}{2} \frac{H^2_{\bar{I}^{z,T}}}{r_T^2} h^2 \, d\mathcal{H}^1 + \int_{\bar{I}^{z,T}} \delta_{\mathrm{err}} \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1, \tag{4.138}$$

$$\int_{\mathbb{R}^2} (\chi - \bar{\chi}^{z,T}) \Big(\partial_t \vartheta^{z,T} + (B^{z,T} \cdot \nabla) \vartheta^{z,T} \Big) \, dx$$
$$\leq \int_{\bar{I}^{z,T}} \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1 \tag{4.139}$$

$$
\begin{aligned}
&- \int_{\bar{I}^{z,T}} \frac{1}{r_T^2} H_{\bar{I}^{z,T}} h \dot{\mathfrak{T}} \, d\mathcal{H}^1 - \int_{\bar{I}^{z,T}} \frac{1}{r_T^2} h(\mathrm{n}_{\bar{I}^{z,T}} \cdot \dot{z}) \, d\mathcal{H}^1 \\
&+ \int_{\bar{I}^{z,T}} \delta_{\mathrm{err}} \bigg(\frac{1}{r_T^3} \big| \dot{\mathfrak{T}} h \big| + \frac{1}{r_T^2} \big| (\mathrm{n}_{\bar{I}^{z,T}} \cdot \dot{z}) h \big| + \frac{1}{r_T^4} h^2 \bigg) \, d\mathcal{H}^1.
\end{aligned}
$$

Proof of (4.126)–(4.139). We proceed in several steps.

Step 1: Properties of gradient flow calibration. Thanks to (4.75), the definitions (4.67)–(4.69), and the identities (4.17)–(4.19), it holds on $I \subset \{\mathrm{dist}(\cdot, \bar{I}^{z,T}) < r_T/8\}$

$$\xi^{z,T} = \bar{\mathrm{n}}_{\bar{I}^{z,T}} = \nabla s_{\bar{I}^{z,T}}, \tag{4.140}$$

$$B^{z,T} = \bar{H}_{\bar{I}^{z,T}} \bar{\mathrm{n}}_{\bar{I}^{z,T}}, \tag{4.141}$$

$$\vartheta^{z,T} = -\frac{s_{\bar{I}^{z,T}}}{r_T^2}. \tag{4.142}$$

In particular, because of

$$\nabla P_{\bar{I}^{z,T}} = \mathrm{Id} - \bar{\mathrm{n}}_{\bar{I}^{z,T}} \otimes \bar{\mathrm{n}}_{\bar{I}^{z,T}} - s_{\bar{I}^{z,T}} \nabla \bar{\mathrm{n}}_{\bar{I}^{z,T}},$$

we obtain by direct computation throughout $\{\mathrm{dist}(\cdot, \bar{I}^{z,T}) < r_T/4\}$

$$\nabla \xi^{z,T} = -\frac{\bar{H}_{\bar{I}^{z,T}}}{1 - \bar{H}_{\bar{I}^{z,T}} s_{\bar{I}^{z,T}}} \bar{\tau}_{\bar{I}^{z,T}} \otimes \bar{\tau}_{\bar{I}^{z,T}}, \tag{4.143}$$

$$\nabla \cdot \xi^{z,T} = -\frac{\bar{H}_{\bar{I}^{z,T}}}{1 - \bar{H}_{\bar{I}^{z,T}} s_{\bar{I}^{z,T}}}, \tag{4.144}$$

$$\nabla B^{z,T} = -\frac{\bar{H}^2_{\bar{I}^{z,T}}}{1 - \bar{H}_{\bar{I}^{z,T}} s_{\bar{I}^{z,T}}} \bar{\tau}_{\bar{I}^{z,T}} \otimes \bar{\tau}_{\bar{I}^{z,T}} + \frac{\bar{H}'_{\bar{I}^{z,T}}}{1 - \bar{H}_{\bar{I}^{z,T}} s_{\bar{I}^{z,T}}} \bar{\mathrm{n}}_{\bar{I}^{z,T}} \otimes \bar{\tau}_{\bar{I}^{z,T}}, \tag{4.145}$$

$$\nabla \cdot B^{z,T} = -\frac{\bar{H}^2_{\bar{I}^{z,T}}}{1 - \bar{H}_{\bar{I}^{z,T}} s_{\bar{I}^{z,T}}}, \tag{4.146}$$

$$\nabla \vartheta^{z,T} = -\frac{1}{r_T^2} \bar{\mathrm{n}}_{\bar{I}^{z,T}}. \tag{4.147}$$

Note that these computations are justified thanks to $|1 - \bar{H}_{\bar{I}^{z,T}} s_{\bar{I}^{z,T}}| \geq 1/2$ being valid throughout $\{\mathrm{dist}(\cdot, \bar{I}^{z,T}) < r_T/4\}$, which in turn follows from $\bar{H}_{\bar{I}^{z,T}} \leq 2/r_T$ since, by assumption, $r_T/2$ is an admissible tubular neighborhood width for $\bar{I}^{z,T}$ (cf. Definition 50). Within $\{\mathrm{dist}(\cdot, \bar{I}^{z,T}) < r_T/4\}$, we also record the following simplifications of (4.96) and (4.98):

$$\partial_t \xi^{z,T} = -\big(1 + \dot{\mathfrak{T}}\big) \frac{H'_{\bar{I}^{z,T}} \circ P_{\bar{I}^{z,T}}}{1 - \bar{H}_{\bar{I}^{z,T}} s_{\bar{I}^{z,T}}} \bar{\tau}_{\bar{I}^{z,T}} + \big(\bar{\tau}_{\bar{I}^{z,T}} \cdot \dot{z}\big) \frac{\bar{H}_{\bar{I}^{z,T}}}{1 - \bar{H}_{\bar{I}^{z,T}} s_{\bar{I}^{z,T}}} \bar{\tau}_{\bar{I}^{z,T}}, \tag{4.148}$$

$$\partial_t \vartheta^{z,T} = -\frac{1 + \dot{\mathfrak{T}}}{r_T^2} \Big(2 \frac{s_{\bar{I}^{z,T}}}{r_T^2} - \bar{H}_{\bar{I}^{z,T}} \Big) + \frac{\bar{\mathrm{n}}_{\bar{I}^{z,T}} \cdot \dot{z}}{r_T^2}. \tag{4.149}$$

Step 2: Identities for geometric quantities of perturbed interface. First, we define $(\mathfrak{h}, \mathfrak{h}', \mathfrak{h}'') := (h, h', h'') \circ P_{\bar{I}^{z,T}}$. Then, denoting by $o(1)$ any continuous function $f(\bar{H}_{\bar{I}^{z,T}} \mathfrak{h}, \mathfrak{h}')$ such that $f(\bar{H}_{\bar{I}^{z,T}} \mathfrak{h}, \mathfrak{h}') \to 0$ whenever $|\bar{H}_{\bar{I}^{z,T}} \mathfrak{h}| \to 0$ and $|\mathfrak{h}'| \to 0$, we claim that along I

$$\mathrm{n} = \Big(1 + \Big(-\frac{1}{2} + o(1) \Big) (\mathfrak{h}')^2 \Big) \bar{\mathrm{n}}_{\bar{I}^{z,T}} - \big(1 + o(1) \big) \mathfrak{h}' \bar{\tau}_{\bar{I}^{z,T}}, \tag{4.150}$$

$$V = \frac{\bar{H}_{\bar{I}^{z,T}}}{1-\bar{H}_{\bar{I}^{z,T}}\mathfrak{h}} + \mathfrak{h}'' + o(1)\mathfrak{h}'' + o(1)\mathfrak{h}'\bar{H}_{\bar{I}^{z,T}} + o(1)\mathfrak{h}\bar{H}'_{\bar{I}^{z,T}}. \tag{4.151}$$

For a proof of (4.150)–(4.151), it is computationally convenient to represent the interface I as the image of the curve $\gamma_h := (\mathrm{id} + h\bar{\mathrm{n}}_{\bar{I}^{z,T}}) \circ \bar{\gamma}^{z,T}$, where $\bar{\gamma}$ is an arc-length parametrization of $\bar{I}(T^{-1}(t))$ such that $\bar{\tau}_{\bar{I}^{z,T}} \circ \bar{\gamma}^{z,T} = (\bar{\gamma}^{z,T})'$. Then

$$\gamma_h' = \Big(\Big(1-\bar{H}_{\bar{I}^{z,T}}h\Big)\bar{\tau}_{\bar{I}^{z,T}} + h'\bar{\mathrm{n}}_{\bar{I}^{z,T}}\Big) \circ \bar{\gamma}^{z,T}, \tag{4.152}$$

hence (recall that $J \in \mathbb{R}^{2\times 2}$ denotes counter-clockwise rotation by $90°$)

$$\mathrm{n}_{\gamma_h} = J\frac{\gamma_h'}{|\gamma_h'|} = \left(\frac{\Big(1-\bar{H}_{\bar{I}^{z,T}}h\Big)\bar{\mathrm{n}}_{\bar{I}^{z,T}} - h'\bar{\tau}_{\bar{I}^{z,T}}}{\sqrt{\Big(1-\bar{H}_{\bar{I}^{z,T}}h\Big)^2 + (h')^2}}\right) \circ \bar{\gamma}^{z,T}, \tag{4.153}$$

so that (4.150) follows from Taylor expansion. By virtue of H^2 regularity of the height function h and V being the distributional curvature of I due to [46, Definition 13, item iii)] and $\chi_i \equiv 0$ for all $i \notin \{1, P\}$, we deduce $V = H_{\gamma_h}$. In other words,

$$\begin{aligned} V \circ \gamma_h &= \frac{\gamma_h'' \cdot J\gamma_h'}{|\gamma_h'|^3} \\ &= \left(\frac{h'\Big(\bar{H}'_{\bar{I}^{z,T}}h + 2\bar{H}_{\bar{I}^{z,T}}h'\Big) + (1-\bar{H}_{\bar{I}^{z,T}}h)\Big(h'' + \bar{H}_{\bar{I}^{z,T}}(1-\bar{H}_{\bar{I}^{z,T}}h)\Big)}{\sqrt{\Big(1-\bar{H}_{\bar{I}^{z,T}}h\Big)^2 + (h')^2}^{\,3}}\right) \circ \bar{\gamma}^{z,T}, \end{aligned} \tag{4.154}$$

so that (4.151) again follows from Taylor expansion.

Step 3: Change of variables formula. Let $g\colon I \to \mathbb{R}$ be integrable. Then, by the coarea formula

$$\int_I g\, d\mathcal{H}^1 = \int_{\bar{I}^{z,T}} g \circ (\mathrm{id}{+}h\bar{\mathrm{n}}_{\bar{I}^{z,T}})\sqrt{\Big(1-H_{\bar{I}^{z,T}}h\Big)^2 + (h')^2}\, d\mathcal{H}^1. \tag{4.155}$$

Furthermore, since the Jacobian of the tubular neighborhood diffeomorphism $x \mapsto (P_{\bar{I}^{z,T}}(x), s_{\bar{I}^{z,T}})$ is given by $1/(1-\bar{H}_{\bar{I}^{z,T}}s_{\bar{I}^{z,T}})$, we also obtain for any integrable $G\colon \mathbb{R}^2 \to \mathbb{R}$ with $\operatorname{supp} G \subset \{\mathrm{dist}(\cdot, \bar{I}^{z,T}) < r_T/4\}$ by the area formula

$$\int_{\mathbb{R}^2} (\chi - \bar{\chi}^{z,T})G\, dx = -\int_{\bar{I}^{z,T}} \int_0^h \frac{G\Big(x{+}s\bar{\mathrm{n}}_{\bar{I}^{z,T}}(x)\Big)}{1-\bar{H}_{\bar{I}^{z,T}}(x)s}\, ds d\mathcal{H}^1(x). \tag{4.156}$$

Step 4: Collecting further auxiliary identities. We obtain from (4.150) and (4.140) that, along I,

$$\mathrm{n} - \xi^{z,T} = o(1)\mathfrak{h}'\bar{\mathrm{n}}_{\bar{I}^{z,T}} - \Big(1{+}o(1)\Big)\mathfrak{h}'\bar{\tau}_{\bar{I}^{z,T}}, \tag{4.157}$$

$$1 - \mathrm{n} \cdot \xi^{z,T} = \Big(\frac{1}{2} + o(1)\Big)(\mathfrak{h}')^2. \tag{4.158}$$

In addition, we infer from (4.151) and (4.144) that, along I,

$$V + \nabla \cdot \xi^{z,T} = \mathfrak{h}'' + o(1)\mathfrak{h}'' + o(1)\mathfrak{h}'\bar{H}_{\bar{I}^{z,T}} + o(1)\mathfrak{h}\bar{H}'_{\bar{I}^{z,T}}, \tag{4.159}$$

as well as from (4.150), (4.151) and (4.140)–(4.141)

$$\begin{aligned} &V\mathrm{n} - (B^{z,T}\cdot\xi^{z,T})\xi^{z,T} \\ &= \Big(\bar H^2_{\bar I^{z,T}}\mathfrak{h}{+}\mathfrak{h}''\Big)\bar{\mathrm{n}}_{\bar I^{z,T}} - \bar H_{\bar I^{z,T}}\mathfrak{h}'\bar\tau_{\bar I^{z,T}} \\ &\quad + \Big(o(1)\mathfrak{h}'' + o(1)\bar H_{\bar I^{z,T}}\mathfrak{h}' + o(1)(\bar H'_{\bar I^{z,T}}\circ P_{\bar I^{z,T}})\mathfrak{h} + o(1)\bar H^2_{\bar I^{z,T}}\mathfrak{h}\Big)\bar{\mathrm{n}}_{\bar I^{z,T}} \\ &\quad + \Big(o(1)\mathfrak{h}'' + o(1)\bar H_{\bar I^{z,T}}\mathfrak{h}' + o(1)(\bar H'_{\bar I^{z,T}}\circ P_{\bar I^{z,T}})\mathfrak{h} + o(1)\bar H^2_{\bar I^{z,T}}\mathfrak{h}\Big)\bar\tau_{\bar I^{z,T}} \end{aligned} \tag{4.160}$$

and

$$B\cdot\xi - V = -\bar H^2_{\bar I^{z,T}}\mathfrak{h} - \mathfrak{h}'' + o(1)\mathfrak{h}'' + o(1)\mathfrak{h}'\bar H_{\bar I^{z,T}} + o(1)\mathfrak{h}\bar H'_{\bar I^{z,T}} + o(1)\mathfrak{h}\bar H^2_{\bar I^{z,T}}. \tag{4.161}$$

Next, we exploit the information gathered so far to express the terms originating from the stability estimate of E_{int} in terms of the height function h (and its derivatives). First, we get from combining (4.155), (4.148), (4.140)–(4.141), (4.143), (4.145) and (4.157),

$$\begin{aligned} &-\int_I \Big(\partial_t\xi^{z,T}{+}(B^{z,T}\cdot\nabla)\xi^{z,T}{+}(\nabla B^{z,T})^{\mathsf T}\xi^{z,T}\Big)\cdot(\mathrm{n}{-}\xi^{z,T})\,d\mathcal{H}^1 \\ &\le \int_{\bar I^{z,T}} H_{\bar I^{z,T}}h'(\tau_{\bar I^{z,T}}\cdot\dot z)\,d\mathcal{H}^1 - \int_{\bar I^{z,T}} H'_{\bar I^{z,T}}\dot{\mathfrak{T}}h'\,d\mathcal{H}^1 \\ &\quad + \int_{\bar I^{z,T}} |o(1)|\Big(\Big|H_{\bar I^{z,T}}(\tau_{\bar I^{z,T}}\cdot\dot z)h'\Big|{+}\Big|H'_{\bar I^{z,T}}\dot{\mathfrak{T}}h'\Big|\Big)\,d\mathcal{H}^1 \\ &= -\int_{\bar I^{z,T}} H^2_{\bar I^{z,T}}h(\mathrm{n}_{\bar I^{z,T}}\cdot\dot z)\,d\mathcal{H}^1 \\ &\quad - \int_{\bar I^{z,T}} H'_{\bar I^{z,T}}(\tau_{\bar I^{z,T}}\cdot\dot z)h\,d\mathcal{H}^1 - \int_{\bar I^{z,T}} H'_{\bar I^{z,T}}\dot{\mathfrak{T}}h'\,d\mathcal{H}^1 \\ &\quad + \int_{\bar I^{z,T}} |o(1)|\Big(\Big|H_{\bar I^{z,T}}(\tau_{\bar I^{z,T}}\cdot\dot z)h'\Big|{+}\Big|H'_{\bar I^{z,T}}\dot{\mathfrak{T}}h'\Big|\Big)\,d\mathcal{H}^1, \end{aligned} \tag{4.162}$$

where in the last step we also integrated by parts. Next, it directly follows from (4.155), (4.140)–(4.141) and (4.144)

$$\int_I \frac{1}{2}\Big|\nabla\cdot\xi^{z,T}{+}B^{z,T}\cdot\xi^{z,T}\Big|^2\,d\mathcal{H}^1 \le \int_{\bar I^{z,T}} \Big(1{+}|o(1)|\Big)\frac{1}{2}H^4_{\bar I^{z,T}}h^2\,d\mathcal{H}^1, \tag{4.163}$$

and exploiting in addition (4.158)

$$\begin{aligned} &-\int_I (1-\mathrm{n}\cdot\xi^{z,T})\nabla\cdot\xi^{z,T}(B^{z,T}\cdot\xi^{z,T})\,d\mathcal{H}^1 \\ &\le \int_{\bar I^{z,T}} \Big(1{+}|o(1)|\Big)\frac{1}{2}H^2_{\bar I^{z,T}}(h')^2\,d\mathcal{H}^1. \end{aligned} \tag{4.164}$$

Analogously, recalling also (4.145) and (4.146),

$$\int_I (1-\mathrm{n}\cdot\xi^{z,T})\nabla\cdot B^{z,T}\,d\mathcal{H}^1 \le -\int_{\bar I^{z,T}} \Big(1{-}|o(1)|\Big)\frac{1}{2}H^2_{\bar I^{z,T}}(h')^2\,d\mathcal{H}^1 \tag{4.165}$$

as well as

$$\begin{aligned} &-\int_I (\mathrm{n}{-}\xi^{z,T})\otimes(\mathrm{n}{-}\xi^{z,T}):\nabla B^{z,T}\,d\mathcal{H}^1 \\ &\le \int_{\bar I^{z,T}} \Big(1{+}|o(1)|\Big)H^2_{\bar I^{z,T}}(h')^2\,d\mathcal{H}^1 + \int_{\bar I^{z,T}} |o(1)|\Big|H'_{\bar I^{z,T}}\Big|(h')^2\,d\mathcal{H}^1. \end{aligned} \tag{4.166}$$

Just plugging in (4.159) and estimating by Young's inequality yields

$$
\begin{aligned}
&-\int_I \frac{1}{2}\Big|V+\nabla\cdot\xi^{z,T}\Big|^2\,d\mathcal{H}^1 \\
&\quad\leq -\int_{\bar{I}^{z,T}}\Big(1-|o(1)|\Big)\frac{1}{2}(h'')^2\,d\mathcal{H}^1 \\
&\qquad+\int_{\bar{I}^{z,T}}|o(1)|\Big(H^2_{\bar{I}^{z,T}}(h')^2+\Big((H'_{\bar{I}^{z,T}})^2+H^4_{\bar{I}^{z,T}}\Big)h^2\Big)\,d\mathcal{H}^1,
\end{aligned}
\tag{4.167}
$$

and analogously based on (4.160)

$$
\begin{aligned}
&-\int_I \frac{1}{2}\Big|V-(B^{z,T}\cdot\xi^{z,T})\xi^{z,T}\Big|^2\,d\mathcal{H}^1 \\
&\quad\leq -\int_{\bar{I}^{z,T}}\Big(1-|o(1)|\Big)\frac{1}{2}\Big((h'')^2+H^2_{\bar{I}^{z,T}}(h')^2+H^4_{\bar{I}^{z,T}}h^2\Big)\,d\mathcal{H}^1 \\
&\qquad-\int_{\bar{I}^{z,T}}H^2_{\bar{I}^{z,T}}hh''\,d\mathcal{H}^1 \\
&\qquad+\int_{\bar{I}^{z,T}}|o(1)|(H'_{\bar{I}^{z,T}})^2h^2\,d\mathcal{H}^1 \\
&\quad\leq -\int_{\bar{I}^{z,T}}\Big(1-|o(1)|\Big)\frac{1}{2}\Big((h'')^2+H^4_{\bar{I}^{z,T}}h^2\Big)\,d\mathcal{H}^1 \\
&\qquad+\int_{\bar{I}^{z,T}}\frac{1}{2}H^2_{\bar{I}^{z,T}}(h')^2\,d\mathcal{H}^1 \\
&\qquad+\int_{\bar{I}^{z,T}}2H_{\bar{I}^{z,T}}H'_{\bar{I}^{z,T}}hh'\,d\mathcal{H}^1 \\
&\qquad+\int_{\bar{I}^{z,T}}|o(1)|\Big(H^2_{\bar{I}^{z,T}}(h')^2+(H'_{\bar{I}^{z,T}})^2h^2\Big)\,d\mathcal{H}^1,
\end{aligned}
\tag{4.168}
$$

where in the last step we also carried out an integration by parts and estimated by Young's inequality.

We continue with the terms originating from the stability estimate of E_{bulk}. First, by means of (4.155), (4.142), (4.161) and an integration by parts we obtain

$$
\begin{aligned}
&\int_I \vartheta^{z,T}(B^{z,T}\cdot\xi^{z,T}-V)\,d\mathcal{H}^1 \\
&\quad\leq \int_{\bar{I}^{z,T}}\frac{H^2_{\bar{I}^{z,T}}}{r_T^2}h^2\,d\mathcal{H}^1-\int_{\bar{I}^{z,T}}\frac{1}{r_T^2}(h')^2\,d\mathcal{H}^1 \\
&\qquad+\int_{\bar{I}^{z,T}}|o(1)|\bigg(\Big(\frac{H^2_{\bar{I}^{z,T}}}{r_T^2}+\frac{1}{r_T^4}+(H'_{\bar{I}^{z,T}})^2\Big)h^2+H^2_{\bar{I}^{z,T}}(h')^2+(h'')^2\bigg)\,d\mathcal{H}^1.
\end{aligned}
\tag{4.169}
$$

Next, just plugging in (4.141)–(4.142) and (4.157) into (4.155) and applying Young's inequality entails

$$
\int_I \vartheta^{z,T}B^{z,T}\cdot(\mathrm{n}-\xi^{z,T})\,d\mathcal{H}^1\leq\int_{\bar{I}^{z,T}}|o(1)|\Big(\frac{1}{r_T^4}h^2+H^2_{\bar{I}^{z,T}}(h')^2\Big)\,d\mathcal{H}^1.
\tag{4.170}
$$

In addition, based on (4.156), (4.142) and (4.146), we may infer

$$
\int_{\mathbb{R}^2}(\chi-\bar{\chi}^{z,T})\vartheta^{z,T}\nabla\cdot B^{z,T}\,dx\leq-\int_{\bar{I}^{z,T}}\Big(1-|o(1)|\Big)\frac{1}{2}\frac{H^2_{\bar{I}^{z,T}}}{r_T^2}h^2\,d\mathcal{H}^1,
\tag{4.171}
$$

whereas it finally follows from (4.156), (4.149), (4.141) and (4.147)

$$\begin{aligned}
&\int_{\mathbb{R}^2}(\chi-\bar{\chi}^{z,T})\Big(\partial_t\vartheta^{z,T}+(B^{z,T}\cdot\nabla)\vartheta^{z,T}\Big)\,dx \\
&\quad\le \int_{\bar{I}^{z,T}}\frac{1}{r_T^4}h^2\,d\mathcal{H}^1-\int_{\bar{I}^{z,T}}\frac{H_{\bar{I}^{z,T}}}{r_T^2}h\dot{\mathfrak{T}}\,d\mathcal{H}^1-\int_{\bar{I}^{z,T}}\frac{1}{r_T^2}h(\mathrm{n}_{\bar{I}^{z,T}}\cdot\dot{z})\,d\mathcal{H}^1 \\
&\qquad+\int_{\bar{I}^{z,T}}|o(1)|\bigg(\frac{1}{r_T^3}\big|\dot{\mathfrak{T}}h\big|+\frac{1}{r_T^2}\big|H_{\bar{I}^{z,T}}\dot{\mathfrak{T}}h\big|+\frac{1}{r_T^2}\big|(\mathrm{n}_{\bar{I}^{z,T}}\cdot\dot{z})h\big|+\frac{1}{r_T^4}h^2\bigg)\,d\mathcal{H}^1.
\end{aligned} \tag{4.172}$$

Step 5: Conclusion. Due to (4.140) and (4.141), the identities (4.127), (4.129) and (4.131) hold true for trivial reasons. The remaining estimates follow from (4.162)–(4.172) and $|H_{\bar{I}^{z,T}}|\le 2/r_T$. □

CHAPTER 5

Weighted Inertia-Dissipation-Energy approach to doubly nonlinear wave equations

Abstract. We discuss a variational approach to doubly nonlinear wave equations of the form $\rho\partial_t^2 u + g(\partial_t u) - \Delta u + f(u) = 0$. This approach hinges on the minimization of a parameter-dependent family of uniformly convex functionals over entire trajectories, the so-called Weighted Inertia-Dissipation-Energy (WIDE) functionals. We prove that the WIDE functionals admit minimizers and that the corresponding Euler-Lagrange system is solvable in the strong sense. Moreover, we check that the parameter-dependent minimizers converge, up to subsequences, to a solution of the target doubly nonlinear wave equation as the parameter goes to 0. The analysis relies on specific estimates on the WIDE minimizers, on the decomposition of the subdifferential of the WIDE functional, and on the identification of the nonlinearities in the limit. Eventually, we investigate the viscous limit $\rho \to 0$, both at the functional level and on that of the equation.

5.1 Introduction

Semilinear wave equations of the form $\rho\partial_t^2 u - \Delta u + f(u) = 0$ in the space-time domain $\Omega \times (0,T)$ with $\Omega \subset \mathbb{R}^d$, $\rho > 0$, and $f = F'$ monotone can be addressed variationally by considering minimizers of the global-in-time functionals

$$u \mapsto \int_0^T \int_\Omega \mathrm{e}^{-t/\varepsilon} \left(\frac{\varepsilon^2 \rho}{2} |\partial_t^2 u|^2 + \frac{1}{2} |\nabla u|^2 + F(u) \right) \, \mathrm{d}x \, \mathrm{d}t$$

under given initial and boundary conditions. Indeed, minimizers u_ε of the latter converge up to subsequences to solutions of the semilinear wave equation as $\varepsilon \to 0$. This is the content of a celebrated conjecture by De Giorgi on the variational resolution of hyperbolic problems, which was proved in [120] for $T < \infty$ and in [109] for its original formulation with $T = \infty$. The interest in this possibility relies in reformulating the differential problem in terms of a uniformly convex minimization problem, combined with a limit passage. This ultimately delivers a novel approximation methodology for nonlinear wave equations.

Starting from this first positive results, the reach of the De Giorgi conjecture has been extended to other classes of nonlinear hyperbolic problems [110]. These include nonhomogeneous forcing terms [123, 124], general mechanical systems [77], and time-dependent domains [87]. The aim of this paper is to consider the extension of the variational approach to the case of a nonlinearly damped wave equation of the form

$$\rho\partial_t^2 u + g(\partial_t u) - \Delta u + f(u) = 0. \tag{5.1}$$

In addition to the nonlinearity $f(u)$ on u, the latter equation features a second nonlinear dissipation term $g(\partial_t u)$ with g monotone, making it a *doubly nonlinear* wave equation. Correspondingly, the global-in-time functionals take the form

$$I_{\rho\varepsilon} : u \mapsto \int_0^T \int_\Omega \mathrm{e}^{-t/\varepsilon} \left(\frac{\varepsilon^2\rho}{2}|\partial_t^2 u|^2 + \varepsilon G(\partial_t u) + \frac{1}{2}|\nabla u|^2 + F(u) \right) \,\mathrm{d}x \,\mathrm{d}t$$

where $G' = g$. These functionals feature the weighted sum (via the exponential weight $t \mapsto \mathrm{e}^{-t/\varepsilon}$ and powers of the parameter ε) of an *inertial* term $\rho|\partial_t^2 u|^2/2$, a *dissipation* term $G(\partial_t u)$, and an *energy* term $F(u)$. These global-in-time functionals are hence usually referred to as being of *Weighted Inertia-Dissipation-Energy* (WIDE) type. Correspondingly, the above-mentioned variational approximation strategy of minimizing the WIDE functionals and then passing to the limit $\varepsilon \to 0$ is called the *WIDE approach*. The relation between the minimization of $I_{\rho\varepsilon}$ and the solution to (5.1) is revealed by computing the Euler-Lagrange equation for $I_{\rho\varepsilon}$. Postponing all necessary details to the coming sections, we anticipate that it takes the form of the following fourth-order elliptic-in-time equation

$$\varepsilon^2\rho\partial_t^4 u - 2\varepsilon\rho\partial_t^3 u + \rho\partial_t^2 u - \varepsilon\partial_t(g(\partial_t u)) + g(\partial_t u) - \Delta u + f(u) = 0. \tag{5.2}$$

In particular, by formally taking the limit $\varepsilon \to 0$ one recovers the doubly nonlinear equation (5.1). The minimization of $I_{\rho\varepsilon}$ hence corresponds to an elliptic-in-time regularization of (5.1). Note that the Euler-Lagrange equation (5.2) is *not causal*, as its solution u_ε at a given time t depends on its values on the interval (t, T) as well. Causality is restored in the limit $\varepsilon \to 0$, which is hence referred to as *causal limit* in this context.

The WIDE approach for (5.1) has already been investigated for quadratic ψ. In this case, the resulting limiting problem (5.1) is a linearly damped semilinear wave equation. The amenability of this variational approximation procedure has been ascertained both in the case $T < \infty$ [78] and for $T = \infty$ [110]. In taking the limit $\varepsilon \to 0$, the linearity of the dissipation term $g(\partial_t u)$ is crucially used in [78, 110]. In particular, the identification of the nonlinearity $f(u)$ follows by compactness.

The focus of this paper is in extending the WIDE theory to the genuinely doubly nonlinear setting by letting G be not quadratic. We assume G to be convex and of p-growth, for some $2 \le p < 4$. This calls for a number of delicate extensions of the available arguments. First, the problem will be abstractly reformulated in Banach spaces, as opposed to the Hilbertian formulations of [78, 110]. Secondly, in passing to the limit as $\varepsilon \to 0$ one needs to identify two limits. Compactness will still enable to identify the limit in $f(u)$. For the identification of $g(\partial_t u)$ one uses a lower semicontinuity argument instead. This is challenging due to the hyperbolic nature of the problem. At this point, let us refer to [33], where the case of a positively 1-homogeneous ψ (but with $f = 0$) has been considered in the context of dynamic plasticity, with the help of tools for rate-independent flows [91].

Our first main result is the amenability of the WIDE approach in this doubly nonlinear hyperbolic setting. Theorem 68 states that, under suitable assumptions, the functionals $I_{\rho\varepsilon}$ admit unique minimizers u_ε, that they are strong solutions of the Euler-Lagrange equation

(5.2), and that u_ε converge to solutions of the doubly nonlinear wave equation (5.1) as $\varepsilon \to 0$, up to subsequences.

We then turn to the investigation of the so-called viscous limit $\rho \to 0$. This can be alternatively discussed at the level of the functionals $I_{\rho\varepsilon}$ or at the level of their minimizers, which we now indicate with $u_{\rho\varepsilon}$. Theorem 69 states that one can take any limit $(\rho, \varepsilon) \to (\rho_0, \varepsilon_0)$ and prove the convergence of the respective trajectories $u_{\rho\varepsilon}$ to the limiting one $u_{\rho_0\varepsilon_0}$. In particular, for $(\rho, \varepsilon) \to (0, 0)$ the minimizers $u_{\rho\varepsilon}$ converge to the unique solution of the doubly nonlinear flow $g(\partial_t u) - \Delta u + f(u) = 0$.

Before closing this introduction, let us mention the literature related to the *parabolic* version of the WIDE approach. In fact, elliptic-regularization nonvariational techniques for nonlinear parabolic PDEs are classical and can be traced back to Lions [80], see also Kohn & Nirenberg [69], Oleinĭk [97], and the book by Lions & Magenes [81]. An early result in a nonlinear setting is by Barbu [16].

The WIDE approach in the parabolic setting has been pioneered by Ilmanen [63], for the mean curvature flow of varifolds, and Hirano [60], for periodic solutions of gradient flows. Note that WIDE functionals are mentioned in the classical textbook by Evans [39, Problem 3, p. 487].

A variety of different parabolic abstract problems have been tackeld by the WIDE approach, including gradient flows [10, 19, 93], rate-independent flows [90, 92], doubly-nonlinear flows [6, 7, 8, 9, 88], nonpotential perturbations [5, 89] and variational approximations [76], curves of maximal slope in metric spaces [102, 103, 108], and parabolic SPDEs [105]. On the more applied side, the WIDE approach has been applied to microstructure evolution [31], crack propagation [71], mean curvature flow [63, 119], dynamic plasticity [33], and the incompressible Navier-Stokes system [17, 98].

The plan of this chapter is the following. We formulate the problem in abstract spaces, collect assumptions, and formulate our main results, Theorems 68-69 in Section 5.2. After collecting some preliminary material in Section 5.3, we prove the existence of a solution to the Euler-Lagrange problem in Section 5.4. This calls for an approximation of the WIDE functionals based on the Moreau-Yosida regularization, the characterization of their subdifferential, and a limiting procedure with respect to the approximation parameter. We eventually prove in Subsection 5.4.6 that the WIDE functional admits minimizers. The passage to the causal limit $\varepsilon \to 0$ is detailed in Section 5.5. Eventually, the viscous limit $\rho \to 0$ and its combination with the causal limit $\varepsilon \to 0$ are discussed in Section 5.6.

5.2 Assumptions and main results

In this section, we present an abstract formulation for the doubly nonlinear wave equation (5.1) and state our main results Theorems 68 and 69. Let us start by fixing some assumptions, which will hold throughout the paper.

Let $d \in \{2, 3\}$ and $\Omega \subset \mathbb{R}^d$ be a nonempty, open, bounded, and Lipschitz domain and $V \equiv L^p(\Omega)$ for $2 \le p < 4$. Moreover, let $X \equiv H^1_0(\Omega)$, so that $X \subset V$ densely and compactly. We indicate by V^* and X^* the dual spaces, by $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_X$ the corresponding duality pairings. Finally, in the following we denote by u' the time-derivative $\partial_t u$.

We are concerned with the analysis of WIDE approach to the abstract nonlinear hyperbolic Cauchy problem defined as

$$\rho u'' + \xi(t) + \eta(t) = 0 \qquad \text{in } V^* \text{ for a.e. } t \in (0, T), \tag{5.3}$$

$$\xi(t) = \mathrm{d}_V\psi(u'(t)) \quad \text{in } V^* \text{ for a.e. } t \in (0,T), \tag{5.4}$$
$$\eta(t) \in \partial\phi(u(t)) \quad \text{in } V^* \text{ for a.e. } t \in (0,T), \tag{5.5}$$
$$u(0) = u_0, \tag{5.6}$$
$$\rho u'(0) = \rho u_1. \tag{5.7}$$

Here, $T > 0$ is some reference final time, the prime denotes time differentiation, and ρ is a positive parameter ($\rho \to 0$ will be considered in Section 5.6 below). The convex functionals $\psi, \phi : V \to [0,\infty)$ are given as

$$\psi(v) = \begin{cases} \int_\Omega G(v)\,\mathrm{d}x & \text{if } G \circ v \in L^1(\Omega), \\ \infty & \text{otherwise,} \end{cases} \tag{5.8}$$
$$\phi(u) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 + F(u)\Big)\,\mathrm{d}x \quad \forall u \in V. \tag{5.9}$$

We denote by d_V the Gâteaux derivative and by ∂ the subdifferential in the sense of convex analysis. Finally, $u_0 \in X$ and $u_1 \in X \cap L^{q'}(\Omega)$ are given initial data, where $q' = 2p/(4-p)$.

The WIDE approach to the Cauchy problem (5.3)-(5.7) consists in defining the parameter-dependent family of WIDE functionals $I_{\rho\varepsilon} : L^p(0,T;V) \to (-\infty,\infty]$ over entire trajectories as

$$I_{\rho\varepsilon}(u) = \begin{cases} \int_0^T \mathrm{e}^{-t/\varepsilon}\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |u''(t)|^2\,\mathrm{d}x + \varepsilon\psi(u'(t)) + \phi(u(t))\Big)\mathrm{d}t & \text{if } u \in K(u_0,u_1), \\ \infty & \text{else,} \end{cases} \tag{5.10}$$

where we let

$$K(u_0,u_1) = \{u \in H^2(0,T;L^2(\Omega)) \cap W^{1,p}(0,T;L^p(\Omega)) \cap L^2(0,T;H^1_0(\Omega)) : \\ u(0) = u_0,\ \rho u'(0) = \rho u_1\}.$$

In the following, we assume $\psi : V \to [0,\infty)$ to be twice Gateaux differentiable, convex, and of p-growth. In particular, we assume that there exists a constant $C_1 > 0$ such that the following conditions hold

$$|u|_V^p \le C_1(\psi(u) + 1)\,, \quad \forall u \in V,\ \psi(0) = 0; \tag{5.11}$$
$$|\mathrm{d}_V\psi(u)|_{V^*}^{p'} \le C_1(|u|_V^p + 1)\,, \quad \forall u \in V,\ p' = p/(p-1)\,; \tag{5.12}$$
$$|\operatorname{Hes}(\psi(u))|_{L^{p''}(\Omega)}^{p''} \le C_1(|u|_V^p + 1)\,, \quad \forall u \in V,\ p'' = p/(p-2),\ \text{for } p > 2; \tag{5.13}$$
$$|\operatorname{Hes}(\psi(u))|_{L^\infty(\Omega)} \le C_1\,, \quad \forall u \in V,\ \text{for } p = 2. \tag{5.14}$$

As a consequence, there exists a constant $C_2 > 0$ such that

$$|u|_V^p \le C_2(\langle \mathrm{d}_V\psi(u), u\rangle_V + 1) \quad \forall u \in V, \tag{5.15}$$
$$\psi(u) \le \psi(0) + \langle \mathrm{d}_V\psi(u), u\rangle_V \le C_2\,(|u|_V^p + 1) \quad \forall u \in V. \tag{5.16}$$

We assume $F \in C^1(\mathbb{R})$ to be convex and $f = F' \in C(\mathbb{R})$ to have polynomial growth of order $r-1$, for $r \in [1,p]$. In particular, we ask for some constant $C_3 > 0$ such that

$$\frac{1}{C_3}|v|^r \le F(v) + C_3 \quad \text{and} \quad |f(v)|^{r'} \le C_3(1 + |v|^r) \quad \text{for all } v \in \mathbb{R}, \tag{5.17}$$

where $1/r + 1/r' = 1$. The growth assumptions in (5.17) imply that F has at most r-growth. We indicate by $\partial_X \phi_X$ the subdifferential from X to X^* of the restriction ϕ_X of ϕ to X. Note that $\partial_X \phi_X$ is a single-valued. More precisely, we have $\eta = \partial_X \phi_X = -\Delta u + f(u)$ in X^*. Furthermore, we can deduce the existence of a constant $C_4 > 0$ such that the following conditions hold:

$$D(\phi) \subset X\,; \ \ |u|_X^2 \le C_4(\phi(u) + 1) \ \ \forall u \in D(\phi), \tag{5.18}$$
$$|\eta|_{X^*} \le C_4(|u|_X + |u|_{L^r(\Omega)}^{r-1} + 1) \ \ \forall \eta = \partial_X \phi_X(u), \tag{5.19}$$
$$|\eta|_{X^*}^2 \le C_4(|u|_X^2 + \ell(|u|_V) + 1) \ \ \forall \eta = \partial_X \phi_X(u), \tag{5.20}$$

where ℓ is a nondecreasing function in $\mathbb{R}$. Note that (5.19) requires that $r \le 2^*$, whereas (5.20) requires that $r \le p$.

The Euler-Lagrange equation for $I_{\rho\varepsilon}$ under the constraints $u_\varepsilon(0) = u_0$ and $\rho u_\varepsilon'(0) = \rho u_1$ reads

$$\rho\varepsilon^2 u_\varepsilon'''' - 2\rho\varepsilon u_\varepsilon''' + \rho u_\varepsilon'' - \varepsilon\xi_\varepsilon' + \xi_\varepsilon + \eta_\varepsilon = 0 \quad \text{in } X^*, \text{ a.e. in } (0,T), \tag{5.21}$$
$$\xi_\varepsilon = \mathrm{d}_V\psi(u_\varepsilon') \quad \text{in } V^*, \text{ a.e. in } (0,T), \tag{5.22}$$
$$\eta_\varepsilon = \partial_X \phi_X(u_\varepsilon) \quad \text{in } X^*, \text{ a.e. in } (0,T), \tag{5.23}$$
$$u_\varepsilon(0) = u_0, \tag{5.24}$$
$$\rho u_\varepsilon'(0) = \rho u_1, \tag{5.25}$$
$$\rho u_\varepsilon''(T) = 0, \tag{5.26}$$
$$\varepsilon\rho u_\varepsilon'''(T) - \xi_\varepsilon(T) = 0. \tag{5.27}$$

In particular, the minimizers of the WIDE functionals u_ε solve a regularization of the target problem (5.3)-(5.7).

Our first main result reads as follows.

Theorem 68 (WIDE variational approach)**.** *Assume* (5.9)-(5.8), (5.11)-(5.14), and (5.17). *Then,*

i) The WIDE *functional $I_{\rho\varepsilon}$ admits a unique global minimizer $u_\varepsilon \in K(u_0, u_1)$.*

ii) For the unique minimizer u_ε of $I_{\rho\varepsilon}$, by letting $\xi_\varepsilon = \mathrm{d}_V\psi(u_\varepsilon')$ and $\eta_\varepsilon = \partial_X\phi_X(u_\varepsilon)$, the triple $(u_\varepsilon, \xi_\varepsilon, \eta_\varepsilon)$ belongs to

$$[W^{4,p'}(0,T;X^*) \cap H^2(0,T;L^2(\Omega)) \cap W^{1,p}(0,T;V) \cap L^2(0,T;X)]$$
$$\times\, W^{1,p'}(0,T;X^*) \times L^2(0,T;X^*),$$

and is a strong solution of the Euler-Lagrange problem (5.21)-(5.27)*. Moreover, the global minimizer of $I_{\rho\varepsilon}$ and the strong solution of the Euler-Lagrange system coincide.*

iii) For any sequence $\varepsilon_k \to 0$, there exists a subsequence (denoted by the same symbol) such that $(u_{\varepsilon_k}, \xi_{\varepsilon_k}, \eta_{\varepsilon_k}) \to (u, \xi, \eta)$ weakly in

$$[W^{1,p}(0,T;V) \cap L^2(0,T;X)] \times L^{p'}(0,T;V^*) \times L^2(0,T;X^*),$$

where (u, η, ξ) is a strong solution to the doubly nonlinear hyperbolic problem (5.3)-(5.7)*.*

Theorem 68.i-ii is proved in Section 5.4 by means of a regularization procedure whereas the $\varepsilon \to 0$ limit in Theorem 68.iii is obtained in Section 5.5.

Furthermore, we study the viscous limit for $\rho \to 0$ of the doubly nonlinear hyperbolic problem (5.3)-(5.7), recovering the doubly nonlinear parabolic problem studied in [8], namely

$$\xi(t) + \eta(t) = 0 \quad \text{in } V^* \text{ for a.e. } t \in (0,T), \tag{5.28}$$
$$\xi(t) = \mathrm{d}_V \psi(u'(t)) \quad \text{in } V^* \text{ for a.e. } t \in (0,T), \tag{5.29}$$
$$\eta(t) \in \partial\phi(u(t)) \quad \text{in } V^* \text{ for a.e. } t \in (0,T), \tag{5.30}$$
$$u(0) = u_0. \tag{5.31}$$

In this regard, our results are the following:

Theorem 69 (Viscous limit)**.** *Assume* (5.9)-(5.8), (5.11)-(5.14), and (5.17). *Then,*

i) The WIDE functionals $I_{\rho\varepsilon}$ Γ- converge as $\rho \to 0$ with respect to the strong topology of $L^p(0,T;V)$ to

$$\bar{I}_\varepsilon(u) = \begin{cases} \displaystyle\int_0^T \mathrm{e}^{-t/\varepsilon}\Big(\varepsilon\psi(u'(t)) + \phi(u(t))\Big)\mathrm{d}t & \text{if } u \in \bar{K}(u_0), \\ \infty & \text{else,} \end{cases}$$

where

$$\bar{K}(u_0) = \{W^{1,p}(0,T;V) \cap L^2(0,T;X) : u(0) = u_0\}.$$

ii) Let $(u_\rho, \xi_\rho, \eta_\rho)$ be a strong solution to the doubly nonlinear hyperbolic problem (5.3)-(5.7) *in*

$$[W^{1,p}(0,T;V) \cap L^2(0,T;X)] \times L^{p'}(0,T;V^*) \times L^2(0,T;X^*).$$

For any sequence $\rho_k \to 0$, there exists a subsequence (denoted by the same symbol) such that $(u_{\rho_k}, \xi_{\rho_k}, \eta_{\rho_k}) \to (\bar{u}, \bar{\xi}, \bar{\eta})$ weakly in*

$$[W^{1,p}(0,T;V) \cap L^\infty(0,T;X)] \times L^{p'}(0,T;V^*) \times L^2(0,T;X^*),$$

where $(\bar{u}, \bar{\xi}, \bar{\eta})$ is a strong solution to the doubly nonlinear parabolic problem (5.28)-(5.31).

iii) Let $(u_{\varepsilon\rho}, \xi_{\varepsilon\rho}, \eta_{\varepsilon\rho})$ be a strong solution of the Euler-Langrange problem (5.21)-(5.27) *belonging to the regularity class of Theorem 68.ii. For any pair of sequences $(\varepsilon_k, \rho_k) \to (0,0)$, there exists a not relabeled subsequence such that $(u_{\varepsilon_k\rho_k}, \xi_{\varepsilon_k\rho_k}, \eta_{\varepsilon_k\rho_k}) \to (\tilde{u}, \tilde{\xi}, \tilde{\eta})$ weakly in*

$$[W^{1,p}(0,T;V) \cap L^2(0,T;X)] \times L^{p'}(0,T;V^*) \times L^2(0,T;X^*),$$

where $(\tilde{u}, \tilde{\xi}, \tilde{\eta})$ is a strong solution to the doubly nonlinear parabolic problem (5.28)-(5.31).

Theorem 69 is proved in Section 5.6. More precisely, Theorem 69.i is obtained in Subsection 5.6.1, whereas the proof of Theorem 69.ii is given in Subsection 5.6.2. Eventually, Theorem 69.iii can be proved by combining the arguments of Sections 5.5-5.6.

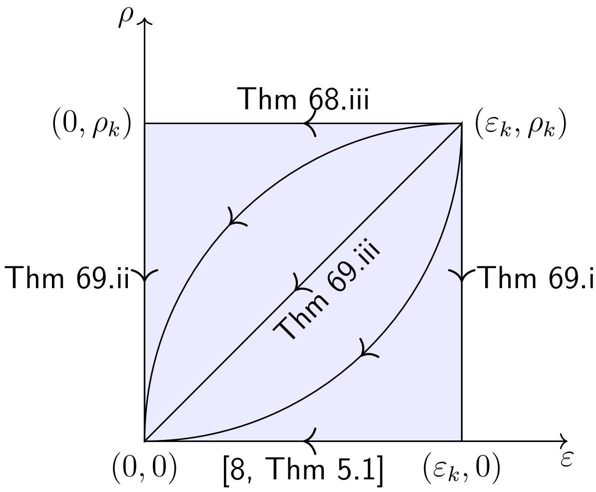


Figure 5.1: Theorems 68-69: illustration of the various limits with respect to ε and ρ.

5.3 Preliminary material

In this section, we present a tool for further use.

Lemma 70 (Weighted limsup tool)**.** *Let $A : D(A) \subset V \to 2^{V^*}$ be a maximal monotone operator. Let $v_\varepsilon \to v$ weakly in $L^p(0,T;V)$ as $\varepsilon \to 0$. Let $w_\epsilon(t) \in Av_\varepsilon(t)$ for almost every $t \in (0,T)$ be such that $w_\epsilon \to w$ weakly in $L^{p'}(0,T;V^*)$ as $\varepsilon \to 0$. Moreover, assume that the following inequality holds*

$$\limsup_{\varepsilon\to 0} \int_0^T (T-t)\langle w_\varepsilon, v_\varepsilon\rangle_V \,\mathrm{d}t \le \int_0^T (T-t)\langle w, v\rangle_V \,\mathrm{d}t. \tag{5.32}$$

Then, $w(t) \in Av(t)$ for almost every $t \in (0,T)$.

Remark 71. *Note that*

$$\int_0^T \int_0^t f(s)\,\mathrm{d}s\,\mathrm{d}t = \int_0^T (T-t) f(t)\,\mathrm{d}t\,, \qquad \text{for any } f \in L^1(0,T).$$

As a consequence, assumption (5.32) *of Lemma 70 is equivalent to*

$$\limsup_{\varepsilon\to 0} \int_0^T \int_0^t \langle w_\varepsilon, v_\varepsilon\rangle_V \,\mathrm{d}s\,\mathrm{d}t \le \int_0^T \int_0^t \langle w, v\rangle_V \,\mathrm{d}s\,\mathrm{d}t\,. \tag{5.33}$$

Proof of Lemma 70. Fix $t_0 \in (0,T)$. Let $\delta > 0$ be such that $t_0 > \delta$ and $T - t_0 > \delta$, and let $B_\delta(t_0) := (t_0 - \delta, t_0 + \delta)$. For any $\tilde{v} \in V$ and $\tilde{w} \in A\tilde{v}$, we define

$$\hat{v}(t) := \begin{cases} v(t) & \text{for } t \notin B_\delta(t_0), \\ \tilde{v} & \text{for } t \in B_\delta(t_0), \end{cases}$$

and

$$\hat{w}(t) := \begin{cases} z(t) & \text{for } t \notin B_\delta(t_0), \\ \tilde{w} & \text{for } t \in B_\delta(t_0), \end{cases}$$

where $z(t) \in Av(t)$.

Using the monotonicity of A, (5.32) and the weak convergence properties of w_ε and of v_ε, we deduce that

$$0 \leq \limsup_{\varepsilon \to 0} \int_0^T (T-t)\langle w_\varepsilon - \hat{w}, v_\varepsilon - \hat{v}\rangle_V \,\mathrm{d}t \leq \int_0^T (T-t)\langle w - \hat{w}, v - \hat{v}\rangle_V \,\mathrm{d}t.$$

From the definition of $\hat{v}$ we deduce

$$\int_0^T (T-t)\langle w - \hat{w}, v - \hat{v}\rangle_V \,\mathrm{d}t = \int_{B_\delta(t_0)} (T-t)\langle w - \tilde{w}, v - \tilde{v}\rangle_V \,\mathrm{d}t.$$

Since the left-hand side is nonnegative, it follows that

$$0 \leq \frac{1}{2\delta} \int_{B_\delta(t_0)} (T-t)\langle w - \tilde{w}, v - \tilde{v}\rangle_V \,\mathrm{d}t,$$

where the integral in the right-hand side converges to $(T-t_0)\langle w(t_0) - \tilde{w}, v(t_0) - \tilde{v}\rangle_V$ as $\delta \to 0$ for almost all $t_0 \in (0,T)$, due to the Lebesgue Differentiation Theorem. As a consequence, we obtain

$$\langle w(t_0) - \tilde{w}, v(t_0) - \tilde{v}\rangle_V \geq 0$$

for almost all $t_0 \in (0,T)$. Finally, recalling that $\tilde{w} \in A\tilde{v}$, by using the maximal monotonicity of A in $V \times V^*$, we can conclude that $w(t) \in Av(t)$ for almost all $t \in (0,T)$. □

5.4 Existence of solutions to the Euler-Lagrange problem

This section focuses on the solution of the Euler-Lagrange problem (5.21)-(5.27) and on the minimization of the WIDE functionals. At first, we introduce some approximation of the functionals in Subsection 5.4.1 and investigate their subdifferential in Subsection 5.4.2. These approximated functionals admit minimizers. The corresponding Euler-Lagrange problem is given in Subsection 5.4.3. We derive a priori estimate in Subsection 5.4.4 which allows to pass to the limit in the approximation in Subsection 5.4.5 and find a solution to the Euler-Lagrange problem (5.21)-(5.27). Eventually, such solutions are checked to correspond to minimizers of the WIDE functionals in Subsection 5.4.6.

5.4.1 Approximating functional $I^\lambda_{\rho\varepsilon}$

Define the spaces

$$\mathcal{V} = L^p(0,T;V)\,, \quad \mathcal{W} = H^2(0,T;L^2(\Omega)) \cap W^{1,p}(0,T;V)\,,$$

and the approximating functional $I^\lambda_{\rho\varepsilon} : \mathcal{V} \to (-\infty, \infty]$ as

$$I^\lambda_{\rho\varepsilon}(u) = \begin{cases} \displaystyle\int_0^T \mathrm{e}^{-t/\varepsilon}\Big(\frac{\varepsilon^2\rho}{2}|u''(t)|^2_{L^2(\Omega)} + \varepsilon\psi(u'(t)) + \phi_\lambda(u)\Big)\mathrm{d}t & \text{if } u \in K_\lambda(u_0,u_1), \\ \infty & \text{else,} \end{cases} \tag{5.34}$$

$$K_\lambda(u_0,u_1) = \{u \in \mathcal{W} : u(0) = u_0, \rho u'(0) = \rho u_1\}.$$

Here, $\lambda > 0$ and ϕ_λ denotes the Moreau-Yosida regularization of ϕ, namely

$$\phi_\lambda(u) := \inf_{v \in V}\left(\frac{1}{p\lambda}|u-v|^p_V + \phi(v)\right) = \frac{1}{p\lambda}|u - J_\lambda u|^p_V + \phi(J_\lambda u), \tag{5.35}$$

where J_λ is the p-resolvent of $\partial_V \phi_\lambda$ at level λ, namely the solution operator $J_\lambda : u \mapsto J_\lambda u$ to

$$F_V(u - J_\lambda u) \in \lambda \partial_V \phi(J_\lambda u) \quad \text{in } V^*, \tag{5.36}$$

for any $u \in V$, where $F_V : V \to V^*$ denotes the p-duality map between V and V^* (namely, $\langle F_V(u), u\rangle_V = |u|_V^p = |F_V(u)|_{V^*}^{p'}$ for any $u \in V$). Recall that $\partial \phi_\lambda(u) := F_V(u - J_\lambda u)/\lambda$. As

$$D(I_{\rho\varepsilon}^\lambda) = K_\lambda(u_0, u_1),$$

$I_{\rho\varepsilon}^\lambda$ can be decomposed as

$$I_{\rho\varepsilon}^\lambda = \bar{I}_{\rho\varepsilon} + \Phi_{\varepsilon\lambda}, \tag{5.37}$$

where the functionals $\bar{I}_{\rho\varepsilon}, \Phi_{\varepsilon\lambda} : \mathcal{V} \to [0, \infty]$ are defined by

$$\bar{I}_{\rho\varepsilon}(u) = \begin{cases} \displaystyle\int_0^T \mathrm{e}^{-t/\varepsilon}\Big(\frac{\varepsilon^2\rho}{2}|u''(t)|_{L^2(\Omega)}^2 + \varepsilon\psi(u'(t))\Big)\mathrm{d}t & \text{if } u \in K_\lambda(u_0, u_1), \\ \infty & \text{else,} \end{cases} \tag{5.38}$$

and

$$\Phi_{\varepsilon\lambda}(u) = \int_0^T \mathrm{e}^{-t/\varepsilon}\phi_\lambda(u)\,\mathrm{d}t \tag{5.39}$$

with domains

$$D(\bar{I}_{\rho\varepsilon}) = K_\lambda(u_0, u_1), \quad D(\Phi_{\varepsilon\lambda}) = \mathcal{V}.$$

The functional $I_{\rho\varepsilon}^\lambda$ is proper, lower semicontinuous, and convex in $\mathcal{V}$. Moreover, thanks to the Poincaré inequality, it is coercive on $\mathcal{V}$. The Direct Method ensures that $I_{\rho\varepsilon}^\lambda$ admits a minimizer $u_{\varepsilon\lambda} \in K_\lambda(u_0, u_1)$.

5.4.2 Representation of subdifferentials

In order to derive the Euler-Lagrange equation for $I_{\rho\varepsilon}^\lambda$, we prepare here some representation results. Recalling that

$$\mathcal{V} = L^p(0, T; V), \quad \mathcal{W} = H^2(0, T; L^2(\Omega)) \cap W^{1,p}(0, T; V),$$

we denote by $\partial_{\mathcal{V}}$ and $\partial_{\mathcal{W}}$ the subdifferentials in the sense of convex analysis from $\mathcal{V}$ to $\mathcal{V}^*$ and from $\mathcal{W}$ to $\mathcal{W}^*$, respectively.

First, note that we can further decompose the functional $\bar{I}_{\rho\varepsilon} : \mathcal{V} \to [0, +\infty]$ as

$$\bar{I}_{\rho\varepsilon} = Y_{\rho\varepsilon} + \bar{Y},$$

where

$$Y_{\rho\varepsilon}(u) = \begin{cases} \displaystyle\int_0^T \mathrm{e}^{-t/\varepsilon}\Big(\frac{\varepsilon^2\rho}{2}|u''(t)|_{L^2(\Omega)}^2 + \varepsilon\psi(u'(t))\Big)\,\mathrm{d}t & \text{if } u \in \mathcal{W}, \\ \infty & \text{else,} \end{cases}$$

$$\bar{Y}(u) = \begin{cases} 0 & \text{if } u \in \mathcal{W},\ u(0) = u_0,\ \text{and } \rho u'(0) = \rho u_1, \\ \infty & \text{else.} \end{cases}$$

The functional $Y_{\rho\varepsilon}$ is Gateaux differentiable in $\mathcal{W}$, in particular we have

$$\langle \mathrm{d}_{\mathcal{W}} Y_{\rho\varepsilon}(u), e\rangle_{\mathcal{W}} = \int_0^T \mathrm{e}^{-t/\varepsilon}\varepsilon^2\rho\langle u''(t), e''(t)\rangle_{L^2(\Omega)} + \mathrm{e}^{-t/\varepsilon}\varepsilon\langle \mathrm{d}_V\psi(u'(t)), e'(t)\rangle_V \,\mathrm{d}t\,, \quad \forall e\in\mathcal{W}\,.$$

On the other hand, we have

$$\langle f, e\rangle_{\mathcal{W}} = 0 \quad \text{for all } [u,f]\in\partial_{\mathcal{W}}\bar{Y} \text{ and } e\in\mathcal{W} \text{ with } e(0)=e'(0)=0.$$

Since $D(Y_{\rho\varepsilon}) = \mathcal{W}$, we deduce that

$$\partial_{\mathcal{W}}\bar{I}_{\rho\varepsilon} = \mathrm{d}_{\mathcal{W}}Y_{\rho\varepsilon} + \partial_{\mathcal{W}}\bar{Y}\,,$$

with domain

$$D(\partial_{\mathcal{W}}\bar{I}_{\rho\varepsilon}) = \{u\in\mathcal{W} : u(0)=u_0, \rho u'(0) = \rho u_1\}\,.$$

Let $u\in D(\partial_{\mathcal{W}}\bar{I}_{\rho\varepsilon})$. As we have $D(\bar{I}_{\rho\varepsilon})\subset\mathcal{W}\subset\mathcal{V}$, we conclude that $\partial_{\mathcal{V}}\bar{I}_{\rho\varepsilon}\subset\partial_{\mathcal{W}}\bar{I}_{\rho\varepsilon}$. Letting $f\in\partial_{\mathcal{V}}\bar{I}_{\rho\varepsilon}(u)$, since $f = (\mathrm{d}_{\mathcal{W}}Y_{\rho\varepsilon}(u) + \partial_{\mathcal{W}}\bar{Y}(u))\in\mathcal{V}^*$, we obtain

$$\int_0^T e^{-t/\varepsilon}\varepsilon^2\rho\left\langle u''(t), e''(t)\right\rangle_{L^2(\Omega)} + e^{-t/\varepsilon}\varepsilon\left\langle \mathrm{d}_V\psi\left(u'(t)\right), e'(t)\right\rangle_V \,\mathrm{d}t = \int_0^T \langle f(t), e(t)\rangle_V\,\mathrm{d}t\,, \tag{5.40}$$

$$\forall e\in\mathcal{W} \text{ such that } e(0)=e'(0)=0\,.$$

By setting

$$g(t) := -\int_t^T f(s)\,\mathrm{d}s,$$

we have $g\in W^{1,p'}(0,T;V^*)$ and, for all $\varphi\in C_c^\infty(0,T)$,

$$\begin{aligned}\int_0^T \varphi(t)g(t)\,\mathrm{d}t &= -\int_0^T\Big(\int_0^s\varphi(t)\;\mathrm{d}t\Big)f(s)\;\mathrm{d}s\\ &= -\int_0^T\Big(e^{-t/\varepsilon}\varepsilon^2\rho u''(t)\varphi'(t) + e^{-t/\varepsilon}\varepsilon\mathrm{d}_V\psi\left(u'(t)\right)\varphi(t)\Big)\,\mathrm{d}t,\end{aligned}$$

where in the last equality we used (5.40). As a next step, we observe that $\mathrm{d}_V\psi(u'(t))\in W^{1,q}(0,T;L^q(\Omega))$ with $q=\frac{2p}{3p-4}$ due to the assumptions on ψ, in particular (5.13)-(5.14). Note that $q\geq 1$ being $p\leq 4$. Hence, $\Big(g + e^{-t/\varepsilon}\varepsilon\mathrm{d}_V\psi(u')\Big)\in W^{1,q}(0,T;L^q(\Omega))$ due to $q\leq p'$, being $p\geq 2$. As a consequence, we obtain

$$\int_0^T\Big(g(t) + e^{-t/\varepsilon}\varepsilon\mathrm{d}_V\psi\left(u'(t)\right)\Big)\varphi(t)\,\mathrm{d}t = -\int_0^T e^{-t/\varepsilon}\varepsilon^2\rho u''(t)\varphi'(t)\,\mathrm{d}t,$$

whence $e^{-t/\varepsilon}\varepsilon^2\rho u''\in W^{1,q}(0,T;L^q(\Omega))$ with $\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{-t/\varepsilon}\varepsilon^2\rho u''\Big) = g + e^{-t/\varepsilon}\varepsilon\mathrm{d}_V\psi\left(u'\right)$. In particular, $\Big(e^{-t/\varepsilon}\varepsilon^2\rho u''' - e^{-t/\varepsilon}\varepsilon\rho u''\Big)\in L^q(0,T;L^q(\Omega))$ and $e^{-t/\varepsilon}\varepsilon^2\rho u'''\in L^q(0,T;L^q(\Omega))$ as $u''\in L^2(0,T;L^2(\Omega))$. It follows from (5.40) that

$$\begin{aligned}&-\int_0^T \mathrm{e}^{-t/\varepsilon}\varepsilon^2\rho u'''(t)\varphi'(t)\;\mathrm{d}t\\ &= \int_0^T f(t)\varphi(t)\;\mathrm{d}t - \int_0^T \mathrm{e}^{-t/\varepsilon}\varepsilon\mathrm{d}_V\psi\left(u'(t)\right)\varphi'(t)\;\mathrm{d}t - \int_0^T \mathrm{e}^{-t/\varepsilon}\varepsilon\rho u''(t)\varphi'(t)\;\mathrm{d}t,\end{aligned}$$

$$\text{for all } \varphi \in C_0^\infty(0,T),$$

which yields $\mathrm{e}^{-t/\varepsilon}\varepsilon^2\rho u''' - \mathrm{e}^{-t/\varepsilon}\varepsilon \mathrm{d}_V\psi(u') - \mathrm{e}^{-t/\varepsilon}\varepsilon\rho u'' \in W^{1,p'}(0,T;V^*)$, and hence, due to the facts obtained so far, we infer that $u \in W^{4,q}(0,T;L^q(\Omega))$.

Therefore, integrating by parts in (5.40), we have

$$f(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(e^{-t/\varepsilon}\varepsilon^2\rho u''(t)\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t/\varepsilon}\varepsilon \mathrm{d}_V\psi\left(u'(t)\right)\right) \in V^*$$

for almost all $t \in (0,T)$, $f \in \mathcal{V}^*$ and also $\varepsilon^2\rho u''(T) = \varepsilon^2\rho u'''(T) - \varepsilon\mathrm{d}_V\psi(u'(T)) = 0$, due to the arbitrariness of $e(T), e'(T) \in V$, as well as $\rho u''(T) = \varepsilon\rho u'''(T) - \mathrm{d}_V\psi(u'(T)) = 0$.

Finally, we can conclude that

$$\begin{aligned} D(\partial_{\mathcal{V}}\bar{I}_{\rho\varepsilon}) \subset \{u \in \mathcal{W} :\ & u \in W^{4,q}(0,T;L^q(\Omega)), u(0) = u_0, \rho u'(0) = \rho u_1, \\ & \rho u''(T) = 0, \varepsilon\rho u'''(T) - \mathrm{d}_V\psi(u'(T)) = 0\}, \end{aligned} \tag{5.41}$$

and

$$\partial_{\mathcal{V}}\bar{I}_{\rho\varepsilon}(u)(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(e^{-t/\varepsilon}\varepsilon^2\rho u''(t)\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t/\varepsilon}\varepsilon\mathrm{d}_V\psi\left(u'(t)\right)\right) \quad \text{for a.e. } t \in (0,T). \tag{5.42}$$

We observe that the inclusion $\subset$ in (5.41) can be replaced by an equality, as the reverse inclusion $\supset$ is straight forward.

Note that $\partial_{\mathcal{V}}\Phi_{\varepsilon\lambda} : \mathcal{V} \to \mathbb{R}$ is demicontinuous (i.e., strong-weak continuous) and single-valued. As a consequence, we have that $D(\Phi_{\varepsilon\lambda}) = \mathcal{V}$. As $\partial_{\mathcal{V}}\bar{I}_{\rho\varepsilon} + \partial_{\mathcal{V}}\Phi_{\varepsilon\lambda}$ is maximal monotone in $\mathcal{V} \times \mathcal{V}^*$, we obtain

$$\partial_{\mathcal{V}} I^\lambda_{\varepsilon\rho} = \partial_{\mathcal{V}}\bar{I}_{\rho\varepsilon} + \partial_{\mathcal{V}}\Phi_{\varepsilon\lambda}. \tag{5.43}$$

Note that the minimizer of $I^\lambda_{\varepsilon\rho}$ is unique. It hence coincides with the strong solution of (5.44).

5.4.3 Euler-Lagrange equation for $I^\lambda_{\rho\varepsilon}$

Thanks to the decomposition (5.43), the minimizer $u_{\varepsilon\lambda}$ of $I^\lambda_{\rho\varepsilon}$ fulfills

$$0 \in \partial_{\mathcal{V}}\bar{I}_{\rho\varepsilon}(u_{\varepsilon\lambda}) + \partial_{\mathcal{V}}\Phi_{\varepsilon\lambda}(u_{\varepsilon\lambda}).$$

In particular, the following holds

$$\rho\varepsilon^2 u''''_{\varepsilon\lambda} - 2\rho\varepsilon u'''_{\varepsilon\lambda} + \rho u''_{\varepsilon\lambda} + \eta_{\varepsilon\lambda} - \varepsilon\xi'_{\varepsilon\lambda} + \xi_{\varepsilon\lambda} = 0 \ \text{ in } V^*, \ \text{ a.e. in } (0,T), \tag{5.44}$$

$$\eta_{\varepsilon\lambda} = \partial_V\phi_\lambda(u_{\varepsilon\lambda}) = -\Delta J_\lambda u_{\varepsilon\lambda} + f(J_\lambda u_{\varepsilon\lambda}) \ \text{ in } V^*, \ \text{ a.e. in } (0,T), \tag{5.45}$$

$$\xi_{\varepsilon\lambda} = \mathrm{d}_V\psi(u'_{\varepsilon\lambda}) \text{ in } V^*, \ \text{ a.e. in } (0,T), \tag{5.46}$$

$$u_{\varepsilon\lambda}(0) = u_0, \tag{5.47}$$

$$\rho u'_{\varepsilon\lambda}(0) = \rho u_1, \tag{5.48}$$

$$\rho u''_{\varepsilon\lambda}(T) = 0, \tag{5.49}$$

$$\varepsilon\rho u'''_{\varepsilon\lambda}(T) - \xi_{\varepsilon\lambda}(T) = 0. \tag{5.50}$$

5.4.4 A priori estimates

We now derive a priori estimates for $u_{\varepsilon\lambda}$ which will eventually allow us to pass to the limit for $\lambda \to 0$ in Subsection 5.4.5, then as $\varepsilon \to 0$ in Section 5.5, and finally $\rho \to 0$ in Section 5.6. In what follows, the symbol C will denote a generic positive constant independent on λ, ε, and ρ, possibly varying from line to line. Furthermore, for the sake of brevity, we write (u, ξ, η) instead of $(u_{\varepsilon\lambda}, \xi_{\varepsilon\lambda}, \eta_{\varepsilon\lambda})$ in the following. Occasionally, specific dependencies of the constant C will be indicated.

Testing (5.44) with $u' - u_1$, integrating over $\Omega \times (0,t)$ for an arbitrary $t \in (0,T]$ and then integrating by parts, we obtain

$$\begin{aligned}
&\frac{\rho\varepsilon^2}{2}|u''(0)|^2_{L^2(\Omega)} - \frac{\rho\varepsilon^2}{2}|u''(t)|^2_{L^2(\Omega)} + \rho\varepsilon^2\langle u'''(t), u'(t) - u_1\rangle \\
&- 2\rho\varepsilon\langle u''(t), u'(t) - u_1\rangle + 2\rho\varepsilon \int_0^t |u''(s)|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&+ \frac{\rho}{2}|u'(t) - u_1|^2_{L^2(\Omega)} - \varepsilon\langle \xi(t), u'(t) - u_1\rangle + \int_0^t \langle \xi(s), u'(s) - u_1\rangle\,\mathrm{d}s + \varepsilon\psi(u'(t)) + \phi_\lambda(u(t)) \\
&= \varepsilon\psi(u_1) + \phi_\lambda(u_0) + \int_0^t \langle \eta(s), u_1\rangle\,\mathrm{d}s\,. \qquad (5.51)
\end{aligned}$$

Observe that for $t = T$, the estimate (5.51) reduces to

$$\begin{aligned}
&\int_0^T \langle \xi(t), u'(t) - u_1\rangle\,\mathrm{d}t + \frac{\rho\varepsilon^2}{2}|u''(0)|^2_{L^2(\Omega)} + 2\rho\varepsilon\int_0^T |u''(t)|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{\rho}{2}|u'(T) - u_1|^2_{L^2(\Omega)} \\
&+ \varepsilon\psi(u'(T)) + \phi_\lambda(u(T)) = \varepsilon\psi(u_1) + \phi_\lambda(u_0) + \int_0^T \langle \eta(t), u_1\rangle\,\mathrm{d}t\,. \qquad (5.52)
\end{aligned}$$

Integrating (5.51) over $(0,T)$, then integrating by parts and summing it to (5.52), we obtain

$$\begin{aligned}
&\frac{\rho\varepsilon^2(T+1)}{2}|u''(0)|^2_{L^2(\Omega)} + \frac{\rho(4-3\varepsilon)\varepsilon}{2}\int_0^T |u''(t)|^2_{L^2(\Omega)}\,\mathrm{d}t + 2\rho\varepsilon\int_0^T\int_0^t |u''(s)|^2_{L^2(\Omega)}\,\mathrm{d}s\,\mathrm{d}t \\
&+ \frac{\rho(1-2\varepsilon)}{2}|u'(T) - u_1|^2_{L^2(\Omega)} + \int_0^T \frac{\rho}{2}|u'(t) - u_1|^2_{L^2(\Omega)}\,\mathrm{d}t + (1-\varepsilon)\int_0^T \langle \xi(t), u'(t) - u_1\rangle\,\mathrm{d}t \\
&+ \int_0^T\int_0^t \langle \xi(s), u'(s) - u_1\rangle\,\mathrm{d}s\,\mathrm{d}t + \varepsilon\psi(u'(T)) + \phi_\lambda(u(T)) \\
&+ \varepsilon\int_0^T \psi(u'(t))\,\mathrm{d}t + \int_0^T \phi_\lambda(u(t))\,\mathrm{d}t \\
&= \varepsilon(T+1)\psi(u_1) + (T+1)\phi_\lambda(u_0) + \int_0^T \langle \eta(t), u_1\rangle\,\mathrm{d}t + \int_0^T\int_0^t \langle \eta(s), u_1\rangle\,\mathrm{d}s\,\mathrm{d}t\,. \qquad (5.53)
\end{aligned}$$

Remark 72. *Note that above computations are only formal. Indeed, the terms u'''', u''' and ξ' are only in $L^q(0,T;L^q(\Omega))$, whereas $u' \in \mathcal{V}^*$. This argument can be made rigorous by means of a time-discretization technique. For more details about a rigorous derivation of* (5.53)*, see Section 5.7.*

Using the fact that $\psi, \phi \geq 0$ and $\langle \xi(t), u'(t) - u_1\rangle \geq \psi(u'(t)) - \psi(u_1)$, we get from (5.53)

$$\frac{\rho(4-3\varepsilon)\varepsilon}{2}\int_0^T |u''(t)|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{\rho(1-2\varepsilon)}{2}|u'(T) - u_1|^2_{L^2(\Omega)}$$

$$
\begin{aligned}
&+ (1-\varepsilon)\int_0^T \psi(u'(t))\,\mathrm{d}t + \int_0^T \phi_\lambda(u(t))\,\mathrm{d}t \\
&\le \Big(\varepsilon + T + \frac{T^2}{2}\Big)\psi(u_1) + (T+1)\phi_\lambda(u_0) + \int_0^T \langle \eta(t), u_1\rangle\,\mathrm{d}t + \int_0^T\int_0^t \langle \eta(s), u_1\rangle\,\mathrm{d}s\,\mathrm{d}t\,.
\end{aligned}
$$

The last two terms in the previous inequality can be treated as follows

$$
\begin{aligned}
&\int_0^T \langle \eta(t), u_1\rangle\,\mathrm{d}t + \int_0^T\int_0^t \langle \eta(s), u_1\rangle\,\mathrm{d}s\,\mathrm{d}t = \int_0^T (1+T-t)\langle \eta(t), u_1\rangle\,\mathrm{d}t \\
&\le (1+T)\int_0^T |\eta(t)|_{X^*}|u_1|_X\,\mathrm{d}t \\
&\le C(T)\int_0^T (|J_\lambda u|_X + |J_\lambda u|^{r-1}_{L^r(\Omega)} + 1)|u_1|_X\,\mathrm{d}t \\
&\le \delta C(T)\int_0^T (|J_\lambda u|^2_X + |J_\lambda u|^r_{L^r(\Omega)})\,\mathrm{d}t + C(T,\delta)(|u_1|^2_X + |u_1|^r_X + |u_1|_X) \\
&\le \delta C(T)\int_0^T \phi(J_\lambda u)\,\mathrm{d}t + C(T,\delta,u_1) \\
&\le \delta C(T)\int_0^T \phi_\lambda(u)\,\mathrm{d}t + C(T,\delta,u_1)
\end{aligned}
$$

where we used (5.19), the Young inequality for a sufficiently small $\delta > 0$ to be fixed below, as well as (5.17) and definition (5.35). As a consequence, we obtain the following estimate

$$
\begin{aligned}
&\frac{\rho(4-3\varepsilon)\varepsilon}{2}\int_0^T |u''(t)|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{\rho(1-2\varepsilon)}{2}|u'(T)-u_1|^2_{L^2(\Omega)} \\
&+ (1-\varepsilon)\int_0^T \psi(u'(t))\,\mathrm{d}t + c\int_0^T \phi_\lambda(u(t))\,\mathrm{d}t \\
&\le \Big(\varepsilon + T + \frac{T^2}{2}\Big)\psi(u_1) + (T+1)\phi_\lambda(u_0) + C(T,\delta,u_1)
\end{aligned} \tag{5.54}
$$

for some strictly positive constant $c = c(T,\delta) < 1$. For a sufficently small $\varepsilon > 0$, from (5.54) we can deduce

$$\varepsilon\rho\|u''\|^2_{L^2(0,T;L^2(\Omega))} \le C, \tag{5.55}$$
$$\rho|u'(T)-u_1|^2_{L^2(\Omega)} \le C, \tag{5.56}$$
$$\|u'\|^p_{L^p(0,T;V)} \le C, \tag{5.57}$$
$$\|\,\mathrm{d}_V\psi(u')\|^{p'}_{L^{p'}(0,T;V^*)} \le C \tag{5.58}$$

due to (5.11) and (5.12), and

$$\int_0^T \phi_\lambda(u(t))\,\mathrm{d}t \le C\,. \tag{5.59}$$

Hence, using (5.17)-(5.18) and (5.35), we get

$$\|J_\lambda u\|_{L^2(0,T;X)} \le C\,, \tag{5.60}$$
$$\|J_\lambda u\|_{L^r(0,T;L^r(\Omega))} \le C\,. \tag{5.61}$$

Furthermore, it follows that

$$\sup_{t\in(0,T)} |u(t)|_V \leq C, \tag{5.62}$$

$$\sqrt{\rho\varepsilon}\|u'\|_{C([0,T];L^2(\Omega))} \leq C\,, \tag{5.63}$$

whence we obtain

$$\sqrt{\rho\varepsilon}\|\mathrm{d}_V\psi(u'(t))\|_{W^{1,q}(0,T;L^q(\Omega))} \leq C \quad \text{with } q = \frac{2p}{3p-4}, \tag{5.64}$$

due to (5.13)-(5.14). Since $\eta \in \partial_V\phi(J_\lambda u) \subset \partial_X\phi_X(J_\lambda u)$, assumption (5.20) together with, (5.60), $|J_\lambda u|_V \leq C(|u|_V + 1)$ (see [22, Lemma 1.3]) and (5.62) give

$$\|\eta\|^2_{L^2(0,T;X^*)} \leq C. \tag{5.65}$$

We close this subsection by deriving additional a priori estimates. A comparison in equation (5.44) yields

$$\|\rho\varepsilon^2 u'''' - 2\rho\varepsilon u''' - \varepsilon\xi'\|_{L^2(0,T;X^*)+L^{p'}(0,T;V^*)} \leq C(T,\rho),$$
$$\|\rho\varepsilon^2 u'''' - 2\rho\varepsilon u''' + \rho u''\|_{L^2(0,T;X^*)+L^q(0,T;L^q(\Omega))+L^{p'}(0,T;V^*)} \leq C(T,\rho).$$

Moreover, we deduce additional regularity for the first two terms in (5.44). Observe that, setting $v := u'''$, we can write the following ODE

$$\rho\varepsilon^2 v'(t) - 2\rho\varepsilon v(t) + w(t) = 0 \quad \text{ in } V^*\,,$$

where $w(t) := \rho u''(t) + \eta(t) - \varepsilon\xi'(t) + \xi(t)$ for a.a. $t \in (0,T)$. Note that we have a uniform bound (only in λ, not in ε) in the $L^q(0,T;L^q(\Omega)+X^*)$ norm for w thanks to (5.55), (5.64), and (5.65). Hence, solving the ODE we can deduce

$$\|v\|_{L^q(0,T;L^q(\Omega)+X^*)} \leq \varepsilon^{-5/2}C(\rho,T)\,,$$

then, by comparison,

$$\|v'\|_{L^q(0,T;L^q(\Omega)+X^*)} \leq \varepsilon^{-7/2}C(\rho,T)\,,$$

which eventually ensures that

$$\|u'''\|_{L^q(0,T;L^q(\Omega)+X^*)} + \|u''''\|_{L^q(0,T;L^q(\Omega)+X^*)} \leq C(\varepsilon,\rho,T)\,.$$

5.4.5 Passage to the limit as $\lambda \to 0$

Let $u_{\varepsilon\lambda}$ be a minimizer of $I^\lambda_{\rho\varepsilon}$, $\eta_{\varepsilon\lambda} = \partial_V\phi_\lambda(u_{\varepsilon\lambda})$, and $\xi_{\varepsilon\lambda} = \mathrm{d}_V\psi(u'_{\varepsilon\lambda})$. We have proved that $(u_{\varepsilon\lambda}, \eta_{\varepsilon\lambda}, \xi_{\varepsilon\lambda})$ solves (5.44)–(5.50). From the uniform estimates of Subsection 5.4.4, we deduce the following convergences as $\lambda \to 0$ (up to not relabeled subsequences)

$$u_{\varepsilon\lambda} \to u_\varepsilon \text{ weakly in } W^{4,q}(0,T;L^q(\Omega)+X^*) \cap H^2(0,T;L^2(\Omega)) \cap W^{1,p}(0,T;V), \tag{5.66}$$
$$J_\lambda u_{\varepsilon\lambda} \to v_\varepsilon \text{ weakly in } L^2(0,T;X), \tag{5.67}$$
$$\xi_{\varepsilon\lambda} \to \xi_\varepsilon \text{ weakly in } W^{1,q}(0,T;L^q(\Omega)+X^*) \cap L^{p'}(0,T;V^*), \tag{5.68}$$
$$\eta_{\varepsilon\lambda} \to \eta_\varepsilon \text{ weakly in } L^2\,(0,T;X^*)\,. \tag{5.69}$$

Convergences (5.66)-(5.69) are sufficient in order to pass to the limit in equation (5.44) and obtain

$$\rho\varepsilon^2 u_\varepsilon'''' - 2\rho\varepsilon u_\varepsilon''' + \rho u_\varepsilon'' - \varepsilon\xi_\varepsilon' + \xi_\varepsilon + \eta_\varepsilon = 0 \quad \text{in } L^q(\Omega) + X^* \text{ for a.e. } t \in (0,T). \tag{5.70}$$

Proceeding as in [8, Subsec. 3.3], we can deduce that

$$J_\lambda u_{\varepsilon\lambda} \to v_\varepsilon \text{ strongly in } C([0,T];V), \tag{5.71}$$

$v_\varepsilon = u_\varepsilon$, and that the strong convergences

$$u_{\varepsilon\lambda} \to u_\varepsilon \text{ strongly in } L^p(0,T;V)\,, \tag{5.72}$$

$$u_{\varepsilon\lambda}(t) \to u_\varepsilon(t) \text{ strongly in } V \text{ for a.a. } t \in (0,T)\,, \tag{5.73}$$

hold. Moreover, due to (5.62) we have that $u_{\varepsilon\lambda} \to u_\varepsilon$ strongly in any $L^s(0,T;V)$ for $s \in [0,\infty)$.

Since $V^* \subset X^*$ compactly and $L^2(\Omega) \subset X^*$ compactly, we have

$$\xi_{\varepsilon\lambda} \to \xi_\varepsilon \text{ strongly in } C([0,T];X^*), \tag{5.74}$$

$$u_{\varepsilon\lambda}'' \to u_\varepsilon'' \text{ strongly in } C([0,T];X^*), \tag{5.75}$$

hence $u_\varepsilon''(T) = 0$. Furthermore, let $q(t) := \liminf_{\lambda\to 0} |u_{\varepsilon\lambda}'(t)|_{L^2(\Omega)}$, and note that $q \in C[0,T]$ due to (5.63). Since $q(t) < \infty$ for all $t \in [0,T]$, we can take a subsequence $\lambda_n^t \to 0$ (possibly depending on t) such that

$$u_{\varepsilon\lambda}'(t) \to u_\varepsilon'(t) \text{ weakly in } L^2(\Omega). \tag{5.76}$$

Finally, the initial data (5.47)-(5.48) and the final datum (5.49) can be recovered in the limit $\lambda \to 0$ thanks to convergence (5.66). The final datum (5.50) can be recovered in the limit arguing as it follows. From (5.50) and (5.74) one can deduce that

$$\varepsilon\rho u_{\varepsilon\lambda}'''(T) = \xi_{\varepsilon\lambda}(T) \to \xi_\varepsilon(T) \text{ strongly in } X^*.$$

On the other hand, it follows from (5.66) that

$$\varepsilon\rho u_{\varepsilon\lambda}'''(T) \to \varepsilon\rho u_\varepsilon'''(T) \text{ weakly in } L^q(\Omega) + X^*,$$

hence $\varepsilon\rho u_\varepsilon'''(T) = \xi_\varepsilon(T)$ in $L^q(\Omega) + X^*$.

Identification of the nonlinearities

Identification of η_ε. Define the operator $A : X \to X^*$ and recall $B : X \to X^*$ as

$$\langle Au, v\rangle_X := \int_\Omega \nabla u\cdot\nabla v \, \mathrm{d}x, \quad \langle B(u), v\rangle_V := \int_\Omega f(u)v \, \mathrm{d}x \tag{5.77}$$

so that $\langle Au, u\rangle = \|u\|_X^2$ and $B(u) = f(u)$ almost everywhere. At first, convergence (5.67) entails that $AJ_\lambda u_{\varepsilon\lambda} \to Au_\varepsilon$ weakly in $L^2(0,T;X^*)$. On the other hand, the strong convergence (5.73) and the continuity of f ensure that $f(u_{\varepsilon\lambda}) \to f(u_\varepsilon)$ almost everywhere. As f is bounded by (5.17), we readily get that $f(u_{\varepsilon\lambda}) \to f(u_\varepsilon)$ strongly in $L^s(0,T;L^s(\Omega))$ for any $s < r'$. As

$r \leq p < 2^*$, this in particular implies that $\eta_\varepsilon = -\Delta u_\varepsilon + f(u_\varepsilon)$ in X^* almost everywhere in $(0,T)$.

Identification of ξ_ε. Recall that $u_{\lambda\varepsilon} \in W^{4,q}(0,T;X^*) \cap H^2(0,T;L^2(\Omega)) \cap W^{1,p}(0,T;V)$ is such that $\rho u''_{\lambda\varepsilon}(T) = 0$ and $u_{\lambda\varepsilon}(0) = u_0$, $\rho u'_{\lambda\varepsilon}(0) = \rho u_1$, and $\xi_{\varepsilon\lambda} \in W^{1,q}(0,T;L^q(\Omega)) \cap L^{p'}(0,T;V^*)$. Moreover, note that we have $\rho\varepsilon^2 u''''_{\lambda\varepsilon} - 2\rho\varepsilon u'''_{\lambda\varepsilon} - \varepsilon\xi'_{\lambda\varepsilon} \in L^{p'}(0,T;V^*)$. Consider a sequence of mollifiers $g_{\tilde\rho}$, namely $g_{\tilde\rho}(x) = \tilde\rho^{-d} g(x/\tilde\rho)$ with $g \in C_c^\infty(\Omega)$ and $\int_{\mathbb{R}} g(t)\,\mathrm{d}t = 1$, and define $u^{\tilde\rho}_{\varepsilon\lambda} := g_{\tilde\rho} * u_{\varepsilon\lambda}$. Then, we have $(u^{\tilde\rho}_{\varepsilon\lambda})' = g_{\tilde\rho} * u'_{\varepsilon\lambda}$ and

$$
\begin{aligned}
&\int_0^T \langle \varepsilon\xi_{\varepsilon\lambda}(t), u'_{\varepsilon\lambda}(t)\rangle_V \,\mathrm{d}t = \lim_{\tilde\rho\to 0} \int_0^T \langle \varepsilon\xi_{\varepsilon\lambda}(t), (u^{\tilde\rho}_{\varepsilon\lambda})'(t)\rangle_V \,\mathrm{d}t \\
&= \lim_{\tilde\rho\to 0} \langle \varepsilon\xi_{\varepsilon\lambda}(T), u^{\tilde\rho}_{\varepsilon\lambda}(T) - u_0\rangle - \langle \varepsilon\xi_{\varepsilon\lambda}(0), u^{\tilde\rho}_{\varepsilon\lambda}(0) - u_0\rangle - \int_0^T \langle \varepsilon\xi'_{\varepsilon\lambda}(t), u^{\tilde\rho}_{\varepsilon\lambda}(t) - u_0\rangle \,\mathrm{d}t
\end{aligned}
$$

where we integrated by parts. Note that the dualities in the last line are with respect the dual space of $X^* + L^q(\Omega)$, namely $X \cap L^{q'}(\Omega)$ with $q' = \frac{2p}{4-p}$ conjugate exponent of $q = \frac{2p}{3p-4}$.

As a next step, we exploit (5.44), obtaining

$$
\begin{aligned}
\int_0^T \langle \varepsilon\xi_{\varepsilon\lambda}(t), u'_{\varepsilon\lambda}(t)\rangle_V \,\mathrm{d}t = \lim_{\tilde\rho\to 0} \Big(&\langle \varepsilon\xi_{\varepsilon\lambda}(T), u^{\tilde\rho}_{\varepsilon\lambda}(T) - u_0\rangle - \langle \varepsilon\xi_{\varepsilon\lambda}(0), u^{\tilde\rho}_{\varepsilon\lambda}(0) - u_0\rangle \\
&- \int_0^T \langle \rho\varepsilon^2 u''''_{\varepsilon\lambda}(t) - 2\rho\varepsilon u'''_{\varepsilon\lambda}(t), u^{\tilde\rho}_{\varepsilon\lambda}(t) - u_0\rangle \,\mathrm{d}t \Big) \\
- \int_0^T \langle \rho u''_{\varepsilon\lambda}(t) + \xi_{\varepsilon\lambda} &+ \eta_{\varepsilon\lambda}, u_{\varepsilon\lambda}(t) - u_0\rangle_V \,\mathrm{d}t.
\end{aligned}
$$

We integrate by parts the term

$$
\begin{aligned}
&\langle \varepsilon\xi_{\varepsilon\lambda}(T), u^{\tilde\rho}_{\varepsilon\lambda}(T) - u_0\rangle - \langle \varepsilon\xi_{\varepsilon\lambda}(0), u^{\tilde\rho}_{\varepsilon\lambda}(0) - u_0\rangle \\
&- \int_0^T \langle \rho\varepsilon^2 u''''_{\varepsilon\lambda}(t) - 2\rho\varepsilon u'''_{\varepsilon\lambda}(t), u^{\tilde\rho}_{\varepsilon\lambda}(t) - u_0\rangle \,\mathrm{d}t \\
&= \langle \varepsilon\xi_{\varepsilon\lambda}(T) - \rho\varepsilon^2 u'''_{\varepsilon\lambda}(T), u^{\tilde\rho}_{\varepsilon\lambda}(T) - u_0\rangle - \langle \varepsilon\xi_{\varepsilon\lambda}(0) - \rho\varepsilon^2 u'''_{\varepsilon\lambda}(0), u^{\tilde\rho}_{\varepsilon\lambda}(0) - u_0\rangle \\
&\quad + \langle 2\rho\varepsilon u''_{\varepsilon\lambda}(T), u^{\tilde\rho}_{\varepsilon\lambda}(T) - u_0\rangle - \langle 2\rho\varepsilon u''_{\varepsilon\lambda}(0), u^{\tilde\rho}_{\varepsilon\lambda}(0) - u_0\rangle \\
&\quad + \int_0^T \langle \rho\varepsilon^2 u'''_{\varepsilon\lambda}(t), (u^{\tilde\rho}_{\varepsilon\lambda})'(t)\rangle \,\mathrm{d}t - \int_0^T \langle 2\rho\varepsilon u''_{\varepsilon\lambda}(t), (u^{\tilde\rho}_{\varepsilon\lambda})'(t)\rangle \,\mathrm{d}t \\
&= -\langle \varepsilon\xi_{\varepsilon\lambda}(0) - \rho\varepsilon^2 u'''_{\varepsilon\lambda}(0), u^{\tilde\rho}_{\varepsilon\lambda}(0) - u_0\rangle - \langle 2\rho\varepsilon u''_{\varepsilon\lambda}(0), u^{\tilde\rho}_{\varepsilon\lambda}(0) - u_0\rangle \\
&\quad + \int_0^T \langle \rho\varepsilon^2 u'''_{\varepsilon\lambda}(t), (u^{\tilde\rho}_{\varepsilon\lambda})'(t)\rangle \,\mathrm{d}t - \int_0^T \langle 2\rho\varepsilon u''_{\varepsilon\lambda}(t), (u^{\tilde\rho}_{\varepsilon\lambda})'(t)\rangle \,\mathrm{d}t \\
&= -\langle \varepsilon\xi_{\varepsilon\lambda}(0) - \rho\varepsilon^2 u'''_{\varepsilon\lambda}(0), u^{\tilde\rho}_{\varepsilon\lambda}(0) - u_0\rangle - \langle 2\rho\varepsilon u''_{\varepsilon\lambda}(0), u^{\tilde\rho}_{\varepsilon\lambda}(0) - u_0\rangle - \langle \rho\varepsilon^2 u''_{\varepsilon\lambda}(0), (u^{\tilde\rho}_{\varepsilon\lambda})'(0)\rangle \\
&\quad - \int_0^T \langle \rho\varepsilon^2 u''_{\varepsilon\lambda}(t), (u^{\tilde\rho}_{\varepsilon\lambda})''(t)\rangle \,\mathrm{d}t - \int_0^T \langle 2\rho\varepsilon u''_{\varepsilon\lambda}(t), (u^{\tilde\rho}_{\varepsilon\lambda})'(t)\rangle \,\mathrm{d}t,
\end{aligned}
$$

due to (5.49) and (5.50). Hence, we obtain

$$
\begin{aligned}
\lim_{\tilde\rho\to 0} \Big(&\langle \varepsilon\xi_{\varepsilon\lambda}(T), u^{\tilde\rho}_{\varepsilon\lambda}(T) - u_0\rangle - \langle \varepsilon\xi_{\varepsilon\lambda}(0), u^{\tilde\rho}_{\varepsilon\lambda}(0) - u_0\rangle \\
&- \int_0^T \langle \rho\varepsilon^2 u''''_{\varepsilon\lambda}(t) - 2\rho\varepsilon u'''_{\varepsilon\lambda}(t), u^{\tilde\rho}_{\varepsilon\lambda}(t) - u_0\rangle \,\mathrm{d}t \Big)
\end{aligned}
$$

$$
\begin{aligned}
&= -\langle \rho\varepsilon^2 u''_{\varepsilon\lambda}(0), u_1\rangle_{X\cap L^{q'}(\Omega)} - \int_0^T \rho\varepsilon^2 |u''_{\varepsilon\lambda}(t)|^2_{L^2(\Omega)}\,\mathrm{d}t - \int_0^T \langle 2\rho\varepsilon u''_{\varepsilon\lambda}(t), u'_{\varepsilon\lambda}(t)\rangle_V\,\mathrm{d}t \\
&= -\langle \rho\varepsilon^2 u''_{\varepsilon\lambda}(0), u_1\rangle_{X\cap L^{q'}(\Omega)} - \int_0^T \rho\varepsilon^2 |u''_{\varepsilon\lambda}(t)|^2_{L^2(\Omega)}\,\mathrm{d}t \\
&\quad - \rho\varepsilon |u'_{\varepsilon\lambda}(T)|^2_{L^2(\Omega)} + \rho\varepsilon |u_1|^2_{L^2(\Omega)}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_0^T \langle \varepsilon\xi_{\varepsilon\lambda}(t), u'_{\varepsilon\lambda}(t)\rangle_V\,\mathrm{d}t \\
&= -\langle \rho\varepsilon^2 u''_{\varepsilon\lambda}(0), u_1\rangle_{X\cap L^{q'}(\Omega)} - \int_0^T \rho\varepsilon^2 |u''_{\varepsilon\lambda}(t)|^2_{L^2(\Omega)}\,\mathrm{d}t \\
&\quad - \rho\varepsilon |u'_{\varepsilon\lambda}(T)|^2_{L^2(\Omega)} + \rho\varepsilon |u_1|^2_{L^2(\Omega)} - \int_0^T \langle \rho u''_{\varepsilon\lambda}, u_{\varepsilon\lambda}(t) - u_0\rangle_V\,\mathrm{d}t \\
&\quad - \int_0^T \langle \xi_{\varepsilon\lambda}, u_{\varepsilon\lambda}(t) - u_0\rangle_V\,\mathrm{d}t - \int_0^T \langle \eta_{\varepsilon\lambda}, u_{\varepsilon\lambda}(t) - u_0\rangle_V\,\mathrm{d}t\,. \qquad (5.78)
\end{aligned}
$$

Observe now that

$$
\begin{aligned}
\int_0^T \langle \eta_{\varepsilon\lambda}(t), J_\lambda u_{\varepsilon\lambda}(t) - u_0\rangle_X\,\mathrm{d}t &= \int_0^T \langle \eta_{\varepsilon\lambda}(t), u_{\varepsilon\lambda}(t) - u_0\rangle_V\,\mathrm{d}t - \lambda^{p'-1}\int_0^T |\eta_{\varepsilon\lambda}(t)|^{p'}_{V^*}\,\mathrm{d}t \\
&\le \int_0^T \langle \eta_{\varepsilon\lambda}(t), u_{\varepsilon\lambda}(t) - u_0\rangle_V\,\mathrm{d}t\,,
\end{aligned}
$$

and we have

$$
\limsup_{\lambda\to 0}\left(- \int_0^T \langle \eta_{\varepsilon\lambda}(t), J_\lambda u_{\varepsilon\lambda}(t) - u_0\rangle_X\,\mathrm{d}t\right) \le - \int_0^T \langle \eta_\varepsilon(t), u_\varepsilon(t) - u_0\rangle_X\,\mathrm{d}t\,.
$$

Using convergences (5.75), (5.66) and (5.76), we have

$$
\begin{aligned}
\lim_{\lambda\to 0}\left(-\langle \rho\varepsilon^2 u''_{\varepsilon\lambda}(0), u_1\rangle_{X\cap L^{q'}(\Omega)}\right) &= -\langle \rho\varepsilon^2 u''_\varepsilon(0), u_1\rangle_X, \\
\limsup_{\lambda\to 0}\left(- \int_0^T \rho\varepsilon^2 |u''_{\varepsilon\lambda}(t)|^2_{L^2(\Omega)}\,\mathrm{d}t\right) &\le - \int_0^T \rho\varepsilon^2 |u''_\varepsilon(t)|^2_{L^2(\Omega)}\,\mathrm{d}t, \\
\limsup_{\lambda\to 0}\left(- \rho\varepsilon |u'_{\varepsilon\lambda}(T)|^2_{L^2(\Omega)}\right) &\le -\rho\varepsilon |u'_\varepsilon(T)|^2_{L^2(\Omega)}\,.
\end{aligned}
$$

Hence, from convergences (5.66), (5.72) and (5.68), we can deduce that

$$
\begin{aligned}
&\limsup_{\lambda\to 0} \int_0^T \langle \varepsilon\xi_{\varepsilon\lambda}(t), u'_{\varepsilon\lambda}(t)\rangle_V\,\mathrm{d}t \\
&\le -\langle \rho\varepsilon^2 u''_\varepsilon(0), u_1\rangle_{X\cap L^{q'}(\Omega)} - \int_0^T \rho\varepsilon^2 |u''_\varepsilon(t)|^2_{L^2(\Omega)}\,\mathrm{d}t - \rho\varepsilon |u'_\varepsilon(T)|^2_V + \rho\varepsilon |u_1|^2_V \\
&\quad - \int_0^T \langle \rho u''_\varepsilon, u_\varepsilon(t) - u_0\rangle_V\,\mathrm{d}t - \int_0^T \langle \xi_\varepsilon, u_\varepsilon(t) - u_0\rangle_V\,\mathrm{d}t - \int_0^T \langle \eta_\varepsilon, u_\varepsilon(t) - u_0\rangle_X\,\mathrm{d}t\,.
\end{aligned}
$$

Note that the same mollification argument that we used to deduce (5.78) for $u_{\varepsilon\lambda}$ works also for u_ε. As a consequence, we obtain

$$\limsup_{\lambda\to 0}\int_0^T \langle \varepsilon\xi_{\varepsilon\lambda}(t), u'_{\varepsilon\lambda}(t)\rangle_V\,\mathrm{d}t = \int_0^T \langle \varepsilon\xi_\varepsilon(t), u'_\varepsilon(t)\rangle_V\,\mathrm{d}t\,.$$

From the demiclosedness of the maximal monotone operator $u\mapsto \mathrm{d}_V\psi(u(\cdot))$ in $L^p(0,T;V)\times L^{p'}(0,T;V^*)$, we can conclude by Lemma 70 that $\xi_\varepsilon(t)$ coincides with $\mathrm{d}_V\psi(u'_\varepsilon(t))$ for a.a. $t\in(0,T)$, as well as

$$\lim_{\lambda\to 0}\int_0^T \langle \xi_{\varepsilon\lambda}(t), u'_{\varepsilon\lambda}(t)-u_1\rangle\,\mathrm{d}t = \int_0^T \langle \xi_\varepsilon(t), u'_\varepsilon(t)-u_1\rangle\,\mathrm{d}t\,. \qquad (5.79)$$

5.4.6 Minimization of the WIDE functional $I_{\rho\varepsilon}$

Our next aim is to show that the above-determined limit u_ε is the unique minimizer of $I_{\rho\varepsilon}$ on $K(u_0,u_1)$. More precisely, we prove that the strong solution $(u_\varepsilon,\xi_\varepsilon,\eta_\varepsilon)$ of the Euler-Lagrange problem (5.21)-(5.27) is a minimizer of $I_{\rho\varepsilon}$ in $\mathcal{V}$.

Note that $K(u_0,u_1)\subset K_\lambda(u_0,u_1)$ and $\phi_\lambda\le\phi$. By passing to the limit as $\lambda\to 0$ and using the dominated convergence theorem, we have

$$I^\lambda_{\rho\varepsilon}(v)\to I_{\rho\varepsilon}(v)\quad \forall v\in K(u_0,u_1).$$

As $u_{\varepsilon\lambda}$ is a global minimizer of $I^\lambda_{\rho\varepsilon}$, we have

$$I^\lambda_{\rho\varepsilon}(v)\ge I^\lambda_{\rho\varepsilon}(u_{\varepsilon\lambda})\quad \forall v\in K(u_0,u_1).$$

Convergences (5.66)-(5.67) and the lower semicontinuity of the convex integrals $u\mapsto \int_0^T \mathrm{e}^{-t/\varepsilon}\frac{\varepsilon^2\rho}{2}|u''(t)|^2_{L^2(\Omega)}\mathrm{d}t$, $u\mapsto\int_0^T \mathrm{e}^{-t/\varepsilon}\varepsilon\psi(u'(t))\mathrm{d}t$, and $u\mapsto\int_0^T\mathrm{e}^{-t/\varepsilon}\phi\,(u(t))\,\mathrm{d}t$ in $L^p(0,T;V)$ give

$$\begin{aligned}\liminf_{\lambda\to0} I^\lambda_{\rho\varepsilon}(u_{\varepsilon\lambda}) &= \liminf_{\lambda\to 0}\int_0^T \mathrm{e}^{-t/\varepsilon}\Big(\frac{\varepsilon^2\rho}{2}|u''_{\varepsilon\lambda}(t)|^2_{L^2(\Omega)}+\varepsilon\psi(u'_{\varepsilon\lambda}(t))+\phi_\lambda\,(u_{\varepsilon\lambda}(t))\Big)\mathrm{d}t\\ &\ge \liminf_{\lambda\to 0}\int_0^T \mathrm{e}^{-t/\varepsilon}\Big(\frac{\varepsilon^2\rho}{2}|u''_{\varepsilon\lambda}(t)|^2_{L^2(\Omega)}+\varepsilon\psi(u'_{\varepsilon\lambda}(t))+\phi\,(J_\lambda u_{\varepsilon\lambda}(t))\Big)\mathrm{d}t\\ &\ge \int_0^T \mathrm{e}^{-t/\varepsilon}\Big(\frac{\varepsilon^2\rho}{2}|u''_{\varepsilon}(t)|^2_{L^2(\Omega)}+\varepsilon\psi(u'_{\varepsilon}(t))+\phi\big(u_\varepsilon(t)\big)\Big)\mathrm{d}t.\end{aligned}$$

As a consequence, we have

$$I_{\rho\varepsilon}(v)\ge I_{\rho\varepsilon}(u_\varepsilon)\quad\forall v\in K(u_0,u_1).$$

Namely, u_ε minimizes $I_{\rho\varepsilon}$ on $K(u_0,u_1)$, hence $u_\varepsilon\in D(\partial I_{\rho\varepsilon})$ and $0\in\partial_{\mathcal{V}}I_{\rho\varepsilon}(u_\varepsilon)$.

5.5 The causal limit

In this section, we proceed to the proof of Theorem 68 by checking that, up to subsequences, u_ε converges to a strong solution of the target problem (5.3)-(5.7).

Starting from the uniform estimates derived in Subsection 5.4.4 and using the lower semicontinuity of norms and of ϕ, we deduce the following bounds, valid for all $T > 0$,

$$\begin{aligned}&\|u_\varepsilon'\|_{L^p(0,T;V)} + \|\xi_\varepsilon\|_{L^{p'}(0,T;V^*)}\\&+ \|\rho\varepsilon^2 u_\varepsilon'''' - 2\rho\varepsilon u_\varepsilon''' + \rho u_\varepsilon'' - \varepsilon\xi_\varepsilon'\|_{L^2(0,T;X^*)+L^{p'}(0,T;V^*)} \leq C(T),\end{aligned} \tag{5.80}$$

$$\|u_\varepsilon\|_{L^2(0,T;X)} + \int_0^T \phi(u_\varepsilon(t))\,\mathrm{d}t + \|\eta_\varepsilon\|_{L^2(0,T;X^*)} \leq C(T). \tag{5.81}$$

Up to not relabeled subsequences, these bounds entail the following convergences as $\varepsilon \to 0+$:

$$u_\varepsilon \to u \text{ weakly in } W^{1,p}(0,T;V) \text{ and strongly in } C([0,T];V), \tag{5.82}$$

$$u_\varepsilon \to u \text{ weakly in } L^2(0,T;X), \tag{5.83}$$

$$\xi_\varepsilon \to \xi \text{ weakly in } L^{p'}(0,T;V^*), \tag{5.84}$$

$$\rho\varepsilon^2 u_\varepsilon'''' - 2\rho\varepsilon u_\varepsilon''' + \rho u_\varepsilon'' - \varepsilon\xi_\varepsilon' \to \zeta \text{ weakly in } L^2(0,T;X^*) + L^{p'}(0,T;V^*), \tag{5.85}$$

$$\eta_\varepsilon \to \eta \text{ weakly in } L^2\left(0,T;X^*\right). \tag{5.86}$$

Convergences (5.84)-(5.86) are sufficient in order to pass to the limit in equation (5.70) and obtain

$$\zeta + \xi + \eta = 0 \quad \text{in } X^* \text{ a.e. in } (0,T). \tag{5.87}$$

Initial conditions

Note that the strong convergence in (5.82) follows from the compact embedding $W^{1,p}(0,T;V) \cap L^2(0,T;X) \hookrightarrow C([0,T];B)$ for some Banach space B such that $X \subset B$ compactly and from an application of the Aubin-Lions Lemma. The initial condition $u(0) = u_0$ is thus a direct consequence of (5.82).

Furthermore, by defining $w_\varepsilon := \rho\varepsilon^2 u_\varepsilon''' - 2\rho\varepsilon u_\varepsilon'' + \rho u_\varepsilon' - \varepsilon\xi_\varepsilon$, we observe that the uniform bound $\|w_\varepsilon\|_{C([0,T];X^*)} \leq C$ follows from (5.80). In particular, we can take a subsequence $\varepsilon_n \to 0$ such that

$$w_\varepsilon(0) \to w_0 \text{ weakly in } X^*.$$

Finally, by testing $w_\varepsilon(0)$ with $v \in X$, integrating by parts and letting $\varepsilon \to 0$, we can deduce that $w_0 = \rho u_1$.

Identification of the nonlinearities

As a final step, we need to identify the limits ζ, ξ, and η fulfilling (5.87).

Identification of the limit $\zeta \equiv \rho u''$. Let $\varphi \in C_c^\infty([0,T] \times \Omega)$. By integrations by parts we obtain

$$\begin{aligned}&\int_0^T \langle \rho\varepsilon^2 u_\varepsilon'''' - 2\rho\varepsilon u_\varepsilon''' + \rho u_\varepsilon'' - \varepsilon\xi_\varepsilon', \varphi\rangle\,\mathrm{d}t\\&= \int_0^T \langle \rho\varepsilon^2 u_\varepsilon'', \varphi''\rangle\,\mathrm{d}t + \int_0^T \langle 2\rho\varepsilon u_\varepsilon'', \varphi'\rangle\,\mathrm{d}t - \int_0^T \langle \rho u_\varepsilon', \varphi'\rangle\,\mathrm{d}t - \int_0^T \langle \varepsilon\xi_\varepsilon, \varphi'\rangle\,\mathrm{d}t\\&\to -\int_0^T \langle \rho u', \varphi'\rangle_X\,\mathrm{d}t = \int_0^T \langle \rho u'', \varphi\rangle_X\,\mathrm{d}t \quad \text{as } \varepsilon \to 0,\end{aligned}$$

where we used (5.55), (5.58), and (5.82) to pass to the limit. Hence, $\zeta \equiv \rho u''$ almost everywhere.

Identification of the limit $\eta \equiv -\Delta u + f(u)$. Arguing as in Subsection 5.4.5, we exploit the weak convergence (5.83) and the a.e. pointwise convergence (5.82) in order to obtain that $\eta_\varepsilon = -\Delta u_\varepsilon + f(u_\varepsilon) \to -\Delta u + f(u)$ almost everywhere in X^*. Hence, $\eta \equiv -\Delta u + f(u)$ in X^*.

Identification of the limit $\xi \equiv \mathrm{d}_V \psi(u')$. Integrating (5.51) over $(0,T)$ and integrating by parts, we obtain

$$\begin{aligned}
&\int_0^T \int_0^t \langle \xi_{\varepsilon\lambda}(s), u'_{\varepsilon\lambda}(s) - u_1 \rangle \,\mathrm{d}s\,\mathrm{d}t \\
&\leq \frac{\rho\varepsilon^2}{2} \int_0^T |u''_{\varepsilon\lambda}(t)|^2_{L^2(\Omega)} \,\mathrm{d}t + \rho\varepsilon |u'_{\varepsilon\lambda}(T) - u_1|^2_{L^2(\Omega)} - \frac{\rho}{2} \int_0^T |u'_{\varepsilon\lambda}(t) - u_1|^2_{L^2(\Omega)} \,\mathrm{d}t + \varepsilon T \psi(u_1) \\
&\quad + \int_0^T \varepsilon \langle \xi_{\varepsilon\lambda}(t), u'_{\varepsilon\lambda}(t) - u_1 \rangle \,\mathrm{d}t - \int_0^T \phi_\lambda(u_{\varepsilon\lambda}(t)) \,\mathrm{d}t + T\phi_\lambda(u_0) + \int_0^T \int_0^t \langle \eta_{\varepsilon\lambda}(s), u_1 \rangle \,\mathrm{d}s\,\mathrm{d}t \\
&\leq \varepsilon C - \frac{\rho}{2} \int_0^T |u'_{\varepsilon\lambda}(t) - u_1|^2_{L^2(\Omega)} \,\mathrm{d}t - \int_0^T \phi_\lambda(u_{\varepsilon\lambda}(t)) \,\mathrm{d}t + T\phi_\lambda(u_0) + \int_0^T \int_0^t \langle \eta_{\varepsilon\lambda}(s), u_1 \rangle \,\mathrm{d}s\,\mathrm{d}t
\end{aligned}$$

due to (5.55), (5.56), (5.57), and (5.58). Then, we take the $\limsup$ as $\lambda \to 0$. On the left-hand side we pass to the limit thanks to (5.79), while on the right-hand side we use (5.66) and (5.69) together with

$$\liminf_{\lambda\to 0} \phi_\lambda(u_{\varepsilon\lambda}) \geq \liminf_{\lambda\to 0} \phi(J_\lambda u_{\varepsilon\lambda}) \geq \phi(u_\varepsilon)\,,$$

which is given by the weak lower continuity of ϕ in X, (5.67) and $v_\varepsilon = u_\varepsilon$. In particular, we get

$$\begin{aligned}
&\int_0^T \int_0^t \langle \xi_\varepsilon(s), u'_\varepsilon(s) - u_1 \rangle \,\mathrm{d}s\,\mathrm{d}t \\
&\leq -\frac{\rho}{2} \int_0^T |u'_\varepsilon(t) - u_1|^2_{L^2(\Omega)} \,\mathrm{d}t - \int_0^T \phi(u_\varepsilon(t)) \,\mathrm{d}t + T\phi(u_0) + \int_0^T \int_0^t \langle \eta_\varepsilon(s), u_1 \rangle \,\mathrm{d}s\,\mathrm{d}t\,.
\end{aligned}$$

By arguing as above, convergences (5.82), (5.83), and (5.86) give

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_0^T \int_0^t \langle \xi_\varepsilon(s), u'_\varepsilon(s) - u_1 \rangle \,\mathrm{d}s\,\mathrm{d}t \\
&\leq -\frac{\rho}{2} \int_0^T |u'(t) - u_1|^2_{L^2(\Omega)} \,\mathrm{d}t - \int_0^T \phi(u(t)) \,\mathrm{d}t + T\phi(u_0) + \int_0^T \int_0^t \langle \eta(s), u_1 \rangle \,\mathrm{d}s\,\mathrm{d}t \\
&\leq - \int_0^T \int_0^t \langle \rho u''(s) + \eta(s), u'(s) - u_1 \rangle \,\mathrm{d}s\,\mathrm{d}t = \int_0^T \int_0^t \langle \xi(s), u'(s) - u_1 \rangle \,\mathrm{d}s\,\mathrm{d}t
\end{aligned} \tag{5.88}$$

where we integrated by parts and we used (5.87). In particular, recalling Remark 71, we have

$$\limsup_{\varepsilon\to 0} \int_0^T (T-t) \langle \xi_\varepsilon(s), u'_\varepsilon(s) - u_1 \rangle \,\mathrm{d}t \leq \int_0^T (T-t) \langle \xi(s), u'(s) - u_1 \rangle \,\mathrm{d}t$$

From the demiclosedness of the maximal monotone operator $u \mapsto \mathrm{d}_V \psi(u(\cdot))$ in $\mathcal{V} \times \mathcal{V}^*$ we can conclude by Lemma 70 that $\xi(t)$ coincides with $\mathrm{d}_V \psi(u'(t))$ for a.a. $t \in (0,T)$, and

$$\lim_{\varepsilon\to 0} \int_0^T \int_0^t \langle \xi_\varepsilon(s), u'_\varepsilon(s) - u_1 \rangle \,\mathrm{d}s\,\mathrm{d}t = \int_0^T \int_0^t \langle \xi(s), u'(t) - u_1 \rangle \,\mathrm{d}s\,\mathrm{d}t\,. \tag{5.89}$$

5.6 The viscous limit

In this section we discuss the viscous limit $\rho \to 0$. In particular, we prove Theorem 69.i and Theorem 69.ii in Subsection 5.6.1 and Subsection 5.6.2, respectively. Furthermore, we observe that the statement of Theorem 69.iii can be established by arguing as done in Section 5.5, but obtaining $\zeta \equiv 0$ as also $\rho \to 0$.

5.6.1 Gamma-convergence of $I_{\rho\varepsilon}$ in the viscous limit

We consider the WIDE functionals $I_{\rho\varepsilon}$ as defined in (5.10), namely $I_{\rho\varepsilon} : \mathcal{V} \to (-\infty, \infty]$ such that

$$I_{\rho\varepsilon}(u) = \begin{cases} \displaystyle\int_0^T \mathrm{e}^{-t/\varepsilon}\Big(\frac{\varepsilon^2\rho}{2}\int_\Omega |u''(t)|^2\,\mathrm{d}x + \varepsilon\psi(u'(t)) + \phi(u(t))\Big)\mathrm{d}t & \text{if } u \in K(u_0,u_1), \\ \infty & \text{else,} \end{cases}$$

where

$$K(u_0,u_1) = \{u \in H^2(0,T;L^2(\Omega)) \cap W^{1,p}(0,T;V) \cap L^2(0,T;X) : u(0) = u_0, \rho u'(0) = \rho u_1\},$$

hence $D(I_{\rho\varepsilon}) = K(u_0,u_1)$. The aim of this subsection is to prove that, by letting $\rho \to 0$, the sequence of functionals $I_{\rho\varepsilon}$ to Γ- converges with respect to the strong topology in $\mathcal{V}$ to

$$\bar{I}_\varepsilon(u) = \begin{cases} \displaystyle\int_0^T \mathrm{e}^{-t/\varepsilon}\Big(\varepsilon\psi(u'(t)) + \phi(u(t))\Big)\mathrm{d}t & \text{if } u \in \bar{K}(u_0), \\ \infty & \text{else,} \end{cases}$$

where

$$\bar{K}(u_0) = \{W^{1,p}(0,T;V) \cap L^2(0,T;X) : u(0) = u_0\},$$

hence $D(\bar{I}_\varepsilon) = \bar{K}(u_0)$.

Proof of Theorem 69.i. $\Gamma - \liminf$ inequality: Consider a sequence u_ρ in $\mathcal{V}$ converging to u in $\mathcal{V}$. With no loss of generality we can assume that $\sup_\rho I_{\rho\varepsilon}(u_\rho) < \infty$. Then, there exists a subsequence $\rho_k \to 0$ such that $\liminf_{k\to\infty} I_{\rho_k\varepsilon}(u_{\rho_k}) = \lim_{k\to\infty} I_{\rho_k\varepsilon}(u_{\rho_k}) < \infty$. As $\sup_\rho I_{\rho_k\varepsilon}(u_{\rho_k}) < \infty$, we have a uniform estimate for u_{ρ_k} in $W^{1,p}(0,T;V) \cap L^2(0,T;X)$, hence u_{ρ_k} weakly converges to u in $W^{1,p}(0,T;V) \cap L^2(0,T;X)$ up to a subsequence. As a consequence, $\liminf_{\rho\to 0} I_{\rho\varepsilon}(u_\rho) \geq \liminf_{k\to\infty} \bar{I}_\varepsilon(u_{\rho_k}) \geq \bar{I}_\varepsilon(u)$.

Existence of a recovery sequence: Let $u \in \mathcal{V}$. If $u \notin \bar{K}(u_0)$ or if $u \in K(u_0,u_1)$, we can choose $u_\rho = u$ and trivially conclude that $I_{\rho\varepsilon}(u_\rho) \to \bar{I}_\varepsilon(u)$ as $\rho \to 0$. If $u \in \bar{K}(u_0) \setminus K(u_0,u_1)$, we consider a sequence of mollifiers $g_{\tilde\rho}$, namely $g_{\tilde\rho}(t) = \tilde\rho^{-1}g(t/\tilde\rho)$ with $g \in C_c^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} g(t)\,\mathrm{d}t = 1$. We define $(g_{\tilde\rho} * u)(t) := \int_{-1}^T g_{\tilde\rho}(t-s)u(s)\,\mathrm{d}s$ by setting $u(t) = u_0$ for $t \in (-1,0)$ and $u(t) = 0$ for $t \in \mathbb{R} \setminus (-1,T]$, hence $g_{\tilde\rho} * u(t)$ is well-defined for every $t \in \mathbb{R}$. Then we have that $g_{\tilde\rho} * u \to u$ converges to u in $W^{1,p}(0,T;V) \cap L^2(0,T;X)$ and in particular $(g_{\tilde\rho} * u)(0) \to u_0$ in V as $\tilde\rho \to 0$.

As a next step, we define $u_{\tilde\rho} = g_{\tilde\rho} * u + u_0 - (g_{\tilde\rho} * u)(0) + (u_1 - (g_{\tilde\rho} * u)'(0))\zeta^{\tilde\rho}$, where $\zeta^{\tilde\rho}(t) = t\exp(-t/\tilde\rho)$. Note that $(u_{\tilde\rho})'(0) = u_1$ and $\zeta^{\tilde\rho} \to 0$ in $W^{1,p}(0,T)$ as $\tilde\rho \to 0$. As a consequence, $u_{\tilde\rho} \in K(u_0,u_1)$ and $u_{\tilde\rho} \to u$ in $W^{1,p}(0,T;V) \cap L^2(0,T;X)$ as $\tilde\rho \to 0$. Furthermore, we have $|(\zeta_{\tilde\rho})''|^2_{L^2(0,T)} \leq C/\tilde\rho^3$ and $(g_{\tilde\rho} * u)'' = g'_{\tilde\rho} * u'$, whence we deduce that $\|(g_{\tilde\rho} * u)''\|_{L^2(0,T;L^2(\Omega))} \le \|g'_{\tilde\rho}\|_{L^1(0,T)}\|u'\|_{L^2(0,T;L^2(\Omega))} \le C/\tilde\rho$. By choosing $\tilde\rho = \rho^{1/s}$ for some $s > 3$, we obtain $|I_{\rho\varepsilon}(u_{\tilde\rho}) - \bar I_\varepsilon(u_{\tilde\rho})| \le C\rho^{1-3/s}$, whose right-hand side goes to zero as $\rho \to 0$. On the other hand, having the strong convergences above, by means of the Dominated Convergence Theorem, we can conclude that $I_{\rho\varepsilon}(u_{\tilde\rho}) \to \bar I_\varepsilon(u)$ as $\rho \to 0$. □

5.6.2 Viscous limit of the doubly nonlinear wave equation

Consider any solution $(u_\rho, \xi_\rho, \eta_\rho)$ to our target problem (5.3)-(5.7) which belongs to

$$[W^{1,p}(0,T;V) \cap L^2(0,T;X)] \times L^{p'}(0,T;V^*) \times L^2(0,T;X^*) =: \mathcal{Y}.$$

The aim of this subsection is to show that $(u_\rho, \xi_\rho, \eta_\rho)$ converges (with respect to the weak topology of $\mathcal{Y}$ and up to a subsequence) to $(\bar u, \bar\xi, \bar\eta)$ satisfying

$$\bar\xi + \bar\eta = 0 \quad \text{in } X^* \text{ a.e. in } (0,T), \tag{5.90}$$
$$\bar\xi = \mathrm{d}_V\psi(\bar u') \quad \text{in } V^* \text{ a.e. in } (0,T), \tag{5.91}$$
$$\bar\eta = -\Delta\bar u + f(\bar u) \quad \text{in } X^* \text{ a.e. in } (0,T), \tag{5.92}$$

with initial datum $\bar u(0) = u_0$.

Proof of Theorem 69.ii. First, we will show the following estimates holding for $(u_\rho, \xi_\rho, \eta_\rho) \in \mathcal{Y}$ solving (5.3)-(5.7)

$$\rho^{1/2}\|u'_\rho\|_{L^\infty(0,T;L^2(\Omega))} + \|u'_\rho\|_{L^p(0,T;V)} + \|u_\rho\|_{L^\infty(0,T;X)} + \|u_\rho\|_{L^r(0,T;L^r(\Omega))} \le C, \tag{5.93}$$
$$\|\eta_\rho\|_{L^2(0,T;X^*)} + \|\xi_\rho\|_{L^{p'}(0,T;V^*)} \le C. \tag{5.94}$$

By comparison in (5.3), we also obtain

$$\rho\|u''_\rho\|_{L^2(0,T;X^*)+L^{p'}(0,T;V^*)} \le C. \tag{5.95}$$

In order to prove (5.93)-(5.94), we proceed by arguing on increments. For any arbitrary constant $\tau > 0$, we define a backward difference operator δ_τ by

$$\delta_\tau\chi(\cdot) = \frac{\chi(\cdot) - \chi(\cdot - \tau)}{\tau}$$

for functions χ defined on $[0,T]$ with values in a vector space and $t \ge \tau$. Test (5.3) with $\delta_\tau u_\rho$, integrate on (τ, t) and by parts, obtaining

$$\begin{aligned}
&-\frac{\rho}{2\tau}\int_{t-\tau}^t |u'_\rho(s)|^2_{L^2(\Omega)}\,\mathrm{d}s + \frac{\rho}{2\tau}\int_0^\tau |u'_\rho(s)|^2_{L^2(\Omega)}\,\mathrm{d}s + u'_\rho(t)\frac{\rho}{\tau}(u_\rho(t) - u_\rho(t-\tau))\\
&- u'_\rho(\tau)\frac{\rho}{\tau}(u_\rho(\tau) - u_\rho(0)) - \frac{\rho}{2\tau}\int_\tau^t |u'_\rho(s-\tau) - u'_\rho(s)|^2_{L^2(\Omega)}\,\mathrm{d}s + \int_\tau^t \langle\xi_\rho(s), \delta_\tau u_\rho(s)\rangle\,\mathrm{d}s\\
&+ \frac{1}{2\tau}\int_{t-\tau}^t |\nabla u_\rho(s)|^2_{L^2(\Omega)}\,\mathrm{d}s - \frac{1}{2\tau}\int_0^\tau |\nabla u_\rho(s)|^2_{L^2(\Omega)}\,\mathrm{d}s - \frac{1}{2\tau}\int_\tau^t |\nabla u_\rho(s-\tau) - \nabla u_\rho(s)|^2_{L^2(\Omega)}\,\mathrm{d}s\\
&+ \int_\tau^t \langle f(u_\rho(s)), \delta_\tau u_\rho(s)\rangle\,\mathrm{d}s = 0.
\end{aligned}$$

By letting $\tau \to 0$, the regularity of u_ρ and the Lebesgue Differentiation Theorem allow us to conclude that

$$-\frac{\rho}{2}|u_\rho'(t)|^2_{L^2(\Omega)} + \frac{\rho}{2}|u_\rho'(0)|^2_{L^2(\Omega)} + \rho|u_\rho'(t)|^2_{L^2(\Omega)} - \rho|u_\rho'(0)|^2_{L^2(\Omega)} + 0 + \int_0^t \langle \xi_\rho(s), u_\rho'(s)\rangle \, \mathrm{d}s$$
$$+ \frac{1}{2}|\nabla u_\rho(t)|^2_{L^2(\Omega)} - \frac{1}{2}|\nabla u_\rho(0)|^2_{L^2(\Omega)} + 0 + \int_0^t \langle f(u_\rho(s)), u_\rho'(s)\rangle \, \mathrm{d}s = 0$$

for almost every $t \in (0,T)$. This can be rewritten as

$$\frac{\rho}{2}|u_\rho'(t)|^2_{L^2(\Omega)} + \int_0^t \langle \xi_\rho(s), u_\rho'(s)\rangle \, \mathrm{d}s + \frac{1}{2}|\nabla u_\rho(t)|^2_{L^2(\Omega)} + \int_0^t \langle f(u_\rho(s)), u_\rho'(s)\rangle \, \mathrm{d}s$$
$$= \frac{\rho}{2}|u_1|^2_{L^2(\Omega)} + \frac{1}{2}|\nabla u_0|^2_{L^2(\Omega)},$$

whence we obtain (5.93)-(5.94) by using the assumptions of Section 5.2.

From (5.93)-(5.95), up to not relabeled subsequences, we deduce the following convergences:

$$u_\rho \to \bar{u} \text{ weakly in } W^{1,p}(0,T;V) \text{ and strongly in } C([0,T];V), \tag{5.96}$$
$$u_\rho \to \bar{u} \text{ weakly star in } L^\infty(0,T;X), \tag{5.97}$$
$$\xi_\rho \to \bar{\xi} \text{ weakly in } L^{p'}(0,T;V^*), \tag{5.98}$$
$$\rho u_\rho'' \to \bar{\zeta} \text{ weakly in } L^2(0,T;X^*) + L^{p'}(0,T;V^*), \tag{5.99}$$
$$\eta_\rho \to \bar{\eta} \text{ weakly in } L^2\left(0,T;X^*\right). \tag{5.100}$$

The initial condition $\bar{u}(0) = u_0$ follows directly from (5.96). The identification of the limit $\bar{\zeta} \equiv 0$ can be obtained as in Section 5.5. Using the same argument of Section 5.5, we can also deduce that $\bar{\eta} = -\Delta\bar{u} + f(\bar{u})$. It just remains to identify the limit $\bar{\xi}$. In order to achieve this, we first consider (5.88), namely

$$\int_0^T \int_0^t \langle \xi_\rho(s), u_\rho'(s) - u_1\rangle \, \mathrm{d}s \, \mathrm{d}t \le -\int_0^T \phi(u_\rho(t)) \, \mathrm{d}t + T\phi(u_0) + \int_0^T \int_0^t \langle \eta_\rho(s), u_1\rangle \, \mathrm{d}s \, \mathrm{d}t$$

due to (5.89). Then, convergences (5.97) and (5.100) ensure that

$$\limsup_{\rho\to 0} \int_0^T \int_0^t \langle \xi_\rho(s), u_\rho'(s) - u_1\rangle \, \mathrm{d}s \, \mathrm{d}t$$
$$\le -\int_0^T \phi(\bar{u}(t)) \, \mathrm{d}t + T\phi(u_0) + \int_0^T \int_0^t \langle \bar{\eta}(s), u_1\rangle \, \mathrm{d}s \, \mathrm{d}t$$
$$\le -\int_0^T \int_0^t \langle \bar{\eta}(s), \bar{u}'(s) - u_1\rangle \, \mathrm{d}s \, \mathrm{d}t = \int_0^T \int_0^t \langle \bar{\xi}(s), \bar{u}'(s) - u_1\rangle \, \mathrm{d}s \, \mathrm{d}t\,,$$

where we used (5.90). From the demiclosedness of the maximal monotone operator $u \to \mathrm{d}_V\psi(u(\cdot))$ in $\mathcal{V} \times \mathcal{V}^*$, we can conclude by Remark 71 and Lemma 70 that $\bar{\xi}(t)$ coincides with $\mathrm{d}_V\psi(\bar{u}'(t))$ for a.a. $t \in (0,T)$. □

Remark 73. *Observe that, since we do not assume f to be Lipschitz continuous, solutions u_ρ to* (5.3)-(5.7) *(as well as solutions $\bar{u}$ to* (5.28)-(5.31)*) may be not unique. For this reason, there might exist solutions u_ρ to* (5.3)-(5.7) *which cannot be recovered by means of the WIDE approach, namely, that are not limits as $\varepsilon \to 0$ of sequences of solutions to the regularized* *problem* (5.44)-(5.50). *Note that Theorem 69.ii establishes the convergence (with respect to the weak topology of $\mathcal{Y}$ and up to a subsequence) of any solution $(u_\rho, \xi_\rho, \eta_\rho) \in \mathcal{Y}$ to* (5.3)-(5.7) *towards a solution $(\bar{u}, \bar{\xi}, \bar{\eta})$ to* (5.28)-(5.31) *as $\rho \to 0$. This result implies that the same holds true for any $(u_\rho, \xi_\rho, \eta_\rho) \in \mathcal{Y}$ solution to* (5.3)-(5.7) *obtained as causal limit in Section 5.5. This could be proved also starting from the estimates derived in Subsection 5.4.4 which are uniform in $\rho > 0$.*

5.7 Time-discretization of the WIDE approach

In this section, we introduce a time-discretization technique. We follow the strategy outlined in [78], in particular we investigate a time-discrete version of the WIDE principle. We replace the WIDE functional $I^\lambda_{\rho\varepsilon}$ by a time-discrete WIDE functional $I^{(\tau)}_{\rho\varepsilon}$.

For the sake of notational simplicity, in this Section we omit the subscript λ. In particular, in the following $I_{\rho\varepsilon}$ and ϕ stand for $I^\lambda_{\rho\varepsilon}$ and ϕ_λ, respectively.

We recall the notation for the constant time-step $\tau := T/n$ ($n \in \mathbb{N}$) and we introduce

$$\mathcal{V}_\tau := \Big\{(u^{(0)}, ..., u^{(n)}) \in V^{n+1} \ : \ (u^{(2)}, ..., u^{(n-2)}) \in X^{n-3}\Big\}.$$

We define the time-discrete WIDE functional $I^{(\tau)}_{\rho\varepsilon} : \mathcal{V}_\tau \to \mathbb{R}$ by

$$I^{(\tau)}_{\rho\varepsilon}(u^{(0)}, ..., u^{(n)}) = \frac{\varepsilon^2\rho}{2}\sum_{j=2}^{n} \tau e_{\varepsilon\tau,j}|\delta^2 u^{(j)}|^2 + \frac{\varepsilon}{2}\sum_{j=2}^{n-1}\tau e_{\varepsilon\tau,j+1}\psi(\delta u^{(j)}) + \sum_{j=2}^{n-2}\tau e_{\varepsilon\tau,j+2}\phi(u^{(j)}).$$

Here, given a vector $(w^{(0)}, \dots, w^{(n)})$, δw denotes its discrete derivative, i.e., $\delta w^{(j)} := (w^{(j)} - w^{(j-1)})/\tau$ for $j = 1, \dots, n$ and $\delta^2 w = \delta(\delta w)$, $\delta^3 w = \delta(\delta^2 w)$, and so on. Furthermore, we define the weights $e_{\varepsilon\tau,1}, \dots, e_{\varepsilon\tau,n}$ given by $e_{\varepsilon\tau,i} = \left(\frac{\varepsilon}{\varepsilon+\tau}\right)^i$ for $i = 1, \dots, n$, representing a discrete version of the exponentially decaying weight $t \mapsto \exp(-t/\varepsilon)$ and thus satisfy $\delta e_{\varepsilon\tau,i} + e_{\varepsilon\tau,i}/\varepsilon = 0$. For the sake of notational simplicity, from now on we will drop the subscript $\varepsilon\tau$ from $e_{\varepsilon\tau,j}$. Finally, we introduce the discrete counterpart of $K(u_0, u_1)$, which is

$$K_\tau(u_0, u_1) = \{(u^{(0)}, \dots, u^{(n)}) \in \mathcal{V}_\tau \ : \ u^{(0)} = u_0, \ \rho\delta u^{(1)} = \rho u_1\}.$$

The discrete WIDE functional $I^{(\tau)}_{\rho\varepsilon}$ represents a discrete version of the original time-continuous WIDE functional $I_{\rho\varepsilon}$. Note that $I^{(\tau)}_{\rho\varepsilon}$ is convex. From an application of the Direct Method we obtain the existence of a unique discrete minimizer.

Lemma 74 (Well-posedness of the discrete minimum problem)**.** *For ε and τ small and all $u_0, u_1 \in X$, the discrete* WIDE *functional $I^{(\tau)}_{\rho\varepsilon}$ admits a unique minimizer in $K_\tau(u_0, u_1)$.*

5.7.1 Discrete Euler-Lagrange system

The unique minimizer $(u^{(0)}, \dots, u^{(n)})$ of the time-discrete functional $I^{(\tau)}_{\rho\varepsilon}$ solves

$$\varepsilon^2\rho\sum_{j=2}^{n}\tau e_j(\delta^2 u^{(j)}, \delta^2 v^{(j)}) + \varepsilon\sum_{j=2}^{n-1}\tau e_{j+1}\langle \mathrm{d}_V\psi(\delta u^{(j)}), \delta v^{(j)}\rangle + \sum_{j=2}^{n-2}\tau e_{j+2}\langle \partial_V\phi(u^{(j)}), v^{(j)}\rangle = 0$$
$$\forall (v_0, \dots, v_n) \in K_\tau(0,0).$$

As a next step, we sum by parts. In particular, we obtain

$$
\begin{aligned}
&\varepsilon \sum_{j=2}^{n-1} \tau e_{j+1} \langle \, \mathrm{d}_V \psi(\delta u^{(j)}), \delta v^{(j)} \rangle \\
&= \varepsilon e_n \langle \, \mathrm{d}_V \psi(\delta u^{(n-1)}), v^{(n-1)} \rangle - \varepsilon \sum_{j=2}^{n-2} \tau \langle \delta \Big(e_{j+2} \, \mathrm{d}_V \psi(\delta u^{(j+1)}) \Big), v^{(j)} \rangle \\
&= \varepsilon e_n \langle \, \mathrm{d}_V \psi(\delta u^{(n-1)}), v^{(n-1)} \rangle - \varepsilon \sum_{j=2}^{n-2} \tau e_{j+2} \langle \delta \Big(\, \mathrm{d}_V \psi(\delta u^{(j+1)}) \Big), v^{(j)} \rangle \\
&\quad + \sum_{j=2}^{n-2} \tau e_{j+2} \langle \, \mathrm{d}_V \psi(\delta u^{(j+1)}), v^{(j)} \rangle, \qquad (5.101)
\end{aligned}
$$

where we used the formula

$$
\varepsilon \delta \Big(e_{j+2} \, \mathrm{d}_V \psi(\delta u^{(j+1)}) \Big) = \varepsilon e_{j+2} \delta \Big(\, \mathrm{d}_V \psi(\delta u^{(j+1)}) \Big) - e_{j+2} \, \mathrm{d}_V \psi(\delta u^{(j+1)})
$$

which can be deduced from $\delta e_{j+2} = -e_{j+2}/\varepsilon$.

From here on, one can simply follow [78, Sec. 5.2.] together with (5.101) in order to obtain the following.

Lemma 75 (Discrete Euler-Lagrange system)**.** *Let* $(u^{(0)}, \ldots, u^{(n)}) \in K_\tau(u_0, u_1)$ *be the unique minimizer of the discrete* WIDE *functional* $I_{\rho\varepsilon}^{(\tau)}$. *Then,* $(u^{(0)}, \ldots, u^{(n)})$ *solves*

$$
\begin{aligned}
&\varepsilon^2 \rho \delta^4 u^{(j+2)} - 2\varepsilon \rho \delta^3 u^{(j+1)} + \rho \delta^2 u^{(j)} - \varepsilon \delta \Big(\, \mathrm{d}_V \psi(\delta u^{(j+1)}) \Big) \\
&+ \, \mathrm{d}_V \psi(\delta u^{(j+1)}) + \partial_V \phi(u^{(j)}) = 0, \qquad (5.102)
\end{aligned}
$$

subject to the initial and final conditions

$$
u^{(0)} = u_0, \qquad \rho \delta u^{(1)} = \rho u_1, \qquad (5.103)
$$

$$
\varepsilon^2 \rho \delta^2 u^{(n)} = 0, \qquad \varepsilon^2 \rho \delta^3 u^{(n)} = \varepsilon \rho \delta^2 u^{(n-1)} + \varepsilon \, \mathrm{d}_V \psi(\delta u^{(n-1)}). \qquad (5.104)
$$

The system of equations (5.102)–(5.104) is the discrete analogue of (5.44)–(5.50).

5.7.2 Discrete estimate

The argument of Subsection 5.4.4 can be made rigorous at the time-discrete level. We present here a time-discrete version of the estimate (5.54) by using the time-discrete Euler-Lagrange system (5.102)–(5.104). Namely, we prove the following estimate.

Lemma 76 (Discrete estimate)**.** *Let* $(u^{(0)}, \ldots, u^{(n)})$ *minimize the discrete* WIDE *functional* $I_{\rho\varepsilon}^{(\tau)}$ *over* $K_\tau(u_0, u_1)$. *Then, for all* ε *and* τ *sufficiently small we have*

$$
\rho\varepsilon \frac{4-3\varepsilon}{2} \sum_{k=2}^{n-2} \tau |\delta^2 u^{(k)}|^2 + \rho \frac{1-3\varepsilon}{2} |\delta u^{(n-2)} - u_1|^2 + \sum_{k=2}^{n-2} \tau \psi(\delta u^{(k)}) + c \sum_{k=2}^{n-2} \tau \phi(u^{(k)}) \le C \qquad (5.105)
$$

for some constant $0 < c < 1$.

Proof. We argue by reproducing the estimate of Subsection 5.4.4 at the discrete level, i.e., we perform the following:

$$\sum_{k=2}^{n-2}\tau(5.102)\times(\delta u^{(k)}-u_1)+\sum_{k=2}^{n-2}\tau\left(\sum_{j=1}^{k}\tau(5.102)\times(\delta u^{(j)}-u_1)\right).$$

Most of the computations have been already performed in [78, Subsec. 5.3], hence we refer to the Proof of Lemma 5.3 in [78, Subsec. 5.3] for the contributions deriving from the first three terms in (5.102).

Here, we deal with the terms involving the nonquadratic functional ψ. First, for $k \le n-2$, by adding a null term, we compute

$$\begin{aligned}&-\varepsilon\sum_{j=1}^{k}\tau\langle\delta\Big(\mathrm{d}_V\psi(\delta u^{(j+1)})\Big),\delta u^{(j)}-u_1\rangle\\&=\varepsilon\sum_{j=2}^{k}\langle\,\mathrm{d}_V\psi(\delta u^{(j+1)}),\delta u^{(j+1)}-\delta u^{(j)}\rangle-\varepsilon\langle\,\mathrm{d}_V\psi(\delta u^{(k+1)}),\delta u^{(k+1)}-u_1\rangle\\&\quad+\varepsilon\langle\,\mathrm{d}_V\psi(\delta u^{(2)}),\delta u^{(2)}-u_1\rangle.\end{aligned}$$

Additionally, we have

$$\begin{aligned}&\sum_{j=2}^{k}\langle\,\mathrm{d}_V\psi(\delta u^{(j+1)}),\delta u^{(j+1)}-\delta u^{(j)}\rangle\\&\ge\sum_{j=2}^{k}\psi(\delta u^{(j+1)})-\sum_{j=2}^{k}\psi(\delta u^{(j)})=\psi(\delta u^{(k+1)})-\psi(\delta u^{(2)}).\end{aligned}$$

Combining this together with the argument of Lemma 5.3 in [78, Subsec. 5.3], we obtain the discrete analogous of (5.53), namely

$$\begin{aligned}&\rho\varepsilon\frac{4-3\varepsilon}{2}\sum_{k=2}^{n-2}\tau|\delta^2u^{(k)}|^2+2\varepsilon\rho\sum_{k=2}^{n-2}\tau\sum_{j=2}^{k}\tau|\delta^2u^{(j)}|^2+\rho\frac{1-3\varepsilon}{2}|\delta u^{(n-2)}-u_1|^2+\frac{\rho}{2}\sum_{k=2}^{n-2}\tau|\delta u^{(k)}-u_1|^2\\&+(1-\varepsilon)\sum_{k=2}^{n-2}\tau\langle\,\mathrm{d}_V\psi(\delta u^{(k+1)}),\delta u^{(k+1)}-u_1\rangle+\sum_{k=2}^{n-2}\tau\sum_{j=2}^{k}\tau\langle\,\mathrm{d}_V\psi(\delta u^{(j)}),\delta u^{(j)}-u_1\rangle\\&+\varepsilon\psi(\delta u^{(n-1)})+\varepsilon\sum_{k=2}^{n-2}\tau\psi(\delta u^{(k)})+\phi(\delta u^{(n-2)})+\sum_{k=2}^{n-2}\tau\phi(u^{(k)})\\&\le\varepsilon(1+T)\psi(u_1)+\tau\psi(u_1)+(1+T)\phi(u_0)\\&\quad+\sum_{k=2}^{n-2}\tau\langle\partial_V\phi(u^{(j)}),u_1\rangle+\sum_{k=2}^{n-2}\tau\sum_{j=2}^{k}\tau\langle\partial_V\phi(\delta u^{(j)}),u_1\rangle.\end{aligned}$$

Using the fact that $\langle\,\mathrm{d}_V\psi(\delta u^{(j)}),\delta u^{(j)}-\delta u_1\rangle\ge\psi(\delta u^{(j)})-\psi(\delta u_1)$ together with $\psi,\phi\ge0$, we deduce

$$\begin{aligned}&\rho\varepsilon\frac{4-3\varepsilon}{2}\sum_{k=2}^{n-2}\tau|\delta^2u^{(k)}|^2+\rho\frac{1-3\varepsilon}{2}|\delta u^{(n-2)}-u_1|^2+\sum_{k=2}^{n-2}\tau\psi(\delta u^{(k+1)})+\sum_{k=2}^{n-2}\tau\phi(u^{(k)})\\&\le\left(\varepsilon+\tau+T+\frac{T^2}{2}\right)\psi(u_1)+(1+T)\phi(u_0)\\&\quad+\sum_{k=2}^{n-2}\tau\langle\partial_V\phi(u^{(j)}),u_1\rangle+\sum_{k=2}^{n-2}\tau\sum_{j=2}^{k}\tau\langle\partial_V\phi(\delta u^{(j)}),u_1\rangle.\end{aligned}$$

Finally, the same argument of Subsection 5.4.4 allows us to obtain (5.105). □

5.7.3 Discrete-to-continuum convergence

In order to obtain the inequality (5.54) we need to show that the time-discrete energy estimate (5.105) passes to the limit as $\tau \to 0$ (for fixed $\varepsilon > 0$). To this aim, we need the discrete-to-continuum Γ-convergence $I_{\rho\varepsilon}^{(\tau)} \xrightarrow{\Gamma} I_{\rho\varepsilon}$ with respect to the strong topology of

$$\mathcal{K} := H^2(0,T;L^2(\Omega)) \cap W^{1,p}(0,T;V) \cap L^2(0,T;X).$$

Following the strategy outlined in [78, Subsec. 5.4], one can prove the following

Proposition 77 (Discrete-to-continuum Γ-convergence)**.** *Let*

$$\mathcal{X}_\tau := \{u : [0,T] \to X \ : \ u \textit{ is piecewise affine on the time partition}\}$$

and define the functionals $\bar{I}_{\rho\varepsilon}$*,* $\bar{I}_{\rho\varepsilon}^{(\tau)} : \mathcal{K} \to [0,\infty]$ *as*

$$\bar{I}_{\rho\varepsilon}(u) := \begin{cases} I_{\rho\varepsilon}(u) & \textit{if } u \in K(u_0,u_1), \\ \infty & \textit{elsewhere}, \end{cases}$$

$$\bar{I}_{\rho\varepsilon}^{(\tau)}(u) := \begin{cases} I_{\rho\varepsilon}^{(\tau)}(u(0),u(\tau),\dots,u(T)) & \textit{if } u \in \mathcal{X}_\tau \cap K(u_0,u_1), \\ \infty & \textit{elsewhere}. \end{cases}$$

Then, $\bar{I}_{\rho\varepsilon}^{(\tau)}$ Γ*-converges to* $\bar{I}_{\rho\varepsilon}$ *with respect to the strong topology of* $\mathcal{K}$*.*

Note that Proposition 77 is the analogous of Lemma 5.4 in [78, Subsec. 5.4] in our nonquadratic setting. As a consequence, the proof of Proposition 77 can be obtained by reproducing that of Lemma 5.4 in [78, Subsec. 5.4] to our setting. We hence refrain from providing a discussion of this part.

As last step, we observe that the minimizers $u_{\varepsilon\tau}$ of the discrete functional $\bar{I}_{\rho\varepsilon}^{(\tau)}$ fulfill the estimate (5.105) and are hence weakly precompact in $\mathcal{K}$. Since $\bar{I}_{\rho\varepsilon}^{(\tau)}$ Γ-converges to $\bar{I}_{\rho\varepsilon}$ with respect to the same topology by Lemma 77, from the fundamental theorem of Γ-convergence (see [20, Sec. 1.5]) it follows that $u_{\varepsilon\tau} \to u_\varepsilon$ weakly in $\mathcal{K}$, where u_ε is the unique minimizer of $\bar{I}_{\rho\varepsilon}$. Furthermore, estimate (5.105) passes to the limit, so that the inequality (5.54) is proved.

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