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On a non-isothermal Cahn-Hilliard model based on a microforce balance

Tesi di Laurea di
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A mamma, papà e Marco,
sempre al mio fianco per suppportarmi
e accompagnarmi verso ogni traguardo.

Abstract

This thesis is concerned with a non-isothermal Cahn-Hilliard model based on a microforce balance. The model was derived by A. Miranville and G. Schimperna starting from the two fundamental laws of Thermodynamics, following M. Gurtin's two-scale approach. The main working assumptions are made on the behaviour of the heat flux as the absolute temperature tends to zero and to infinity. A suitable Ginzburg-Landau free energy is considered. Local-in-time existence for the initial-boundary value problem associated to the entropy formulation and, in a subcase, also to the weak formulation of the model is proved by deriving suitable a priori estimates and showing weak sequential stability of families of approximating solutions. At last, some highlights regarding a possible approximation of the system are given.

La tesi tratta un modello di tipo Cahn-Hilliard non-isotermo basato su un bilancio di microforze. Il modello è stato derivato da A. Miranville e G. Schimperna partendo dalle due leggi fondamentali della Termodinamica, seguendo l'approccio a due scale dovuto a M. Gurtin. Le principali ipotesi di lavoro riguardano il comportamento della legge di flusso di calore al tendere della temperatura assoluta a zero e a infinito. L'energia libera considerata ha un'espressione opportuna di tipo Ginzburg-Landau. L'esistenza locale in tempo per il problema ai dati iniziali e al bordo associato alla fomulazione entropica e, in un sottocaso, anche debole del modello è dimostrata derivando opportune stime a priori e mostrando la stabilità debole sequenziale delle famiglie di soluzioni approssimanti. Infine, si presentano alcuni aspetti rilevanti di una possibile approssimazione del sistema.

Contents

Introduction

In material sciences, when a binary alloy (e.g., Aluminium/Zinc or Iron/ Chromium) is cooled down sufficiently, the two components spontaneously separate and form domains pure in each component. In other words, the initially homogeneous material quickly becomes inhomogeneous, forming a fine-grained microstructure, which then coarsens at a slower time scale.
In 1958, J.W. Cahn and J.E. Hilliard derived a model to describe important qualitative features of two-phase systems related with *phase separation* processes (see [11]), obtaining a conservation law which describes the transport of atoms between unit cells. Their approach is based on *diffuse interface models*, i.e., the interface between two coexisting phases is replaced by a thin interfacial region. Therefore, a partial mixing of the two components is allowed. Furthermore, their model assumes isotropy and constant temperature, i.e., that the process, caused by instantaneous quench under a critical temperature, is *isothermal*. Hence, only diffusive phenomena are taken into account.
The *Cahn–Hilliard system* is now quite well understood, at least from a mathematical point of view. In particular, one has a rather complete picture as far as the existence, the uniqueness and the regularity of solutions and the asymptotic behaviour of the associated dynamical system are concerned.
In 1986, M. Gurtin made several remarks on the phenomenological derivations of the Cahn–Hilliard system. From his point of view, such derivations obscure the fundamental nature of balance laws in any general framework that includes dissipation. Based on these observations, Gurtin proposed in [31] several generalizations of the isothermal Cahn-Hilliard equation. More precisely, a relevant feature which distinguishes his approach from other macroscopic theories of order parameters is the separation of balance laws from constitutive equations and the introduction of a new balance law for internal *microforces*, i.e., forces at an atomistic level.
Isothermal Cahn-Hilliard models take into account only diffusive phenomena, assuming that the process is caused by an instantaneous quench under a critical temperature. However, in realistic physical systems, quenches are carried out over a finite period of time and an external thermal activation can be used to control the process. These reasons lead to the introduction of *non-isothermal Cahn-Hilliard models*.
In 1992, H.W. Alt and I. Pawłow proposed and studied the existence of solutions for a mathematical model of phase separation that describes coupled phenomena of mass diffusion and heat conduction in binary systems under thermal activation (see [2]- [3]). Indeed, their aim was to extend the Cahn-Hilliard approach to the

non-isothermal setting by accounting for the dynamics of energy transfer. Constitutive relations for the mass flux, the heat flux and the chemical potential are postulated, and the derivation of the model follows from the mass and the energy balance laws. The consistency with the first and the second principle of Thermodynamics is shown a posteriori and it is essentially a theorem.

In 2005, following M. Gurtin's approach, A. Miranville and G. Schimperna extended the previous non-isothermal model derived by Alt and Pawłow to non-isotropic materials and to systems that are far from equilibrium (see [43]). Their approach is a two-scale one and the two fundamental laws of Thermodynamics are used as a starting point to derive the model. Furthermore, no a priori specification of the constitutive equations for the mass flux, the heat flux and the chemical potential is made. Rather, these quantities are kept in an implicit form and only a list of independent constitutive variables upon which they are allowed to depend is specified. It is only a posteriori that the admissible expressions for the physical parameters are deduced. Thus, this kind of procedure seems to be the one that may allow us to describe the most general class of free energies and of chemical potentials which are compatible with the fundamental laws.

The mathematical analysis of Miranville and Schimperna's system of equations seems rather involved. At least to our knowledge, a result regarding the existence of (weak) solutions to this non-isothermal Cahn-Hilliard model is still lacking. However, once suitable assumptions are made, it is possible to collect some formal *a priori estimates* holding for hypothetical solutions to the *strong formulation* of the model, or more precisely, to a proper *regularization* or *approximation* of it. Therefore, by compactness arguments, one can show that at least a subsequence of approximate solutions converges in a suitable way to a so-called *entropy solution* to the initial-boundary value problem and, in a subcase, also to a *weak* one. This procedure will allow us to prove two *main existence theorems* regarding entropy and weak solutions to Miranville and Schimperna's non-isothermal Cahn-Hilliard model, respectively.

At last, we could wonder about a suitable approximation of the strong formulation of the model. Since the system of equations is rather complex, then the related approximation could be particularly long and technical. However, we will give some highlights regarding a possible way to get an existence result for a regularized scheme. The details of such a procedure may be the object of future work.

A synopsis of the thesis is the following:
In Chapter 1, we shall provide an overview of Cahn-Hilliard models. Section 1.1 concerns the isothermal case. In Subsection 1.1.1, we will introduce the Cahn-Hilliard equation and present its standard phenomenological derivation. Afterwards, in Subsection 1.1.2, we will focus on Gurtin's generalizations of the isothermal Cahn-Hilliard equation, whose derivation is based on a two-scale approach. As for Section 1.2, it concerns non-isothermal Cahn-Hilliard models. In Subsection 1.2.1, we will introduce the model of non-isothermal phase separation proposed by Alt and Pawłow, presenting its derivation and proving its thermodynamic consistency. Following Gurtin's approach, in Subsection 1.2.2, we will derive the non-isothermal Cahn-Hilliard model based on a microforce balance proposed by Miranville and Schimperna. In Subsection 1.2.3, we will provide some thermodynamically reasonable expressions for the Helmholtz free energy density. We will compute Alt and Pawlow's system of equations and Miranville and Schimperna's one for two different specific choices and we will outline the main differences.
Chapter 2 deals with the mathematical analysis of Miranville and Schimperna's non-isothermal Cahn-Hilliard model. Section 2.1 concerns the setting of the problem. In particular, we will specify suitable initial and boundary conditions, which are consistent with the physical derivation of the model and lead to the conservation of mass, the balance of energy and that of entropy. In Section 2.2, we will list the assumptions and the hypotheses made and we will present the so-called entropy and weak formulations of the problem. Afterwards, we will state the main results of the thesis in the form of two main existence theorems. The proof of the theorems begins in Section 2.3, where we will prove some formal a priori bounds holding for a hypothetical solution to the strong formulation of the model, or more precisely, to a proper approximation of it. In Subsection 2.3.1, we will deduce the so-called energy and entropy estimates, while, in Subsection 2.3.2, the so-called key estimates. In Section 2.4, by weak compactness arguments, we will collect convergence properties for a sequence of approximate solutions so that, in Subsection 2.4.2, we will show that at least a subsequence converges to an entropy solution to the initial-boundary value problem and, in a subcase, also to a weak one. As for Section 2.5, we will try to weaken a hypothesis in order to generalize the main existence theorems. However, we will see that some a priori bounds would not hold true anymore, but we will propose a strategy to overcome this problem. At last, in Section 2.6, we will give some highlights regarding a possible approximation of the "strong" system that one could try to develop.

Chapter 1

Cahn-Hilliard models

The aim of this chapter is to provide an overview of models of mathematical physics which describe the phase transition processes occurring, for instance, when a binary alloy is cooled down sufficiently. As a starting point, we introduce the isothermal Cahn-Hilliard equation, as originally derived by J.W. Cahn and J.E. Hilliard in 1958. Subsequently, several generalizations have been proposed such as that by M. Gurtin based on a two-scale approach and thus on a microforce balance. Isothermal Cahn-Hilliard models take into account only diffusive phenomena, assuming that the process is caused by an instantaneous quench under a critical temperature. However, in realistic physical systems, quenches are carried out over a finite period of time and an external thermal activation can be used to control the process. These reasons lead to the introduction of non-isothermal Cahn-Hilliard models. In particular, we will focus on the non-isothermal model derived by H.W. Alt and I. Pawłow and on that by A. Miranville and G. Schimperna, comparing them and outlining the main differences once specific choices of the so-called Ginzburg-Landau free energy are made.

1.1 Isothermal Cahn-Hilliard models

This section is devoted to the introduction of isothermal Cahn-Hilliard models, which play an important role in material sciences in the description of qualitative features of two-phase systems related with phase separation processes. Firstly, in Subsection 1.1.1, we focus on the phenomenological aspects and the standard derivation of the Cahn-Hilliard equation, presenting also one of its immediate generalizations: the case with nonconstant mobility. Then, in Subsection 1.1.2, we present M. Gurtin's objections to the standard derivation and his two-scale approach which lead to the generalized Cahn-Hilliard equation.

1.1.1 Introduction to Cahn-Hilliard equation

In material sciences, when a binary alloy (e.g., Aluminium/Zinc or Iron/ Chromium) is cooled down sufficiently, one can observe a partial nucleation, i.e., the apparition of nuclides in the material, or a total nucleation, the so-called spinodal decomposition. The initially homogeneous material quickly becomes inhomogeneous, forming a fine-grained microstructure. In a second stage, which is known as coarsening and occurs at a slower time scale, these microstructures coarsen. In other words, binary systems under quenching are brought from one-phase equilibrium to non-equilibrium states which are located within the coexistence region of their phase diagrams. Then, activated by composition fluctuations, the systems evolve to a new equilibrium that comprises two coexisting phases. This contributes to a local separation of the phases and creation of a spatially heterogeneous structure. Such phenomena play an essential role in determining the mechanical properties of the material, e.g., strength, hardness, resistance to fracture, toughness and ductility.

In 1958, J.W. Cahn and J.E. Hilliard proposed a system of equations which describes important qualitative features of two-phase systems related with phase separation processes (see [11]). Assuming the interface between two coexisting phases to be a 2-dimensional sufficiently smooth surface, as done in the so-called *sharp interface models*, analytical problems related to the interface singularities could arise. Thus, Cahn and Hilliard adopted an alternative approach based on *diffuse interface models*. In this setting, the sharp interface, represented by a lower-dimensional surface, is replaced by a thin interfacial region, whose thickness is related to a small parameter $\alpha > 0$. Therefore, a partial mixing of the two components is allowed. In order to describe this phenomenon, a new variable u is introduced. This quantity may represent a rescaled density of atoms or concentration of one of the material's components.
The Cahn-Hilliard system consists of a conservation law which describes the transport of atoms between unit cells. The model assumes isotropy and constant temperature, i.e., that the process, caused by instantaneous quench under a critical temperature, is isothermal. Hence, only diffusive phenomena are taken into account.

As a starting point we should underline that we consider a bounded, open subset $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$ as the domain occupied by the material. As for time, it plays the same role as in Classical Mechanics, namely it is an external parameter, not a coordinate. In particular, we consider a finite time interval $[0, T] \subset \mathbb{R}^+$. We denote by (x, t) an arbitrary point belonging to $\Omega \times (0, T)$, while by ∇ and by Δ the spatial gradient and Laplacian, respectively.

The Cahn–Hilliard system can be written as

$$\frac{\partial u}{\partial t} = m\Delta\mu, \quad m > 0, \tag{1.1}$$

$$\mu = -\alpha\Delta u + f(u), \quad \alpha > 0, \tag{1.2}$$

in $\Omega \times (0, T)$, which can be restated, equivalently, as the fourth-order in space parabolic equation

$$\frac{\partial u}{\partial t} + \alpha m\Delta^2 u - m\Delta f(u) = 0 \tag{1.3}$$

in $\Omega\times(0,T)$. Here, u is the so-called *order parameter*, i.e., it represents the difference between the rescaled densities of atoms or concentrations of the two material's components. At least in principle, u should take values between -1 and 1; the values -1 and 1 correspond to the pure states. μ is the chemical potential or, more precisely, the difference of chemical potentials between the two components. $m>0$ is the mobility, while $\alpha>0$ is related to the surface tension at the interface, i.e., capillarity. Furthermore, f is the derivative of "coarse-grained" free energy F, a double-well potential whose wells correspond to pure phases configurations. A thermodynamically relevant potential F is the following logarithmic function which follows from a mean-field model:

$$F(u)=-\theta_c u^2+\theta\left[(1-u)\log(1-u)+(1+u)\log(1+u)\right], \qquad u\in(-1,1),\quad 0<\theta<\theta_c\,, \tag{1.4}$$

i.e.

$$f(u)=-2\theta_c u+\theta\log\frac{1+u}{1-u}\,. \tag{1.5}$$

The logarithmic terms correspond to the entropy of mixing and θ and θ_c are proportional to the absolute temperature (assumed constant during the process) and a critical temperature, respectively. The condition $\theta<\theta_c$ ensures that F has a double-well form and that phase separation can occur. Indeed, at temperatures $\theta\geq\theta_c$, F would be convex in u reflecting a continuous solubility range of the system. On the contrary, under the critical temperature θ_c, a miscibility gap arises, and F assumes a double-well form.

In Figure 1.1, the potential F and its phase diagram are represented as a function of u at fixed temperatures $\theta_1>\theta_c$ and $\theta_2<\theta_c$, and as a function of u and θ, respectively.
For fixed $\theta_1>\theta_c$, F has only one minimum point u_m, that is the mean value of the concentration, i.e., $\frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega u\,dx=u_m$. Observe that, when F is given by (1.4), $u_m=0$.
For fixed $\theta_2<\theta_c$, the concentrations u_{e_1} and u_{e_2} correspond to the minimum points of F, i.e., its equilibrium points. Such concentrations have identical chemical potentials and may coexist. By varying θ_2, the locus of the concentration values u_{e_1} and u_{e_2} determines the coexistence curve in the (u,θ)-plane, referred to as the *binodal*. The region above the binodal represents stable single phase states, while its complement refers to states that are not thermodynamically stable. The concentrations u_{s_1} and u_{s_2} are the inflection points of F. By varying θ_2, the locus of the concentration values u_{s_1} and u_{s_2} determine a curve referred to as the *spinodal*. This curve separates the metastable and unstable subregions where the second derivative of F with respect to u is positive and negative, respectively, namely we have $f'(u)>0$ or $f'(u)<0$.
Referring to the phenomenology, when the initially homogenous system is quenched below the critical temperature θ_c, it undergoes separation into two phases. For inner states of the spinodal, the path to phase separation is traditionally classified as *spinodal decomposition*, whereas in the metastable region as *nucleation*.

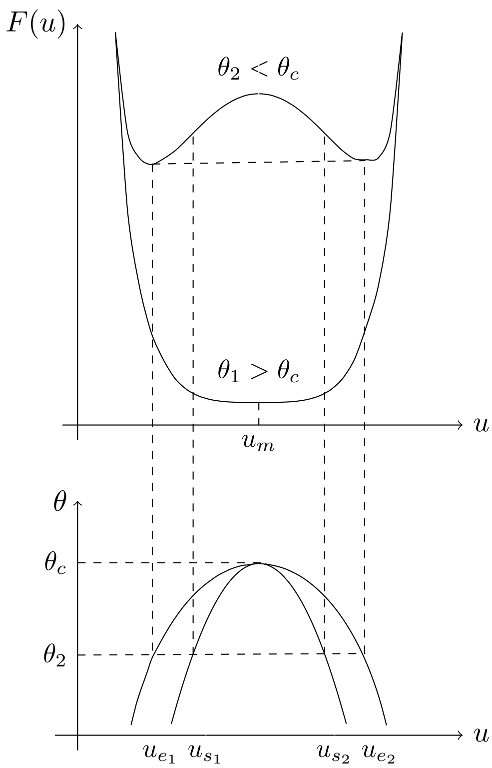

Figure 1.1: F at $\theta_1 > \theta_c$ and $\theta_2 < \theta_c$, and phase diagram of F.

At last, note that, when the absolute temperature is close to the critical one, the double-well potential F can be approximated by more regular ones (see Figure 1.2). Typical choices are given by

$$F(u) = \frac{1}{4}(u^2 - 1)^2 \,, \tag{1.6}$$

i.e.,

$$f(u) = u^3 - u \,; \tag{1.7}$$

or more generally,

$$F(u) = \frac{1}{4}(u^2 - \beta^2)^2 \,, \ \beta \in \mathbb{R}. \tag{1.8}$$

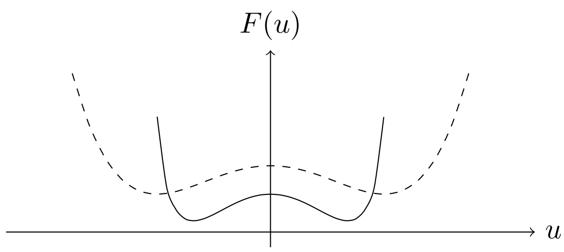

Figure 1.2: logarithmic potential (—) for $\theta < \theta_c$ and polynomial potential (- -).

Derivation of Cahn-Hilliard equation

From a phenomenological point of view, the Cahn–Hilliard system (1.1)-(1.2) can be derived as follows.

Firstly, a total Helmholtz free energy, that is called Ginzburg–Landau free energy, is considered; i.e.,

$$\Psi(u, \nabla u) = \int_\Omega \Big(\frac{\alpha}{2}|\nabla u|^2 + F(u)\Big) dx\,, \tag{1.9}$$

where $\Omega \subset \mathbb{R}^3$ is the domain occupied by the material.
The gradient term in (1.9) has been proposed in [11] in order to model the surface energy of the interface, i.e., capillarity. Indeed, it has a smoothing effect on interfaces between different phases. In consequence, jumps of u are not allowed, instead different phases are separated by (thin) layers, which are small subregions with rapid change of u. The thickness of these layers is related to the value of α (typically it goes as $\sqrt{\alpha}$). As already observed, Cahn-Hilliard model is then thought as a diffuse interface model. Although, the Cahn–Hilliard model approaches a sharp-interface limit when the interfacial thickness is reduced below a threshold while other parameters are fixed (see e.g [52]).
Next, consider the mass balance

$$\frac{\partial u}{\partial t} = -\operatorname{div}\, j\,, \tag{1.10}$$

where j is the mass flux which is related to the chemical potential μ by the following postulated constitutive equation which resembles Fick's law:

$$j = -m\nabla\mu\,, \tag{1.11}$$

where m is the diffusive mobility.
The chemical potential is usually defined as the partial derivative of the Helmholtz free energy with respect to the order parameter u. Here, such a definition is incompatible with the presence of ∇u in the free energy. Instead, μ is defined as a variational derivative of the free energy functional (1.9) with respect to u. This variational derivative can be formally computed by observing that, if δu represents a "small" increment of u,

$$\delta\Psi = \int_\Omega (\alpha\nabla u \cdot \nabla\delta u + f(u)\delta u) dx$$

Assuming proper boundary conditions, it yields

$$\delta\Psi = \int_\Omega (-\alpha\Delta u + f(u))\delta u\, dx\,.$$

Hence,

$$\mu = \frac{\delta\Psi}{\delta u} = -\alpha\Delta u + f(u)\,, \tag{1.12}$$

and the Cahn–Hilliard equation (1.3) follows. Observe that (1.3) can be equivalently rewritten as

$$\frac{\partial u}{\partial t} + \alpha m\Delta^2 u - \nabla\cdot(mf'(u)\nabla u) = 0$$

in $\Omega \times (0, T)$. Thus, the diffusion coefficient is given by $mf'(u)$. Note that, below the critical temperature θ_c, it becomes negative for $u_{s_1} < u < u_{s_2}$, contributing to an unstable development of the process (see Figure 1.1).

Boundary and initial conditions

The Cahn–Hilliard system (1.1)-(1.2), in a bounded and regular domain Ω, is usually associated with Neumann boundary conditions, namely,

$$\nabla \mu \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{1.13}$$

$$\nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{1.14}$$

where ν is the unit outer normal to the boundary. Since $j \cdot \nu = -m\nabla\mu \cdot \nu$, it follows from the first boundary condition that there is no mass flux at the boundary. In particular, the conservation of mass holds, i.e., of the spatial average of the order parameter, obtained by formally integrating (1.1) over Ω:

$$\langle u(t) \rangle \equiv \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega u(x,t)dx = \langle u(0) \rangle, \quad \forall t \in [0,\, T]\,. \tag{1.15}$$

As for the second boundary condition, it is a natural variational boundary condition prescribing that the interface is orthogonal to the boundary. In other words, it allows to write down a convenient variational/weak formulation in view of the mathematical analysis of the problem.
Otherwise, we could consider periodic boundary conditions. In this case $\Omega = \Pi_{i=1}^3 (0, L_i)$, $L_i > 0$, $i = 1, 2, 3$, and the conservation of mass still holds. Periodic boundary conditions are often chosen in order to approximate a large (infinite) system by using a small part called a unit cell. Moreover, they lead to some simplifications from the mathematical point of view.
Note that generally Dirichlet boundary conditions are not associated with the Cahn-Hilliard equation, due precisely to the fact that they do not yield the conservation of mass, although such boundary conditions certainly simplify the mathematical analysis.
The model is completed by appropriate initial conditions:

$$u(x, 0) = u_0(x), \quad \forall x \in \Omega\,,$$

whose mean value

$$\langle u_0 \rangle = u_m$$

is conserved over time if a Neumann or periodic boundary condition for μ is assumed, as already observed above.

The Cahn–Hilliard system is now quite well understood, at least from a mathematical point of view. In particular, one has a rather complete picture as far as the existence, the uniqueness and the regularity of solutions and the asymptotic behavior of the associated dynamical system are concerned. Among the huge literature, we refer the reader to, e.g., [12, 13, 15, 17, 21, 28, 29, 41, 42, 44–46, 67].

Cahn-Hilliard equation with nonconstant mobility

Note that we have assumed so far the mobility m to be a positive constant. Actually, m is often expected to depend on the order parameter and to degenerate at the singular points of f in the case of a logarithmic nonlinear term (see [19, 20, 26]). This essentially restricts the diffusion process to the interfacial region, which is observed, typically, in physical situations in which the movements of atoms are confined to this region (see [54]). In that case, (1.1) reads

$$\frac{\partial u}{\partial t} = \operatorname{div}(m(u)\nabla\mu)$$

in $\Omega \times (0, T)$, where, typically,

$$m(s) = 1 - s^2, \quad \text{or more generally,} \quad m(s) = (1 - s^2)^\delta,\ \delta > 0\,.$$

The existence of solutions to the Cahn–Hilliard equation with degenerate mobilities and logarithmic nonlinearities is proved in [19]. Up to now, only existence of weak solutions is known, whereas existence of strong solutions and uniqueness are still open issues, especially in higher space dimension. The asymptotic behaviour of the Cahn–Hilliard equation with nonconstant and nondegenerating mobilities is studied in [59, 61].

1.1.2 Cahn–Hilliard models based on a microforce balance

This section concerns some models derived by M. Gurtin in [31].
Actually, Gurtin makes several remarks on the phenomenological derivations of the Cahn–Hilliard equation such as that presented in Section 1.1.1. He notes that such derivations should not be regarded as basic, rather as a precursor of more complete theories.

From Gurtin's point of view, while variational derivatives often point the way toward a correct statement of basic laws, such derivations obscure the fundamental nature of balance laws in any general framework that includes dissipation.
In particular, he makes the following objections to the classical isothermal theory:

- It limits the manner in which rate terms enter the equations.
- It requires a priori specification of the constitutive equations for the mass flux (and for the heat flux in the non-isothermal case). Gurtin further notes that such postulated constitutive equations may no longer be valid if the system is far from equilibrium. Indeed, the mass flux (1.11) resembles Fick's law, which is essentially valid close to equilibrium.
- It is not clear how it can be generalized in the presence of processes such as deformations, e.g., for elastic solids, or heat transfers. Recall indeed that, in real systems, quenches are carried out over a certain period of time and the phase separation can occur before the final temperature is reached.

- There is no clear separation between basic balance laws, such as those for mass and forces, and constitutive equations, such as those for elastic solids and viscous fluids, which delineate specific classes of material behaviours. Such a separation is one of the major advances in nonlinear continuum mechanics over the past thirty years.

- It is not clear whether or not there is an underlying balance law which can form a basis for more complete theories.

Based on these observations, Gurtin proposes in [31] several generalizations of the isothermal Cahn-Hilliard equation. More precisely, a relevant feature which distinguishes his approach from other macroscopic theories of order parameters is the separation of balance laws from constitutive equations and the introduction of a new balance law for internal microforces, i.e., forces at an atomistic level.

His approach is based on the belief that fundamental physical laws involving energy should account for the working associated to each operative kinematical process. In Cahn-Hilliard theory the kinematics is associated with the order-parameter u. Therefore, it seems plausible that there should be "microforces" whose working accompanies changes in u. Indeed, if the only manifestation of atomistic kinematics is the order-parameter u, then it seems reasonable that such interatomic forces may be characterized macroscopically by fields that perform work when u undergoes changes. This working is described through terms of the form (force)$\frac{\partial u}{\partial t}$, so that microforces are scalar rather than vector quantities. Specifically, the microforce system is characterized by a vector stress ζ together with scalar body force π and γ that represent, respectively, internal and external forces distributed over the volume of $\Omega \subset \mathbb{R}^3$, the domain occupied by the material. To describe the precise manner in which these fields expend power it is useful to consider the body as a lattice or network together with atoms that move, microscopically, relative to the lattice (see [35]). Note that it is important to focus attention not on individual atoms but on configurations (i.e., arrangements or densities) of atoms as characterized by the order parameter u.

Given an arbitrary control volume $\mathcal{R} \subset \Omega$, with ν the outward unit normal to the boundary $\partial\mathcal{R}$, each of

$$\int_{\partial\mathcal{R}} (\zeta \cdot \nu) \frac{\partial u}{\partial t}\, d\sigma\,, \quad \int_{\mathcal{R}} \pi \frac{\partial u}{\partial t}\, dx\,, \quad \int_{\mathcal{R}} \gamma \frac{\partial u}{\partial t}\, dx\,,$$

represents an expenditure of power on the atomic configurations within $\mathcal{R}$. In particular, $(\zeta \cdot \nu)\frac{\partial u}{\partial t}$ describes power expended across $\partial\mathcal{R}$ by configurations exterior to $\mathcal{R}$, but neighboring. $\pi\frac{\partial u}{\partial t}$ represents power expended on the atoms of $\mathcal{R}$ by the lattice; for example, in the ordering of atoms within unit cells of the lattice or the transport of atoms between unit cells of the lattice. As for $\gamma\frac{\partial u}{\partial t}$, it describes power expended on the atoms of $\mathcal{R}$ by sources external to the body. This system of forces is presumed to be consistent with the microforce balance

$$\int_{\partial\mathcal{R}} \zeta \cdot \nu\, d\sigma + \int_{\mathcal{R}} (\pi + \gamma)\, dx \;= 0\,, \tag{1.16}$$

for each control volume $\mathcal{R} \subset \Omega$.
A partial motivation for the microforce balance (1.16) can be given by the fact that, at equilibrium, the first variation of the Helmholtz free energy vanishes, i.e.

$$\delta\Psi = \int_\Omega (\partial_{\nabla u}\psi \cdot \nabla\delta u + \partial_u\psi\delta u)dx = 0 \, .$$

Assuming proper boundary conditions, we then obtain the Euler-Lagrange equation $\operatorname{div}\zeta + \pi = 0$, where $\zeta = \partial_{\nabla u}\psi$ and $\pi = -\partial_u\psi$, which represents a static version of the microforce balance (1.10), with ζ and π being given constitutive representations and $\gamma = 0$. In dynamics with general forms of dissipation there is no such variational principle, and actually the use of a microforce balance can be seen as an attempt to extend to dynamics an essential feature of static theories.

Gurtin's approach is essentially a two-scale approach. If standard forces in continua are associated with macroscopic length scales, microforces describe forces associated with microscopic configurations of atoms. These different length scales explain the need for a separate balance law for microforces. In order to have a full description of the dynamics of the phase separation process, the microforce balance (1.16) has to be complemented with the mass balance and a mechanical version of the second law of Thermodynamics, which has to be expressed in a form which takes into account the action of the internal microforces.
Furthermore, Gurtin does not introduce a constitutive equation for the mass flux and he does not state the explicit form of the physical quantities. Rather, he keeps these quantities in an implicit form and he just specifies a list, which is taken as wide as possible, of independent constitutive variables upon which they are allowed to depend. It is only a posteriori that the admissible expressions for the physical parameters are deduced and this is done by solving a thermodynamic inequality which arises as a direct consequence of the balance laws. In particular, this kind of procedure allows us to describe what seems to be the most general class of free energies and of chemical potentials which are compatible with the fundamental laws.

Derivation of the generalized Cahn-Hilliard equation

Gurtin's development of the Cahn-Hilliard theory begins with balance laws for mass and microforce in conjunction with a dissipation inequality. In this sense, in order to have a full description of the dynamics of the phase separation process, relation (1.16) has to be complemented with the fundamental balance laws.

Let $\Omega \subset \mathbb{R}^3$ be the domain occupied by the material, which is supposed to be open, bounded and with a smooth boundary $\partial\Omega$, and let $[0, T] \subset \mathbb{R}^+$ be a finite time interval. Firstly, we consider the mass balance, namely

$$\frac{d}{dt}\int_{\mathcal{R}} u dx = -\int_{\partial\mathcal{R}} j \cdot \nu \, d\sigma + \int_{\mathcal{R}} h \, dx, \tag{1.17}$$

for every control volume $\mathcal{R} \subset \Omega$, where j is the mass flux and h is the external mass supply. Next, we consider a version of second law of Thermodynamics appropriate to purely mechanical theory, which asserts that the rate at which the free energy increases cannot exceed the sum of the work on $\mathcal{R}$ and the rate at which free energy is carried into the control volume $\mathcal{R}$ by mass transport. Namely,

$$\frac{d}{dt}\int_{\mathcal{R}} \psi\, dx \leq \mathcal{W}(\mathcal{R}) + \mathcal{M}(\mathcal{R}) \tag{1.18}$$

where ψ is the Helmholtz free energy density, $\mathcal{W}(\mathcal{R})$ is the rate of working of all forces exterior to $\mathcal{R}$ and $\mathcal{M}(\mathcal{R})$ is the rate at which free energy is added to $\mathcal{R}$ by mass transport. To motivate this version of the second law, consider the first two laws of Thermodynamics, i.e., the balance of energy

$$\frac{d}{dt}\int_{\mathcal{R}} e\, dx = -\int_{\partial\mathcal{R}} q\cdot\nu\, d\sigma + \int_{\mathcal{R}} r\, dx + \mathcal{W}(\mathcal{R}) + \mathcal{M}(\mathcal{R})\,, \tag{1.19}$$

and the Clausius-Duhem entropy production inequality

$$\frac{d}{dt}\int_{\mathcal{R}} s dx \geq -\int_{\partial\mathcal{R}} \frac{q\cdot\nu}{\theta}\, d\sigma + \int_{\mathcal{R}} \frac{r}{\theta}\, dx\,, \tag{1.20}$$

in which e is the internal energy density, s is the entropy density, θ is the (absolute) temperature, q is the heat flux and r is the heat supply. Since the Helmholtz free energy density is given by the Gibbs relation

$$\psi = e - \theta s\,,$$

assuming isothermal conditions, i.e., θ = constant, then multiplying (1.20) by θ and subtracting the resulting equation from (1.19), we obtain (1.18). In particular, the rate of working of external forces is given by

$$\mathcal{W}(\mathcal{R}) = \int_{\partial\mathcal{R}} (\zeta\cdot\nu)\frac{\partial u}{\partial t} d\sigma + \int_{\mathcal{R}} \gamma\frac{\partial u}{\partial t} dx\,, \tag{1.21}$$

while the mass transport is characterized by the chemical potential and it reads

$$\mathcal{M}(\mathcal{R}) = -\int_{\partial\mathcal{R}} \mu j\cdot\nu d\sigma + \int_{\mathcal{R}} \mu h\, dx\,. \tag{1.22}$$

Observe that (1.21) does not include the work of π. Indeed, inequality (1.18) holds for the material in $\mathcal{R}$, i.e., lattice plus atoms. Therefore, being π a force exerted by the lattice on the atoms, which acts internally to the material in $\mathcal{R}$, its work is not considered. On the other hand, (1.10) represents a force balance for the atomic configurations and thus it includes the action of π.
Because of the arbitrariness of the control volume $\mathcal{R}$, the microforce balance (1.16) reads

$$\operatorname{div}\zeta + \pi + \gamma = 0\,, \tag{1.23}$$

where the vector ζ is the microstress and the scalar π and γ correspond to the internal and external microforces, respectively. On the other hand, using the divergence theorem and the arbitrariness of the control volume $\mathcal{R}$, from (1.17) we recover

$$\frac{\partial u}{\partial t} = -\operatorname{div} j + h\,. \tag{1.24}$$

Note that

$$\int_{\partial\mathcal{R}} (\zeta \cdot \nu) \frac{\partial u}{\partial t} d\sigma = \int_{\mathcal{R}} \operatorname{div} \Big(\frac{\partial u}{\partial t} \zeta \Big) dx$$

and

$$\int_{\partial\mathcal{R}} \mu j \cdot \nu d\sigma = \int_{\mathcal{R}} \operatorname{div}(\mu j) dx$$

Therefore, since the control volume $\mathcal{R}$ is arbitrary, from (1.18) it follows that

$$\begin{aligned} \frac{\partial \psi}{\partial t} &\leq \operatorname{div} \Big(\frac{\partial u}{\partial t} \zeta \Big) - \operatorname{div}(\mu j) \\ &= \frac{\partial u}{\partial t} \operatorname{div} \zeta + \zeta \cdot \nabla \frac{\partial u}{\partial t} - \mu \operatorname{div} j - j \cdot \nabla \mu \end{aligned}$$

and (1.23)-(1.24) finally yield the following dissipation inequality:

$$\frac{\partial \psi}{\partial t} + (\pi - \mu) \frac{\partial u}{\partial t} - \zeta \cdot \nabla \frac{\partial u}{\partial t} + j \cdot \nabla \mu \leq 0 \,. \tag{1.25}$$

In standard theories of diffusion the chemical potential is given, constitutively, as a function of the order parameter u. On the contrary, if one considers systems sufficiently far from equilibrium such that a relation of this type is no longer valid, the chemical potential and its gradient join u and ∇u in the list of independent constitutive variables. Specifically, constitutive equations of the following form are assumed

$$\begin{aligned} \psi &= \psi(u, \nabla u, \mu, \nabla \mu) \,, \quad j = j(u, \nabla u, \mu, \nabla \mu) \,, \\ \zeta &= \zeta(u, \nabla u, \mu, \nabla \mu) \,, \quad \pi = \pi(u, \nabla u, \mu, \nabla \mu) \,. \end{aligned}$$

If we call Z the set of independent constitutive variables, namely,

$$Z = (u, \nabla u, \mu, \nabla \mu) \,,$$

then $\psi = \psi(Z)$, $j = j(Z)$, $\zeta = \zeta(Z)$ and $\pi = \pi(Z)$. It follows that

$$\frac{\partial \psi}{\partial t} = \partial_u \psi \frac{\partial u}{\partial t} + \partial_{\nabla u} \psi \cdot \nabla \frac{\partial u}{\partial t} + \partial_\mu \psi \frac{\partial \mu}{\partial t} + \partial_{\nabla \mu} \psi \cdot \nabla \frac{\partial \mu}{\partial t}$$

and the dissipation inequality (1.25) becomes

$$(\partial_u \psi + \pi - \mu) \frac{\partial u}{\partial t} + (\partial_{\nabla u} \psi - \zeta) \cdot \nabla \frac{\partial u}{\partial t} + \partial_\mu \psi \frac{\partial \mu}{\partial t} + \partial_{\nabla \mu} \psi \cdot \nabla \frac{\partial \mu}{\partial t} + j \cdot \nabla \mu \leq 0 \,. \tag{1.26}$$

Note that (1.26) has to hold for all fields Z. Actually, it is possible to choose Z such that Z, $\frac{\partial u}{\partial t}$, $\nabla \frac{\partial u}{\partial t}$, $\frac{\partial \mu}{\partial t}$ and $\nabla \frac{\partial \mu}{\partial t}$ take arbitrary prescribed values at some chosen point x and time t. Therefore, since $\frac{\partial u}{\partial t}$, $\nabla \frac{\partial u}{\partial t}$, $\frac{\partial \mu}{\partial t}$ and $\nabla \frac{\partial \mu}{\partial t}$ appear linearly in (1.26), necessarily,

$$\partial_u \psi + \pi - \mu = 0 \,, \tag{1.27}$$

$$\partial_{\nabla u} \psi - \zeta = 0 \,, \tag{1.28}$$

$$\partial_\mu \psi = 0, \quad \partial_{\nabla \mu} \psi = 0 \,. \tag{1.29}$$

Indeed, otherwise, $\frac{\partial u}{\partial t}$, $\nabla\frac{\partial u}{\partial t}$, $\frac{\partial \mu}{\partial t}$ and $\nabla\frac{\partial \mu}{\partial t}$ could be chosen to violate (1.26). It follows that $\psi = \psi(u, \nabla u)$, as expected, and (1.26) reads

$$j \cdot \nabla\mu \leq 0\,,$$

which has to hold for every field Z. This yields the existence of a matrix $A = A(Z)$, the so-called mobility tensor, such that (see Appendix A.1)

$$j = -A(Z)\nabla\mu\,,$$

and A is, in some sense, positive semi-definite:

$$Y \cdot A(Z)Y \geq 0\,,\ \forall Y \in \mathbb{R}^3.$$

Hence, (1.24) can be rewritten as

$$\frac{\partial u}{\partial t} = \operatorname{div}(A(Z)\nabla\mu) + h\,. \tag{1.30}$$

Furthermore, combining together (1.23), (1.27) and (1.28), it yields

$$\mu = \partial_u\psi + \pi = \partial_u\psi - \operatorname{div}\zeta - \gamma = \partial_u\psi - \operatorname{div}\partial_{\nabla u}\psi - \gamma\,. \tag{1.31}$$

Substituting (1.31) into (1.30), we obtain the so-called generalized Cahn-Hilliard equation, that is

$$\frac{\partial u}{\partial t} = \operatorname{div}(A(Z)\nabla(\partial_u\psi - \operatorname{div}\partial_{\nabla u}\psi - \gamma)) + h \tag{1.32}$$

in $\Omega \times (0, T)$. If we consider the Ginzburg-Landau free energy given by (1.9), i.e., $\psi = \frac{\alpha}{2}|\nabla u|^2 + F(u)$, then

$$\partial_u\psi = F'(u) = f(u), \quad \partial_{\nabla u}\psi = \alpha\nabla u\,.$$

Assuming null external microforce, i.e., $\gamma = 0$, it follows that the chemical potential μ is given by the variational derivative of the total free energy, as in (1.12). Finally, assuming null external mass supply, i.e., $h = 0$, (1.32) reduces to

$$\frac{\partial u}{\partial t} = \operatorname{div}(A(Z)\nabla(-\alpha\Delta u + f(u))$$

in $\Omega \times (0, T)$. In particular, assuming isotropy, i.e., the mobility tensor of the form $A = mI$, $m > 0$, we recover the standard Cahn–Hilliard equation (1.3).

1.2 Non-isothermal Cahn-Hilliard models

In isothermal Cahn-Hilliard models, which assume that the process is caused by an instantaneous quench below a critical temperature, only diffusive phenomena are taken into account. However, in realistic physical systems, quenches are usually carried out over a finite period of time, so that phase separation can begin before the final quenching is reached. Furthermore, external thermal activation can be used to control the separation process and the resulting spatial structure.

Different models of non-isothermal phase transitions have been proposed. We mention a model for phase separation during continuous cooling proposed by Huston et al. (see [32]), which is merely the Cahn-Hilliard equation with temperature dependent coefficients, and the conserved and nonconserved phase-field models proposed by Caginalp (see [10]). In their discussion of the phase-field theory of solidification, Penrose and Fife [53] replaced the free energy functional with an entropy functional, from which they deduced kinetic equations. However, is not clear how their derivation can be generalized in the presence of heat transfer. In 1998, Zhiqiang Bi and Sekerka proposed in [6] a generalization of the Cahn-Hilliard equation, developing a general thermodynamically consistent phase field model based on a proper postulated entropy functional which accounts for heat transfers. However, this generalization follows the standard theory in the derivation of the equations.

In 1992, H.W. Alt and I. Pawłow proposed and studied the existence of solutions for a mathematical model of phase separation that describes coupled phenomena of mass diffusion and heat conduction in binary systems under thermal activation (see [3]- [2]). Indeed, their aim was to extend the Cahn-Hilliard approach to the non-isothermal setting by accounting for the dynamics of energy transfer. The construction is based on the Landau-Ginzburg free energy functional and on Non-equilibrium Thermodynamics. Firstly, constitutive relations for the mass and heat fluxes are postulated and a definition for the rescaled chemical potential is provided, then the derivation of the model follows from the mass and energy balance laws. The proposed model has the form of a system of nonlinear parabolic partial differential equations, where the concentration and the energy are conserved quantities. The consistency with the first and the second principle of Thermodynamics is shown a posteriori and it is essentially a theorem.
In 1996, H.W. Alt and I. Pawłow obtained more general models, making no a priori specification on the entropy flux and obtaining the Gibbs relation as a consequence of a weaker form of entropy inequality (see [4]). Furthermore, they derived more general forms of the constitutive equations for the mass and heat fluxes.

In 2005, following M. Gurtin's approach (see Subsection 1.1.2), A. Miranville and G. Schimperna extended the previous non-isothermal model derived by Alt and Pawłow to non-isotropic materials and to systems that are far from equilibrium (see [43]). Their approach is a two-scale one and the two fundamental laws of Thermodynamics are used as a starting point to derive the model. Furthermore, no a priori specification of the constitutive equations for the mass flux, the heat flux and the chemical potential is made. Rather, these quantities are kept in an implicit form and only a list of independent constitutive variables upon which they are allowed to depend is specified. It is only a posteriori that the admissible expressions for the physical parameters are deduced. Thus, this kind of procedure seems to be the one that may allow us to describe the most general class of free energies and of chemical potentials which are compatible with the fundamental laws.

1.2.1 A mathematical model of dynamics of non-isothermal phase separation

In this subsection we introduce the model of non-isothermal phase separation proposed by Alt and Pawłow in [3]. In particular, we present its derivation and we prove its thermodynamic consistency.

Derivation of a non-isothermal Cahn-Hilliard model

Let $\Omega \subset \mathbb{R}^3$, be the domain occupied by the material, which is supposed to be open, bounded and with a smooth boundary $\partial\Omega$, and let $[0,T] \subset \mathbb{R}^+$ be a finite time interval. Let u be the order parameter, whereas θ the absolute temperature.
The Helmholtz free energy density is assumed in the form of the Ginzburg-Landau one (see [23]), i.e.,

$$\psi(u, \nabla u, \theta) = \frac{1}{2}\alpha(u,\theta)|\nabla u|^2 + F(u,\theta)\,, \tag{1.33}$$

where F is the volumetric energy density of a homogeneous system and α is a positive surface tension coefficient possibly dependent on the concentration and temperature. In the following, F and α are left in a general form and they are supposed to be regular enough.
Let e denote the internal energy density of the system and s its entropy density. The Gibbs relation

$$\psi = e - \theta s \tag{1.34}$$

is postulated, with the entropy $s = -\partial_\theta \psi$. It follows that

$$e = \psi - \theta\partial_\theta\psi = \tfrac{1}{2}(\alpha - \partial_\theta\alpha)|\nabla u|^2 + F - \theta\partial_\theta F\,. \tag{1.35}$$

In the isothermal case μ is given by the variational derivative of the total free energy (1.9) with respect to u. This can be generalized to the non-isothermal case by introducing the reduced chemical potential and defining it as the variational derivative of the rescaled free energy with respect to u, that is

$$\frac{\mu}{\theta} = \frac{\delta}{\delta u}\Big(\frac{\Psi}{\theta}\Big)\,, \tag{1.36}$$

where $\Psi(u,\nabla u,\theta) = \int_\Omega \psi(u,\nabla u,\theta)\,dx$ and ψ is given by (1.33). In other words, for spatially non-uniform temperature fields θ the free energy is scaled by $\frac{1}{\theta}$.
The variational derivative can be formally computed by noting that, for a small variation δu,

$$\delta\Big(\frac{\Psi}{\theta}\Big) = \int_\Omega \Big(\frac{1}{2}\frac{\partial_u\alpha}{\theta}|\nabla u|^2\delta u + \alpha\nabla u\cdot\nabla\delta u + \frac{\partial_u F}{\theta}\delta u\Big)dx$$

Assuming proper boundary conditions, it yields

$$\delta\Big(\frac{\Psi}{\theta}\Big) = \int_\Omega \Big(\frac{1}{2}\frac{\partial_u\alpha}{\theta}|\nabla u|^2 - \operatorname{div}\Big(\frac{\alpha\nabla u}{\theta}\Big) + \frac{\partial_u F}{\theta}\Big)\delta u\,dx\,,$$

whence

$$\frac{\mu}{\theta} = \frac{1}{2}\frac{\partial_u \alpha}{\theta}|\nabla u|^2 - \operatorname{div}\left(\frac{\alpha \nabla u}{\theta}\right) + \frac{\partial_u F}{\theta}\,. \tag{1.37}$$

The mass balance is still governed by (1.10). However, in the non-isothermal case, the balance of energy is also to be considered, that is

$$\frac{\partial e}{\partial t} + \operatorname{div} q = r\,, \tag{1.38}$$

where q is the energy flux due to heat and mass transfer, while $r = r(x,t)$ represents the rate of distributed heat sources, i.e., the heat supply. By employing the Non-equilibrium Thermodynamics [16], Alt and Pawłow postulate the constitutive relations for the mass and energy fluxes in the forms

$$j = -l_{11}\nabla\frac{\mu}{\theta} + l_{12}\nabla\frac{1}{\theta}\,, \tag{1.39}$$

$$q = l_{22}\nabla\frac{1}{\theta} - l_{21}\nabla\frac{\mu}{\theta}\,, \tag{1.40}$$

where the coefficients $l_{ij} = l_{ij}(u, \frac{\mu}{\theta}, \frac{1}{\theta})$, $i,j = 1,2$, satisfy

$$l_{11} > 0\,, \quad l_{22} > 0\,, \quad l_{11}l_{22} - l_{12}l_{21} > 0\,, \tag{1.41}$$

i.e., the matrix $(l_{i,j})_{i,j=1,2}$ is positive definite and it has positive diagonal elements. Equations (1.10), (1.35), (1.37), (1.38), (1.39) and (1.40) can be summarized in the following system that describe the dynamics of the non-isothermal phase separation

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(l_{11}\nabla\frac{\mu}{\theta} - l_{12}\nabla\frac{1}{\theta}\right) = 0\,, \tag{1.42}$$

$$\frac{\partial e}{\partial t} + \operatorname{div}\left(l_{22}\nabla\frac{1}{\theta} - l_{21}\nabla\frac{\mu}{\theta}\right) = r\,, \tag{1.43}$$

$$\frac{\mu}{\theta} = \frac{1}{2}\frac{\partial_u \alpha}{\theta}|\nabla u|^2 - \operatorname{div}\left(\frac{\alpha \nabla u}{\theta}\right) + \frac{\partial_u F}{\theta}\,, \tag{1.44}$$

$$e = \frac{1}{2}(\alpha - \partial_\theta \alpha)|\nabla u|^2 + F - \theta\partial_\theta F\,, \tag{1.45}$$

in $\Omega \times (0,T)$.

Boundary and initial conditions

Alt and Pawłow assumed the system to be thermodynamically closed, that is, it may exchange only heat with its environment. The condition of mass isolation is given by

$$j \cdot \nu = 0 \ \text{ on } \partial\Omega,$$

where ν is the unit outer normal to the boundary $\partial\Omega$. As for the concentration,

$$\nabla u \cdot \nu = 0 \ \text{ on } \partial\Omega$$

is imposed. For the temperature Alt and Pawłow postulate Newton's heat exchange through the boundary, that is

$$q \cdot \nu + p\Big(u, \frac{\mu}{\theta}, \frac{1}{\theta}\Big) = 0 \ \text{ on } \partial\Omega,$$

where

$$p\Big(u, \frac{\mu}{\theta}, \frac{1}{\theta}\Big) = -p_0 \int_{\theta_{ext}}^{\theta} k_b\Big(u, \frac{\mu}{\theta'}, \frac{1}{\theta'}\Big) d\theta' - r_b \,.$$

$p_0 = p_0(x)$ is the non-negative heat exchange coefficient, θ_{ext} is the external temperature, $k_b = k_b(u, \frac{\mu}{\theta'}, \frac{1}{\theta'})$ represents the effective heat conductivity at the boundary, while $r_b = r_b(x, t)$ is the boundary flux. Summarizing and using constitutive equations (1.39) and (1.40), the boundary conditions imposed by Alt and Pawłow are the following

$$\nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{1.46}$$

$$\Big(- l_{11} \nabla \frac{\mu}{\theta} + l_{12} \nabla \frac{1}{\theta} \Big) \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{1.47}$$

$$\Big(l_{22} \nabla \frac{1}{\theta} - l_{21} \nabla \frac{\mu}{\theta} \Big) \cdot \nu + p\Big(u, \frac{\mu}{\theta}, \frac{1}{\theta}\Big) = 0 \quad \text{on } \partial\Omega. \tag{1.48}$$

The model is completed by appropriate initial conditions ofr the concentration and the temperature:

$$u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \quad \forall x \in \Omega \,,$$

with prescribed mean value

$$\langle u_0 \rangle = u_m \,,$$

which is conserved over time. Indeed, integrating (1.42) over Ω and using boundary condition (1.47), we obtain $\frac{\partial u}{\partial t} = 0$, which implies

$$\langle u(t) \rangle = \langle u(0) \rangle, \quad \forall t \in [0, T] \,. \tag{1.49}$$

Observe that there is a direct correspondence between the system of equations deduced by Alt and Pawłow and the standard Cahn-Hilliard equation. Indeed, assuming $\alpha =$ constant and $r = 0$, if θ is constant, the system of equations (1.42)-(1.45) reduce to (1.3) with $m = \frac{l_{11}}{\theta}$, while the boundary conditions (1.46)-(1.48) reduce to the Neumann boundary conditions (1.13) and (1.14).

Thermodynamic consistency

The model proposed by Alt and Pawłow (1.42)-(1.48) conforms with two basic laws of Thermodynamics.

- Balance of energy:
  Integrating (1.42) over Ω and using the boundary condition (1.48), we obtain

  $$\frac{d}{dt} \int_\Omega e \, dx = \int_\Omega r \, dx + \int_{\partial\Omega} p\Big(u, \frac{\mu}{\theta}, \frac{1}{\theta}\Big) d\sigma \,,$$

  which expresses the balance of energy.

- Clausius-Duhem inequality for the entropy production:
  Multiplying (1.42) by $\frac{\mu}{\theta}$, then integrating over Ω and using (1.47), it yields

  $$\int_\Omega \frac{\mu}{\theta} \frac{\partial u}{\partial t} \, dx + \int_\Omega \Big(l_{11} \nabla \frac{\mu}{\theta} - l_{12} \nabla \frac{1}{\theta} \Big) \cdot \nabla \frac{\mu}{\theta} \, dx = 0 \,. \tag{1.50}$$

On the other hand, multiplying (1.43) by $\frac{1}{\theta}$, then integrating over Ω and using (1.48), we obtain

$$\int_\Omega \frac{1}{\theta}\frac{\partial e}{\partial t}\,dx - \int_\Omega \Big(l_{22}\nabla\frac{1}{\theta} - l_{21}\nabla\frac{\mu}{\theta}\Big)\cdot\nabla\frac{1}{\theta}\,dx + \\ - \int_{\partial\Omega}\frac{1}{\theta}p\Big(u,\frac{\mu}{\theta},\frac{1}{\theta}\Big)d\sigma = \int_\Omega \frac{r}{\theta}\,dx\,. \tag{1.51}$$

Subtracting (1.50) from (1.51), it yields

$$\int_\Omega \Big(\frac{1}{\theta}\frac{\partial e}{\partial t} - \frac{\mu}{\theta}\frac{\partial u}{\partial t}\Big)dx - \int_{\partial\Omega}\frac{1}{\theta}p\Big(u,\frac{\mu}{\theta},\frac{1}{\theta}\Big)d\sigma - \int_\Omega\frac{r}{\theta}\,dx = \\ \int_\Omega \Big(l_{11}\nabla\frac{\mu}{\theta} - l_{12}\nabla\frac{1}{\theta}\Big)\cdot\nabla\frac{\mu}{\theta}\,dx + \int_\Omega\Big(l_{22}\nabla\frac{1}{\theta} - l_{21}\nabla\frac{\mu}{\theta}\Big)\cdot\nabla\frac{1}{\theta}\,dx\,,$$

whence, using conditions (1.41), i.e., that the matrix $(l_{i,j})_{i,j=1,2}$ is positive definite, we deduce

$$\int_\Omega \Big(\frac{1}{\theta}\frac{\partial e}{\partial t} - \frac{\mu}{\theta}\frac{\partial u}{\partial t}\Big)dx - \int_{\partial\Omega}\frac{1}{\theta}p\Big(u,\frac{\mu}{\theta},\frac{1}{\theta}\Big)d\sigma - \int_\Omega\frac{r}{\theta}\,dx \geq 0\,. \tag{1.52}$$

Rewriting the Gibbs relation (1.34) as $s = \frac{e}{\theta} - \frac{\psi}{\theta}$ and taking the partial derivative wih respect to time, we obtain

$$\frac{\partial s}{\partial t} = \frac{1}{\theta}\frac{\partial e}{\partial t} - \frac{1}{\theta}\frac{\partial\psi}{\partial t} - \frac{(e-\psi)}{\theta^2}\frac{\partial\theta}{\partial t}\,.$$

From (1.33) and (1.35) it follows

$$\frac{\partial\psi}{\partial t} = \frac{1}{2}\partial_u\alpha|\nabla u|^2\frac{\partial u}{\partial t} + \frac{1}{2}\partial_\theta\alpha|\nabla u|^2\frac{\partial\theta}{\partial t} + \alpha\nabla u\cdot\nabla\frac{\partial u}{\partial t} + \partial_u F\frac{\partial u}{\partial t} + \partial_\theta F\frac{\partial\theta}{\partial t}$$

and

$$e - \psi = -\tfrac{1}{2}\partial_\theta\alpha|\nabla u|^2 - \theta\partial_\theta F\,.$$

Hence, we can rewrite (1.52) as

$$\int_\Omega\Big(\frac{\partial s}{\partial t} + \frac{1}{2}\frac{\partial_u\alpha}{\theta}|\nabla u|^2\frac{\partial u}{\partial t} + \frac{\alpha}{\theta}\nabla u\cdot\nabla\frac{\partial u}{\partial t} + \frac{\partial_u F}{\theta}\frac{\partial u}{\partial t} - \frac{\mu}{\theta}\frac{\partial u}{\partial t}\Big)dx + \\ - \int_{\partial\Omega}\frac{1}{\theta}p\Big(u,\frac{\mu}{\theta},\frac{1}{\theta}\Big)d\sigma - \int_\Omega\frac{r}{\theta}\,dx \geq 0\,.$$

Using the boundary condition (1.46) and (1.44), we can conclude that

$$\int_\Omega\frac{\partial s}{\partial t}\,dx - \int_{\partial\Omega}\frac{1}{\theta}p\Big(u,\frac{\mu}{\theta},\frac{1}{\theta}\Big)d\sigma - \int_\Omega\frac{r}{\theta}\,dx \geq 0\,.$$

Finally, in view of the constitutive equation (1.40) and the boundary conditions (1.48), we deduce

$$\int_{\partial\Omega}\frac{1}{\theta}p\Big(u,\frac{\mu}{\theta},\frac{1}{\theta}\Big)d\sigma = \int_{\partial\Omega}\frac{q\cdot\nu}{\theta}\,d\sigma\,.$$

It follows that

$$\int_\Omega\frac{\partial s}{\partial t}\,dx \geq -\int_{\partial\Omega}\frac{q\cdot\nu}{\theta}\,d\sigma + \int_\Omega\frac{r}{\theta}\,dx\,,$$

which is the integral form of the Clausius-Duhem inequality for the entropy production.

1.2.2 Non-isothermal phase separation based on a microforce balance

A. Miranville and G. Schimperna in [43] extend the approach of Gurtin to the non-isothermal case in order to derive non-isothermal Cahn-Hilliard models. In particular, these models generalize those derived by Alt and Pawłow to anisotropic materials and to systems that are far from equilibrium.
Following Gurtin's approach (see Subsection 1.1.2), they derive the model by considering the microforce balance (1.16) (or (1.23)) together with the mass balance (1.17) (or (1.24)) complemented with the two fundamental laws of Thermodynamics. Clearly, these laws, and especially the energy equality, have to be expressed in a form which takes into account the action of the internal microforces. Furthermore, they do not introduce a system of constitutive equations for the mass and heat fluxes stating the explicit form of the physical quantities (as done by Alt and Pawłow in [3] and in Subsection 1.2.1). Rather, still following Gurtin, they keep these quantities in an implicit form and just specify a list of independent constitutive variables upon which they are allowed to depend. It is only a posteriori that the admissible expressions for the physical parameters are deduced by solving a system of thermodynamic inequalities which arises as a direct consequence of the balance laws. As already observed, this kind of procedure allows them to describe what seems to be the most general class of free energies, of chemical potentials and also of heat fluxes in this non-isothermal setting, which are compatible with the fundamental laws.

Derivation of a non-isothermal Cahn-Hilliard model based on a microforce balance

Let $\Omega \subset \mathbb{R}^3$ be the domain occupied by the material, which is supposed to be open, bounded and with a smooth boundary $\partial\Omega$, and let $[0,T] \subset \mathbb{R}^+$ be a finite time interval.

In order to derive the model, we consider the two fundamental laws of Thermodynamics:

- Balance of energy:

$$\frac{d}{dt}\int_{\mathcal{R}} e\,dx = -\int_{\partial\mathcal{R}} q\cdot\nu\,d\sigma + \int_{\mathcal{R}} r\,dx + \mathcal{W}(\mathcal{R}) + \mathcal{M}(\mathcal{R})\,, \tag{1.53}$$

  where $\mathcal{R} \subset \Omega$ is an arbitrary control volume, ν is the unit outer normal vector to $\partial\mathcal{R}$, e is internal energy density, q is the heat flux, r is the heat supply, $\mathcal{W}(\mathcal{R})$ is the rate of working of all forces exterior to $\mathcal{R}$ and $\mathcal{M}(\mathcal{R})$ is the rate at which free energy is added to $\mathcal{R}$ by mass transport. The rate of working of external forces and the mass transport are still defined as (1.21) and (1.22), so that (1.53) can be rewritten as

$$\frac{d}{dt}\int_{\mathcal{R}} e\,dx = -\int_{\partial\mathcal{R}} q\cdot\nu\,d\sigma + \int_{\mathcal{R}} r\,dx + \int_{\partial\mathcal{R}} (\zeta\cdot\nu)\frac{\partial u}{\partial t}\,d\sigma + \\ + \int_{\mathcal{R}} \gamma\frac{\partial u}{\partial t}dx - \int_{\partial\mathcal{R}} \mu j\cdot\nu\,d\sigma + \int_{\mathcal{R}} \mu h\,d\sigma\,.$$

Using Green's formula to treat the surface integrals, we obtain

$$\frac{d}{dt}\int_{\mathcal{R}} e\,dx = \int_{\mathcal{R}} \Big(-\operatorname{div} q + r + \frac{\partial u}{\partial t}\operatorname{div}\zeta + \zeta\cdot\nabla\frac{\partial u}{\partial t} + \\ + \gamma\frac{\partial u}{\partial t} - \mu\operatorname{div} j - j\cdot\nabla\mu + \mu h\Big)dx\,.$$

Using the microforce balance (1.23) and the mass balance (1.24), since the control volume $\mathcal{R}$ is arbitrary, we deduce

$$\frac{\partial e}{\partial t} = -\operatorname{div} q + r + (\mu-\pi)\frac{\partial u}{\partial t} + \zeta\cdot\nabla\frac{\partial u}{\partial t} - j\cdot\nabla\mu\,. \tag{1.54}$$

- Clausius-Duhem entropy production inequality:

$$\frac{d}{dt}\int_{\mathcal{R}} s dx \geq -\int_{\partial\mathcal{R}} \frac{q\cdot\nu}{\theta}\,d\sigma + \int_{\mathcal{R}} \frac{r}{\theta}\,dx\,, \tag{1.55}$$

where s is the entropy density. Noting that the control volume $\mathcal{R}$ is arbitrary, (1.55) yields

$$\frac{\partial s}{\partial t} \geq -\operatorname{div}\left(\frac{q}{\theta}\right) + \frac{r}{\theta}\,. \tag{1.56}$$

We now multiply (1.54) by $\frac{1}{\theta}$ to obtain

$$\frac{\partial}{\partial t}\Big(\frac{e}{\theta}\Big) - e\frac{\partial}{\partial t}\Big(\frac{1}{\theta}\Big) = -\operatorname{div}\left(\frac{q}{\theta}\right) + q\cdot\nabla\frac{1}{\theta} + \frac{r}{\theta} + \left(\frac{\mu}{\theta} - \frac{\pi}{\theta}\right)\frac{\partial u}{\partial t} + \frac{\zeta}{\theta}\cdot\nabla\frac{\partial u}{\partial t} - \frac{j}{\theta}\cdot\nabla\mu$$

Since we know from Thermodynamics that the Helmholtz free energy ψ is given by the Gibbs relation, i.e.,

$$\psi = e - \theta s,$$

using (1.56), we then deduce that

$$\frac{\partial}{\partial t}\Big(\frac{\psi}{\theta}\Big) - e\frac{\partial}{\partial t}\Big(\frac{1}{\theta}\Big) \leq q\cdot\nabla\frac{1}{\theta} + \left(\frac{\mu}{\theta} - \frac{\pi}{\theta}\right)\frac{\partial u}{\partial t} + \frac{\zeta}{\theta}\cdot\nabla\frac{\partial u}{\partial t} - \frac{j}{\theta}\cdot\nabla\mu\,. \tag{1.57}$$

Following Gurtin's approach and in view of the equations obtained by Alt and Pawłow, Miranville and Schimperna choose

$$Z = \Big(u, \nabla u, \frac{\mu}{\theta}, \nabla\frac{\mu}{\theta}, \frac{1}{\theta}, \nabla\frac{1}{\theta}\Big) \tag{1.58}$$

as independent constitutive variables. Note that, in order to obtain Alt and Pawłow's equations (1.42)-(1.45), we should consider $Z = \Big(u, \nabla u, \frac{1}{\theta}, \nabla\frac{1}{\theta}\Big)$, with $\frac{\mu}{\theta}$ given constitutively and somehow postulated, as done in (1.36), and we should assume $h = 0$ and $\gamma = 0$. A priori assuming that $\psi = \psi(Z)$, $e = e(Z)$, $j = j(Z)$, $\zeta = \zeta(Z)$ and $\pi = \pi(Z)$, we deduce from (1.57) the following dissipation inequality:

$$\Big(\partial_{\frac{1}{\theta}}\frac{\psi}{\theta} - e\Big)\frac{\partial}{\partial t}\Big(\frac{1}{\theta}\Big) - q\cdot\nabla\frac{1}{\theta} + \frac{1}{\theta}(\pi - \mu + \partial_u\psi)\frac{\partial u}{\partial t} + \frac{1}{\theta}(\partial_{\nabla u}\psi - \zeta)\cdot\frac{\partial\nabla u}{\partial t} + \\ + \frac{1}{\theta}\partial_{\frac{\mu}{\theta}}\psi\frac{\partial}{\partial t}\Big(\frac{\mu}{\theta}\Big) + \frac{1}{\theta}\partial_{\nabla\frac{\mu}{\theta}}\psi\cdot\frac{\partial}{\partial t}\Big(\nabla\frac{\mu}{\theta}\Big) + \frac{1}{\theta}\partial_{\nabla\frac{1}{\theta}}\psi\cdot\frac{\partial}{\partial t}\Big(\nabla\frac{1}{\theta}\Big) + \frac{j}{\theta}\cdot\nabla\mu \leq 0\,, \tag{1.59}$$

which has to hold for every field Z. Actually, it is possible to choose Z such that Z, $\frac{\partial u}{\partial t}$, $\frac{\partial \nabla u}{\partial t}$, $\frac{\partial \frac{\mu}{\theta}}{\partial t}$, $\frac{\partial \nabla \frac{\mu}{\theta}}{\partial t}$, $\frac{\partial \frac{1}{\theta}}{\partial t}$ and $\frac{\partial \nabla \frac{1}{\theta}}{\partial t}$ take arbitrary prescribed values at some chosen point x and time t. Therefore, since $\frac{\partial u}{\partial t}$, $\frac{\partial \nabla u}{\partial t}$, $\frac{\partial \frac{\mu}{\theta}}{\partial t}$, $\frac{\partial \nabla \frac{\mu}{\theta}}{\partial t}$, $\frac{\partial \frac{1}{\theta}}{\partial t}$ and $\frac{\partial \nabla \frac{1}{\theta}}{\partial t}$ appear linearly in (1.59), necessarily,

$$e = \partial_{\frac{1}{\theta}} \frac{\psi}{\theta}\,, \tag{1.60}$$

$$\pi - \mu + \partial_u \psi = 0\,, \tag{1.61}$$

$$\partial_{\nabla u} \psi - \zeta = 0\,, \tag{1.62}$$

$$\partial_{\frac{\mu}{\theta}} \psi = 0\,, \quad \partial_{\nabla \frac{\mu}{\theta}} \psi = 0\,, \quad \partial_{\nabla \frac{1}{\theta}} \psi = 0\,. \tag{1.63}$$

Indeed, otherwise, $\frac{\partial u}{\partial t}$, $\frac{\partial \nabla u}{\partial t}$, $\frac{\partial \frac{\mu}{\theta}}{\partial t}$, $\frac{\partial \nabla \frac{\mu}{\theta}}{\partial t}$, $\frac{\partial \frac{1}{\theta}}{\partial t}$ and $\frac{\partial \nabla \frac{1}{\theta}}{\partial t}$ could be chosen to violate (1.59). It follows that $\psi = \psi(u, \nabla u, \theta)$, and (1.59) reads

$$-\, q \cdot \nabla \frac{1}{\theta} + \frac{j}{\theta} \cdot \nabla \mu \leq 0 \tag{1.64}$$

for every field Z. Combining together (1.23), (1.61) and (1.62), we obtain

$$\mu = \partial_u \psi - \mathrm{div}(\partial_{\nabla u} \psi) - \gamma, \tag{1.65}$$

whence

$$\frac{\mu}{\theta} = \frac{1}{\theta} \partial_u \psi - \frac{1}{\theta} \mathrm{div}\,(\partial_{\nabla u} \psi) - \frac{\gamma}{\theta}.$$

Writing $\frac{j}{\theta} \cdot \nabla \mu = j \cdot \nabla \frac{\mu}{\theta} - \mu j \cdot \nabla \frac{1}{\theta}$, (1.64) becomes

$$-\,(q + \mu j) \cdot \nabla \frac{1}{\theta} + j \cdot \nabla \frac{\mu}{\theta} \leq 0 \tag{1.66}$$

for every field Z, which yields (see Appendix A.1)

$$j = -A \nabla \frac{\mu}{\theta} - B \nabla \frac{1}{\theta}\,, \tag{1.67}$$

$$q + \mu j = C \nabla \frac{\mu}{\theta} + D \nabla \frac{1}{\theta}\,, \tag{1.68}$$

where the matrices A, B, C and D depend on Z and are such that (1.66) is satisfied, i.e., A and D are, in some sense, positive semi-definite.
Using (1.61) and (1.62), we can rewrite the energy balance (1.54) as

$$\frac{\partial e}{\partial t} = -\,\mathrm{div}\, q + r + \partial_u \psi \frac{\partial u}{\partial t} + \partial_{\nabla u} \psi \cdot \nabla \frac{\partial u}{\partial t} - j \cdot \nabla \mu\,,$$

which, using (1.68), reads

$$\frac{\partial e}{\partial t} = -\,\mathrm{div}\left(C \nabla \frac{\mu}{\theta} + D \nabla \frac{1}{\theta} \right) + r + \partial_u \psi \frac{\partial u}{\partial t} + \partial_{\nabla u} \psi \cdot \nabla \frac{\partial u}{\partial t} + \mu\, \mathrm{div}\, j\,.$$

Using (1.24) and (1.65), we finally obtain the energy equation

$$\frac{\partial e}{\partial t} = -\,\mathrm{div}\left(C \nabla \frac{\mu}{\theta} + D \nabla \frac{1}{\theta} - \frac{\partial u}{\partial t} \partial_{\nabla u} \psi \right) + r + \gamma \frac{\partial u}{\partial t} + h(\partial_u \psi - \mathrm{div}(\partial_{\nabla u} \psi) - \gamma). \tag{1.69}$$

Combining together (1.24), (1.67), (1.69), (1.65) and (1.60), one obtains the following non-isothermal generalized Cahn-Hilliard system:

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(A\nabla\frac{\mu}{\theta} + B\nabla\frac{1}{\theta}\right) + h, \tag{1.70}$$

$$\frac{\partial e}{\partial t} = -\,\mathrm{div}\left(C\nabla\frac{\mu}{\theta} + D\nabla\frac{1}{\theta} - \frac{\partial u}{\partial t}\partial_{\nabla u}\psi\right) + r + \gamma\frac{\partial u}{\partial t} + h(\partial_u\psi - \mathrm{div}\,(\partial_{\nabla u}\psi) - \gamma), \tag{1.71}$$

$$\mu = \partial_u\psi - \mathrm{div}\,(\partial_{\nabla u}\psi) - \gamma, \tag{1.72}$$

$$e = \partial_{\frac{1}{\theta}}\frac{\psi}{\theta} = \psi - \theta\partial_\theta\psi, \tag{1.73}$$

in $\Omega \times (0,T)$.
At last, assuming null external microforces and null external mass supply, i.e., $\gamma = 0$ and $h = 0$, we obtain the following non-isothermal Cahn-Hilliard system:

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(A\nabla\frac{\mu}{\theta} + B\nabla\frac{1}{\theta}\right), \tag{1.74}$$

$$\frac{\partial e}{\partial t} = -\,\mathrm{div}\left(C\nabla\frac{\mu}{\theta} + D\nabla\frac{1}{\theta} - \frac{\partial u}{\partial t}\partial_{\nabla u}\psi\right) + r, \tag{1.75}$$

$$\mu = \partial_u\psi - \mathrm{div}\,(\partial_{\nabla u}\psi)\,, \tag{1.76}$$

$$e = \partial_{\frac{1}{\theta}}\frac{\psi}{\theta} = \psi - \theta\partial_\theta\psi, \tag{1.77}$$

in $\Omega \times (0,T)$.

Miranville and Schimperna's model as an extension of Alt and Pawlow's one

Note that the main difference with respect to Alt and Pawłow's system of equations is given by the presence of the term $-\frac{\partial u}{\partial t}\partial_{\nabla u}\psi$ in (1.75). As for the chemical potential, observe that, following Gurtin's approach, in (1.72) we recover the same expression in terms of the free energy density as in the isothermal case (see (1.31)). On the contrary, Alt and Pawłow define the reduced chemical potential as the variational derivative of the rescaled free energy with respect to the order parameter u (see (1.36)), obtaining (1.37), i.e. $\frac{\mu}{\theta} = \partial_u\frac{\psi}{\theta} - \mathrm{div}(\partial_{\nabla u}\frac{\psi}{\theta})$. Hence, Alt and Pawłow's model seems to have a variational structure, at least with respect to u. However, there is no reason why the free energy should obey a variational principle in the non-isothermal setting (see [53]). On the other hand, since Alt and Pawłow assume that the mass and heat fluxes depend linearly on $\nabla\frac{\mu}{\theta}$ and on $\nabla\frac{1}{\theta}$, their theory can be interpreted as a thermodynamic theory near constant equilibria $\frac{\mu}{\theta}$ =constant and $\frac{1}{\theta}$= constant. On the contrary, following Gurtin's approach, expressions for mass and heat fluxes are found a posteriori, without providing any a priori constitutive equation. Hence, his approach is suitable for the description of systems which are far from equilibrium. Furthermore, because the coefficients l_{ij} in (1.42)-(1.43) do not depend on the gradients, Alt and Pawłow's analysis is essentially restricted to the case of isotropic materials. On the contrary, Miranville and Schimperna's model provides matrices whose coefficients may depend on Z (see (1.58)), hence it applies to the case of anisotropic materials. Lastly, observe that, as for Alt and Pawłow's system of equations, there is a direct correspondence between that deduced by Miranville and Schimperna and the standard Cahn-Hilliard equation. Indeed, assuming $\alpha =$ constant, $A = m\theta I$, $m > 0$, I identity matrix, $r = 0$ and the Neumann boundary conditions (1.13) and (1.14), if θ is constant, the system of equations (1.74)-(1.77) reduce to (1.3).

1.2.3 Ginzburg-Landau free energy

In the previous subsections 1.2.1 and 1.2.2 we presented non-isothermal Cahn-Hilliard models proposed by Alt and Pawlow and by Miranville and Schimperna based on Gurtin's approach, respectively. Note that no explicit expression for the Helmholtz free energy density ψ were given. However, the choice of ψ plays a crucial role in determining the explicit expression of the system of equations (1.42)-(1.45) and that of (1.74)-(1.77). In this section, we give some examples of the free energy ψ in the form of the Ginzburg-Landau one, in part proposed by Falk in [23] and by Alt and Pawlow in [3, 4], that match some structural assumptions in order to be physically reasonable.

A general form of the Ginzburg-Landau free energy density ψ is given by

$$\psi(u, \nabla u, \theta) = \frac{\alpha}{2}|\nabla u|^2 + F(u, \theta)\,, \quad \alpha > 0\,, \tag{1.78}$$

where the first term is the so-called inhomogeneous (or gradient) part, while $F(u, \theta)$ is the homogeneous (or volumetric) part, which is given by

$$F(u, \theta) = c_V F_1(\theta) + F_2(\theta, u), \quad c_V > 0\,. \tag{1.79}$$

The first term in (1.79) represents the main concave term of the free energy and it refers to pure heat conduction and it is linked to the specific heat $C_V(\theta) = Q'(\theta)$ by the relation $Q(\theta) = c_V F_1(\theta) - c_V \theta F_1'(\theta)$, where $'$ denotes the derivative with respect to θ. In particular, it can be postulated in one of the following forms:

$$F_1(\theta) = \delta_1\theta + \delta_2 - \theta\log\theta\,, \tag{1.80}$$

$$F_1(\theta) = -\delta_3\theta^2\,, \tag{1.81}$$

$$F_1(\theta) = -\theta\log(\theta + 1)\,. \tag{1.82}$$

where $\delta_1, \delta_2, \delta_3 \geq 0$ are constants.
As for the second term in (1.79), it represents a potential associated with the phase separation process. Indeed, it is in charge of changes in the qualitative behaviour of F below a critical temperature $\theta_c > 0$ from a convex into a non-convex double-well potential in order to assign lower energy values to the pure states, and it determines the behaviour of F for large value of $|u|$. A thermodynamically relevant potential F_2 is the logarithmic one given by (1.4), i.e.,

$$F_2(u, \theta) = -\theta_c u^2 + \theta\left[(1-u)\log(1-u) + (1+u)\log(1+u)\right], \quad u \in (-1, 1)\,.$$

Many polynomial potentials F_2 for $u \in (-\infty, +\infty)$ have been considered in the literature. Such forms arise by expanding the logarithmic free energy density in u at the mean value u_m for θ near the critical temperature θ_c. However, for various solid solutions, for instance, metallic alloys, fourth or sixth order polynomials in u, such as

$$F_2(u,\theta) = \gamma_1(\theta - \theta_c)u^2 + \gamma_2 u^4 + \gamma_3 u^6 \,, \tag{1.83}$$

where $\gamma_1 > 0, \gamma_2, \gamma_3 \geq 0, \gamma_2 + \gamma_3 > 0$, are phenomenologically justified choices for F_2 (see [47, 48, 51]).

According to the above considerations, we present some possible expressions for F. In our setting, a possible choice for the homogeneous part F is then the logarithmic one

$$F_{\log}(u,\theta) = -\frac{c_V}{2}\theta^2 - \theta_c u^2 + \theta\left[(1-u)\log(1-u) + (1+u)\log(1+u)\right], \tag{1.84}$$

or the the polynomial one

$$F_{\text{pol}}(u,\theta) = -\frac{c_V}{2}\theta^2 + (\theta - \theta_c)u^2 + u^4, \tag{1.85}$$

where $c_V, \theta_c > 0$. Observe that for such choices of F there is a change of the qualitative behaviour for θ crossing the critical temperature θ_c, i.e., for $\theta > \theta_c$ both $F_{\log}$ and F_{pol} are convex in u, while for $\theta < \theta_c$ they turn out to be non-convex (see Figures 1.3 and 1.4).

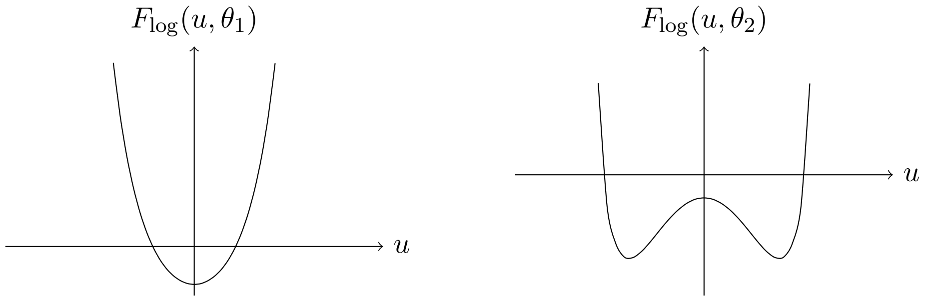

Figure 1.3: convex $F_{\log}$ for $\theta_1 > \theta_c$ and non-convex double-well $F_{\log}$ for $\theta_2 < \theta_c$.

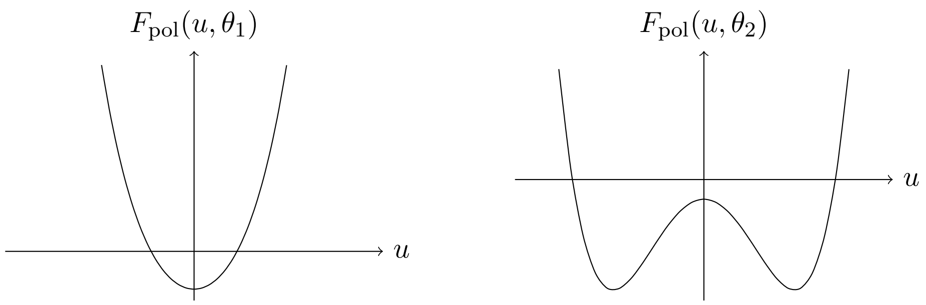

Figure 1.4: convex F_{pol} for $\theta_1 > \theta_c$ and non-convex double-well F_{pol} for $\theta_2 < \theta_c$.

We observe that the coupling term $\gamma_1 \theta u^2$ in (1.83) is related to the latent heat of the phase transition. Indeed, the latent heat is given by the difference in energy at fixed temperature between two pure phases (represented by the two local minima of the homogeneous part F), i.e.,

$$\lambda(u_1, u_2) \equiv \theta(s(u_1) - s(u_2)) .$$

Hence, for (1.83), recalling that $s = -\partial_\theta \psi$, we obtain

$$\lambda(u_1, u_2) = -\gamma_1 \theta u_1^2 + \gamma_1 \theta u_2^2 = 0 ,$$

which means that no latent heat is involved in the process. However, in several concrete physical situations of phase transitions (see [53] for more details), the latent heat is not null. Hence, a more general term of the form $\gamma_1 \theta(\lambda_2 u^2 - \lambda_1 u + \lambda_0)$, $\lambda_2, \lambda_1, \lambda_0 \geq 0$ can be considered. Indeed, in this case, for u_1, u_2 local minima of F at fixed $\theta < \theta_c$,

$$\lambda(u_1, u_2) = -\gamma_1 \lambda_2 \theta u_1^2 + \gamma_1 \lambda_1 \theta u_1 + \gamma_1 \lambda_2 \theta u_2^2 - \gamma_1 \lambda_1 \theta u_2 \neq 0 .$$

At last, notice that some articles have been devoted to the case where $\lambda_2 \equiv 0$, so that the coupling term is linear in u, i.e. $\gamma_1 \theta(-\lambda_1 u + \lambda_0)$, $\lambda_1, \lambda_0 \geq 0$ (see for example [18]). The reason is that, from the mathematical point of view, assuming such a linear coupling term, thus linear latent heat, seems preferable as one looks for the well-posedness of the model. In the following, we will show how the system of equations (1.74)-(1.77) changes and becomes more complicated to study by assuming a quadratic coupling term instead of a linear one.

Non-isothermal Cahn-Hilliard models for a specific choice of the Ginzburg-Landau free energy

We already observed that Miranville and Schimperna's model can be considered as an extension of Alt and Pawłow's one to systems which are far from equilibrium and to anisotropic materials. Furthermore, we noted that the main difference between Miranville and Schimperna's system of equations (1.74)-(1.77) and Alt and Pawłow's one (1.42)-(1.45) is given by the presence of the term $-\frac{\partial u}{\partial t} \partial_{\nabla u} \psi$ in (1.75). We now compute them for two specific choices for the Ginzburg-Landau free energy density ψ.

Firstly, let us assume the homogeneous part F of the Ginzburg-Landau free energy density (1.78) to be polynomial as in (1.85); i.e.,

$$\psi(u, \nabla u, \theta) = \frac{\alpha}{2} |\nabla u|^2 - \frac{c_V}{2} \theta^2 + \lambda(\theta - \theta_c) u^2 + u^4 ,$$

where $\alpha, \lambda, \theta_c > 0$, which can be rewritten as

$$\psi(u, \nabla u, \theta) = \frac{\alpha}{2} |\nabla u|^2 - Q(\theta) + \lambda \theta u^2 + F(u), \tag{1.86}$$

where $Q(\theta) = \frac{c_V}{2}\theta^2$, $F(u) = u^4 - \lambda\theta_c u^2$. For such a choice of the free energy Alt and Pawłow's system of equations (1.42)-(1.45) reduce to

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(l_{11}\nabla\frac{\mu}{\theta} - l_{12}\nabla\frac{1}{\theta}\right), \tag{1.87}$$

$$\frac{\partial e}{\partial t} + \operatorname{div}\left(l_{22}\nabla\frac{1}{\theta} - l_{21}\nabla\frac{\mu}{\theta}\right) = r\,, \tag{1.88}$$

$$\frac{\mu}{\theta} = -\operatorname{div}\left(\frac{\alpha\nabla u}{\theta}\right) + 2\lambda u + \frac{f(u)}{\theta}\,, \tag{1.89}$$

$$e = \frac{\alpha}{2}|\nabla u|^2 + Q(\theta) + F(u)\,, \tag{1.90}$$

in $\Omega\times(0,\,T)$, where f is the derivative of F, i.e. $f(u) = 4u^3 - \lambda\theta_c u$, and the matrix $(l_{i,j})_{i,j=1,2} = (l_{i,j}(u,\,\frac{\mu}{\theta},\,\frac{1}{\theta}))_{i,j=1,2}$ is positive definite with positive diagonal elements. On the other hand, using (1.86), Miranville and Schimperna's system of equations (1.74)-(1.77) become

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(A\nabla\frac{\mu}{\theta} + B\nabla\frac{1}{\theta}\right), \tag{1.91}$$

$$\frac{\partial e}{\partial t} + \operatorname{div}\left(C\nabla\frac{\mu}{\theta} + D\nabla\frac{1}{\theta} - \alpha\frac{\partial u}{\partial t}\nabla u\right) = r, \tag{1.92}$$

$$\mu = -\alpha\Delta u + 2\lambda\theta u + f(u), \tag{1.93}$$

$$e = \frac{\alpha}{2}|\nabla u|^2 + Q(\theta) + F(u), \tag{1.94}$$

in $\Omega\times(0,\,T)$, where the matrices $A,\ B,\ C,\ D$ depend on the variables $(u,\,\nabla u,\,\frac{\mu}{\theta},\,\nabla\frac{\mu}{\theta},\,\frac{1}{\theta},\,\nabla\frac{1}{\theta})$ and, in order to satisfy (1.66), $A,\ D$ are, in some sense, positive semi-definite. Observe that the explicit expressions for the internal energy density (1.90) and (1.94) coincide. As for the expression for the chemical potential, Alt and Pawłow's system of equations and Miranville and Schimperna's one differ (see (1.89) and (1.93)). Indeed, as already observed, Alt and Pawłow's chemical potential is a priori assumed as constitutive equation, whereas Miranville and Schimperna's one is a posteriori deduced from the two fundamental laws of Thermodynamics. Regarding the differential equations, note that the coefficients $l_{i,j}$, $i,j = 1,2$, in (1.87)-(1.88) are replaced in (1.91)-(1.92) by matrices A, B, C, D, which satisfy equivalent properties extended to matrices. Furthermore, as already stated, the main difference between (1.88) and (1.92) is the presence of the term $-\alpha\frac{\partial u}{\partial t}\nabla u$ in (1.92), which plays a crucial role in rewriting the system (1.91)-(1.94) in an equivalent form, as we see in the following. Indeed, once given (1.94), computing $\frac{\partial e}{\partial t}$, we obtain

$$\frac{\partial e}{\partial t} = \alpha\nabla u\cdot\frac{\partial\nabla u}{\partial t} + \frac{\partial Q(\theta)}{\partial t} + f(u)\frac{\partial u}{\partial t}\,,$$

and, inserting it into (1.92), it yields

$$\alpha\nabla u\cdot\frac{\partial\nabla u}{\partial t} + \frac{\partial Q(\theta)}{\partial t} + f(u)\frac{\partial u}{\partial t} + \operatorname{div}\left(C\nabla\frac{\mu}{\theta} + D\nabla\frac{1}{\theta}\right) - \alpha\frac{\partial\nabla u}{\partial t}\cdot\nabla u - \alpha\frac{\partial u}{\partial t}\Delta u = r\,,$$

whence

$$\frac{\partial Q(\theta)}{\partial t} + f(u)\frac{\partial u}{\partial t} + \operatorname{div}\left(C\nabla\frac{\mu}{\theta} + D\nabla\frac{1}{\theta}\right) - \alpha\frac{\partial u}{\partial t}\Delta u = r\,.$$

Finally, using (1.93), we obtain

$$\frac{\partial Q(\theta)}{\partial t} + \frac{\partial u}{\partial t}(\mu - 2\lambda\theta u) + \operatorname{div}\Big(C\nabla\frac{\mu}{\theta} + D\nabla\frac{1}{\theta}\Big) = r\,.$$

Hence, system of equations (1.91)-(1.94) is equivalent to the following

$$\frac{\partial u}{\partial t} = \operatorname{div}\Big(A\nabla\frac{\mu}{\theta} + B\nabla\frac{1}{\theta}\Big)\,, \tag{1.95}$$

$$\mu = -\alpha\Delta u + 2\lambda\theta u + f(u)\,, \tag{1.96}$$

$$\frac{\partial Q(\theta)}{\partial t} + \frac{\partial u}{\partial t}(\mu - 2\lambda\theta u) + \operatorname{div}\Big(C\nabla\frac{\mu}{\theta} + D\nabla\frac{1}{\theta}\Big) = r\,, \tag{1.97}$$

in $\Omega \times (0, T)$. The mathematical analysis of models of the form (1.87)-(1.90) (or more generally of (1.42)-(1.45)) can be found, e.g., in [2] and [62]. The mathematical analysis of (1.91)-(1.94), or equivalently of (1.95)-(1.96) (or more generally of (1.74)-(1.77)), seems however much more involved. To illustrate this, let us consider for instance the case where $A = D = I$ (the identity matrix), $B = C = 0$ and $r = 0$. The system of equations (1.95)-(1.96) reduce to

$$\frac{\partial u}{\partial t} = \Delta\frac{\mu}{\theta}\,, \tag{1.98}$$

$$\mu = -\alpha\Delta u + 2\lambda\theta u + f(u)\,, \tag{1.99}$$

$$\frac{\partial Q(\theta)}{\partial t} + \frac{\partial u}{\partial t}(\mu - 2\lambda\theta u) + \Delta\frac{1}{\theta} = 0\,, \tag{1.100}$$

hence setting $\chi = \frac{\mu}{\theta}$ and inserting (1.98) into (1.100), we obtain

$$\frac{\partial u}{\partial t} = \Delta\chi\,, \tag{1.101}$$

$$\chi\theta = -\alpha\Delta u + 2\lambda\theta u + f(u)\,, \tag{1.102}$$

$$\frac{\partial Q(\theta)}{\partial t} + \theta\Delta\chi(\chi - 2\lambda u) + \Delta\frac{1}{\theta} = 0\,, \tag{1.103}$$

in $\Omega \times (0, T)$. Here, the problem is that we do not know how to treat the term $\theta\Delta\chi(\chi - 2\lambda u)$. However, we note that the conservations of mass and energy and the balance of entropy follow. Indeed, once assumed proper Neumann boundary conditions, integrating (formally) (1.101) over the domain Ω occupied by the material, we obtain the conservation of the spatial average of the order parameter; i.e.,

$$\langle u(t)\rangle = \langle u(0)\rangle = u_m\,, \quad \forall t \in [0, T]\,. \tag{1.104}$$

Multiplying (1.101) by $\chi\theta$ and (1.102) by $\frac{\partial u}{\partial t}$, integrating (formally) over Ω, then taking the difference, it yields

$$\frac{d}{dt}\int_\Omega \Big[\frac{\alpha}{2}|\nabla u|^2 + F(u)\Big]dx - \int_\Omega \lambda\frac{\partial u}{\partial t}\theta u\,dx - \int_\Omega \chi\theta\,\Delta\chi = 0\,. \tag{1.105}$$

Integrating (1.103) over Ω and summing it to (1.105), using (1.101), we obtain the conservation of energy

$$\frac{d}{dt}\int_\Omega e\,dx = \frac{d}{dt}\int_\Omega \Big[\frac{\alpha}{2}|\nabla u|^2 + F(u) + Q(\theta)\Big]dx = 0\,. \tag{1.106}$$

Multiplying (1.103) by $\frac{1}{\theta}$ and using the chain rule, we reduce to

$$\frac{\partial \Lambda(\theta)}{\partial t} + \Delta\frac{\chi^2}{2} - 2\lambda\Delta\chi u + \Delta\frac{1}{\theta} = |\nabla\chi|^2 + k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2,$$

where $\Lambda(\theta) = c_V\theta$, whence, using (1.101) and integrating (formally) it over Ω, the balance of entropy follows

$$\frac{d}{dt}\int_\Omega (\Lambda(\theta) - \lambda u^2)\,dx = \int_\Omega |\nabla\chi|^2\,dx + \int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\,dx\,. \tag{1.107}$$

As already observed, several articles have been devoted to the case where the general coupling term $\gamma_1\theta(\lambda_2 u^2 - \lambda_1 u + \lambda_0)$ is assumed to be linear in u, i.e. $\lambda_2 \equiv 0$. The reason is that, from the mathematical point of view, assuming such a linear coupling term seems preferable as one looks for the well-posedness of the model. For instance, let us assume a Ginzburg-Landau free energy density (1.78) in the following form

$$\psi(u, \nabla u, \theta) = \frac{\alpha}{2}|\nabla u|^2 - Q(\theta) - \lambda\theta u + F(u)\,, \tag{1.108}$$

where $\alpha, \lambda > 0$, $Q(\theta) = \frac{c_V}{2}\theta^2$, $c_V > 0$, and F is possibly a non-convex double-well polynomial potential. For such a choice of the free energy Alt and Pawłow's system of equations (1.42)-(1.45) reduce to

$$\frac{\partial u}{\partial t} = \operatorname{div}\Big(l_{11}\nabla\frac{\mu}{\theta} - l_{12}\nabla\frac{1}{\theta}\Big)\,, \tag{1.109}$$

$$\frac{\partial e}{\partial t} + \operatorname{div}\Big(l_{22}\nabla\frac{1}{\theta} - l_{21}\nabla\frac{\mu}{\theta}\Big) = r\,, \tag{1.110}$$

$$\frac{\mu}{\theta} = -\operatorname{div}\Big(\frac{\alpha\nabla u}{\theta}\Big) - \lambda + \frac{f(u)}{\theta}\,, \tag{1.111}$$

$$e = \frac{\alpha}{2}|\nabla u|^2 + Q(\theta) + F(u)\,, \tag{1.112}$$

in $\Omega \times (0, T)$, where the matrix $(l_{i,j})_{i,j=1,2} = (l_{i,j}(u, \frac{\mu}{\theta}, \frac{1}{\theta}))_{i,j=1,2}$ is positive definite with positive diagonal elements, while f is the derivative of F. On the other hand, using (1.108), (1.74)-(1.77) become

$$\frac{\partial u}{\partial t} = \operatorname{div}\Big(A\nabla\frac{\mu}{\theta} + B\nabla\frac{1}{\theta}\Big), \tag{1.113}$$

$$\frac{\partial e}{\partial t} + \operatorname{div}\Big(C\nabla\frac{\mu}{\theta} + D\nabla\frac{1}{\theta} - \alpha\frac{\partial u}{\partial t}\nabla u\Big) = r, \tag{1.114}$$

$$\mu = -\alpha\Delta u - \lambda\theta + f(u), \tag{1.115}$$

$$e = \frac{\alpha}{2}|\nabla u|^2 + Q(\theta) + F(u), \tag{1.116}$$

in $\Omega \times (0, T)$, where the matrices A, B, C, D depend on the variables $(u, \nabla u, \frac{\mu}{\theta}, \nabla\frac{\mu}{\theta}, \frac{1}{\theta}, \nabla\frac{1}{\theta})$ and, in order to satisfy (1.66), A, D are, in some sense, positive semi-definite.

As already noticed, the explicit expressions for the internal energy density (1.112) and (1.116) coincide. On the contrary, for the expression for the chemical potential, Alt and Pawłow's system of equations and Miranville and Schimperna's one differ (see (1.111) and (1.115)). Furthermore, as already stated, the main difference between (1.110) and (1.114) is the presence of the term $-\alpha\frac{\partial u}{\partial t}\nabla u$ in (1.114), which plays a crucial role in rewriting the system (1.113)-(1.116) in an equivalent form. Proceeding as done in order to deduce (1.95)-(1.96), we can conclude that the system of equations (1.113)-(1.116) is equivalent to the following

$$\frac{\partial u}{\partial t} = \operatorname{div}\Big(A\nabla\frac{\mu}{\theta} + B\nabla\frac{1}{\theta}\Big), \tag{1.117}$$

$$\mu = -\alpha\Delta u - \lambda\theta + f(u)\,, \tag{1.118}$$

$$\frac{\partial Q(\theta)}{\partial t} + \frac{\partial u}{\partial t}(\mu + \lambda\theta) + \operatorname{div}\Big(C\nabla\frac{\mu}{\theta} + D\nabla\frac{1}{\theta}\Big) = r\,, \tag{1.119}$$

in $\Omega \times (0, T)$. The mathematical analysis of models of the form (1.109)-(1.112) (or more generally of (1.42)-(1.45)) can still be found in [2] and [62]. As for that of (1.113)-(1.116), or equivalently (1.117)-(1.118), it seems to be more involved. However, let us assume $r = 0$, $B = C = 0$, $A = mI$, $m > 0$, $D = k(\theta)I$, where k is a suitable function of θ and I is the identity matrix. The system of equations (1.117)-(1.118) then reduce to

$$\frac{\partial u}{\partial t} = m\Delta\frac{\mu}{\theta}\,, \tag{1.120}$$

$$\mu = -\alpha\Delta u - \lambda\theta + f(u)\,, \tag{1.121}$$

$$\frac{\partial Q(\theta)}{\partial t} + \frac{\partial u}{\partial t}(\mu + \lambda\theta) + \operatorname{div}\Big(k(\theta)\nabla\frac{1}{\theta}\Big) = 0\,, \tag{1.122}$$

in $\Omega \times (0, T)$. Setting $\chi = \frac{\mu}{\theta}$ and inserting (1.120) into (1.122), we obtain

$$\frac{\partial u}{\partial t} = m\Delta\chi\,, \tag{1.123}$$

$$\chi\theta = -\alpha\Delta u - \lambda\theta + f(u)\,, \tag{1.124}$$

$$\frac{\partial Q(\theta)}{\partial t} + m\theta\Delta\chi(\chi + \lambda) + \operatorname{div}\Big(k(\theta)\nabla\frac{1}{\theta}\Big) = 0\,, \tag{1.125}$$

in $\Omega \times (0, T)$. Recalling that the mass and the heat fluxes are given by (1.67) and (1.68), we deduce

$$j = -m\nabla\chi, \quad q = m\theta\chi\nabla\chi + k(\theta)\nabla\frac{1}{\theta}\,.$$

Observe that k is related to the heat conductivity and that, in order to recover Fourier's law, we should assume

$$k(\theta) = k_2\theta^2, \quad k_2 > 0\,. \tag{1.126}$$

However, several papers have been devoted to the case where

$$k(\theta) \equiv k_0\,, \tag{1.127}$$

where k_0 is a positive constant (see e.g. [34,36,65]), or more generally

$$k(\theta) = k_1\theta^\delta, \quad \delta \in [0,1)\,, \tag{1.128}$$

where k_1 is a positive constant (see e.g. [37]) Indeed, the law (1.127) (also (1.128)) turns out to be satisfactory for low and intermediate temperatures and offers some advantages from the mathematical point of view, but it does not look acceptable for high temperatures because it does not provide any coerciveness as θ becomes larger and larger. These considerations suggest to combine (1.127) with (1.126), obtaining

$$k(\theta) = k_0 + k_2\theta^2\,, \quad k_0, k_2 > 0\,. \tag{1.129}$$

The meaning of the above expression is that of describing a diffusion law which is singular as the temperature θ approaches zero, while it has approximately a linear structure for large values of θ, as in the classical Fourier relation. Concerning this case, more generally, in [14,27], k is assumed in the following form

$$k(\theta) = \kappa'(\theta)\theta^2\,, \tag{1.130}$$

where κ' is the derivative of a function κ which satisfies the following properties: $\kappa(1) = 0$, κ is strictly increasing and such that $\kappa(\theta) \searrow -\infty$ as $\theta \searrow 0$, $\kappa(\theta) \nearrow +\infty$ as $\theta \nearrow +\infty$.

Concerning the system of equations (1.123)-(1.124), in the following chapter we will be able to conclude about the existence of the so-called entropy solutions (see Def. 2.2.1) assuming

$$k(\theta) = k_0 + k_1\theta^\beta, \quad \beta \in [0,2)\,, \tag{1.131}$$

where $k_0, k_1 > 0$, and also about that of weak solutions (see Def. 2.2.2) for $\beta \in (\frac{5}{3}, 2)$.

Chapter 2

Entropy and weak solutions to a non-isothermal Cahn-Hilliard model

The aim of this chapter is to prove the existence of the so-called entropy solutions to the non-isothermal Cahn-Hilliard model given by equations (1.123)-(1.124). Furthermore, in a subcase, we will also be able to conclude about the existence of weak solutions. Once suitable assumptions are made, we will prove some formal a priori estimates holding for hypothetical solutions to the strong formulation of the model, or more precisely, to a proper regularization or approximation of it. Then, we will show by compactness arguments that at least a subsequence converges in a suitable way to an entropy solution and, in a subcase, also to a weak solution to our problem. At last, we will propose a possible approximation of the "strong" system that one could try to develop.

2.1 Setting of the problem

Let Ω be the domain occupied by the material, which we assume to be a bounded, open subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$, and let $T > 0$ be a given final time, which may be arbitrarily large. Then, let (x, t) be an arbitrary point belonging to $\Omega \times (0, T)$. Let us denote by ∇ and by Δ the spatial gradient and Laplacian, respectively. Moreover, for $j \in \mathbb{N}$, let D^j stand for the j^{th} spatial derivative. As for the time variable, from now on, we will denote by $(\,\cdot\,)_t$ the partial derivative with respect to t.

Consider the *strong formulation* of the non-isothermal Cahn-Hilliard model given by the system equations (1.123)-(1.124), which reads

$$u_t = m\Delta\chi\,, \tag{2.1}$$

$$\chi\theta = -\alpha\Delta u - \lambda\theta + f(u)\,, \tag{2.2}$$

$$(Q(\theta))_t + m\theta\Delta\chi(\chi+\lambda) + \operatorname{div}\Big(k(\theta)\nabla\frac{1}{\theta}\Big) = 0\,, \tag{2.3}$$

in $\Omega \times (0, T)$. Here, $u, \theta : \Omega \times (0, T) \to \mathbb{R}$ represent the so-called order parameter and the (absolute) temperature of the system, while $\chi : \Omega \times (0, T) \to \mathbb{R}$ is an auxiliary variable defined through equation (2.2) which helps particularly for the statement of the weak formulation of the model (see Def.2.2.1). As introduced in the previous chapter, χ stands for the rescaled chemical potential $\frac{\mu}{\theta}$. m, α and λ are positive constants related to the mobility, the thickness of the interface and the latent heat, respectively. Moreover, $f : \mathbb{R} \to \mathbb{R}$ is the derivative of $F : \mathbb{R} \to \mathbb{R}$, a non-convex double-well polynomial potential, while $k : \mathbb{R}^+ \to \mathbb{R}^+$, as a function of θ, is related to the heat conductivity. The expressions for F and k will be specified in Section 2.2. Lastly, let $Q : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $Q(\theta) = \frac{c_V}{2}\theta^2$, where $c_V > 0$.

Boundary and initial conditions

In order to get a well-posed problem, we have to specify suitable initial and boundary conditions. Consistently with the physical derivation presented in the previous chapter, we will essentially assume that the system is insulated from the exterior. This leads to taking the following no mass flux (through the boundary) condition:

$$\nabla\chi \cdot \nu = 0 \ \text{ on } \partial\Omega \,, \tag{2.4}$$

where ν denotes the unit outer normal to the boundary $\partial\Omega$. As already observed in Subsection 1.1.1, thanks to (2.4), integrating (2.1) in space and time, we obtain the mass conservation:

$$\langle u(t)\rangle \equiv \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega u(x,t)dx = \langle u(0)\rangle, \quad \forall t \in [0, T] \,, \tag{2.5}$$

which is a characteristic feature of Cahn-Hilliard-type models. Next, we assume that

$$\nabla u \cdot \nu = 0 \ \text{ on } \partial\Omega \,. \tag{2.6}$$

This condition prescribes that the diffuse interface is orthogonal to the boundary of the domain. Moreover, we take no-flux boundary conditions for the temperature:

$$k(\theta)\nabla\frac{1}{\theta} \cdot \nu = 0 \ \text{ on } \partial\Omega \,. \tag{2.7}$$

Finally, the system is complemented by the initial conditions

$$u(\cdot, 0) = u_0 \,, \quad \theta(\cdot, 0) = \theta_0 \,.$$

Balance laws

We already observed that the boundary condition (2.4) leads to the conservation of mass (2.5). Another conservation law follows from the system of equations (2.1)-(2.3) endowed with the Neumann boundary conditions (2.4)-(2.7): the conservation of internal energy.
Multiplying (2.1) by $\chi\theta$ and (2.2) by u_t, then taking the difference, we obtain

$$\left[\frac{\alpha}{2}|\nabla u|^2 + F(u)\right]_t - \lambda u_t \theta - m\chi\theta\Delta\chi = 0 \,. \tag{2.8}$$

Summing (2.3) to (2.8), then using (2.1), the balance of internal energy equation follows

$$\Big[\frac{\alpha}{2}|\nabla u|^2 + Q(\theta) + F(u)\Big]_t + \operatorname{div}\Big(k(\theta)\nabla\frac{1}{\theta}\Big) = 0\,. \tag{2.9}$$

Indeed, as provided by the original formulation of the problem (1.113)-(1.116), the internal energy density is given by

$$e = \frac{\alpha}{2}|\nabla u|^2 + Q(\theta) + F(u)\,.$$

Hence, integrating (2.9) over Ω and using the boundary condition (2.7), we recover the conservation of energy

$$\frac{d}{dt}\int_\Omega e\,dx = \frac{d}{dt}\int_\Omega \Big[\frac{1}{2}|\nabla u|^2 + F(u) + Q(\theta)\Big]dx = 0\,. \tag{2.10}$$

A key point in the statement of the entropy formulation of our model (see Def. 2.2.1) consists in replacing the "heat" equation (2.3) with the balance of entropy. This relation is, indeed, mathematically more tractable, but it keeps all the main features of the problem.
Multiplying (2.3) by $\frac{1}{\theta}$ and using the chain rule, we obtain an equivalent formulation of it, which is given by

$$(\Lambda(\theta))_t + m\Delta\Big(\frac{\chi^2}{2} + \lambda\chi\Big) + \operatorname{div}\Big(\frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\Big) = m|\nabla\chi|^2 + k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2, \tag{2.11}$$

where $\Lambda(\theta) = c_V\theta$, whence, using (2.1), we deduce the balance of entropy equation

$$(\Lambda(\theta) + \lambda u)_t + m\Delta\Big(\frac{\chi^2}{2}\Big) + \operatorname{div}\Big(\frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\Big) = m|\nabla\chi|^2 + k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2. \tag{2.12}$$

Indeed, being the Helmholtz free energy ψ given by (1.108), the entropy density is then $s = -\partial_\theta\psi = \Lambda(\theta)\ + \lambda u$. Integrating (2.12) over Ω and using the boundary conditions (2.4) and (2.7), the integral form of the balance of entropy follows

$$\frac{d}{dt}\int_\Omega (\Lambda(\theta) + \lambda u)\,dx = \int_\Omega m|\nabla\chi|^2\,dx + \int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\,dx\,. \tag{2.13}$$

Remark 2.1.1. *It is worth noting that (2.11), or equivalently (2.12), is an equality at this level, but it will turn to an inequality in the framework of the rigorous definition of entropy solution that will be introduced later on (see Def. 2.2.1). Of course, this phenomenon is due to the quadratic terms on the right hand side, which do not behave well with respect to weak limits. However, in case we could prove that there exists a smooth solution to the entropy formulation of the model, then for that solution it would be possible to recover the entropy equality (2.11). Furthermore, (2.11) is equivalent to (2.3) in that setting. In this sense, the entropy formulation presented in the next section turns out to be compatible with the strong formulation (2.1)-(2.3) given at the beginning, at least when sufficient regularity holds.*

2.2 Main results

Before presenting the main results of the thesis in the form of rigorous statements, we list the hypotheses imposed on the constitutive functions and we present the entropy formulation and the weak formulation of the problem.

Assumptions on coefficients and data

First of all, just for the sake of simplicity, we assume $m = 1$ and $\alpha = 1$. However, all the following results will hold also for general constants α, $m > 0$. Next, we consider a polynomial double-well potential F, whose expression is given by

$$F(u) = a_1|u|^\rho - a_2u^2, \quad \rho \in \left[3, \tfrac{7}{2}\right], \tag{2.14}$$

where a_1, $a_2 > 0$, so that its derivative reads

$$f(u) = a_1\rho\,\mathrm{sgn}(u)|u|^{\rho-1} - 2a_2u\,, \tag{2.15}$$

where sgn represents the sign function. The condition $\rho \in \left[3, \frac{7}{2}\right]$ will be needed to derive a priori estimates in the next section (cf. for instance (2.42) and (2.44)). As for the function k related to the heat conductivity, we assume it in the following form

$$k(\theta) = k_0 + k_1\theta^\beta, \tag{2.16}$$

where $k_0, k_1 > 0$ and $\beta \in [0, 2)$. The latter condition will be needed to derive a priori estimates in the next section (cf. for instance (2.48), (2.49) and (2.50)). We conclude by specifying our hypotheses on the initial data:

$$\begin{aligned} &u_0 \in H^1(\Omega)\,, \quad \theta_0 > 0 \text{ almost everywhere in } \Omega\,, \\ &\theta_0 \in L^2(\Omega)\,, \quad \frac{1}{\theta_0} \in L^1(\Omega) \text{ and } \frac{f(u_0) - \Delta u_0}{\theta_0^{\frac{1}{2}}} \in L^2(\Omega)\,. \end{aligned} \tag{2.17}$$

Notice that the regularities provided by (2.17) are satisfied for instance when

$$u_0 \in H^2(\Omega), \quad \theta_0 \in L^2(\Omega), \quad \theta_0 \geq \bar{\theta} > 0 \text{ almost everywhere in } \Omega\,,$$

where $\bar{\theta} > 0$ is a given constant.

Entropy and weak formulations

Firstly, we introduce the notion of entropy solution to our problem:

Definition 2.2.1. An *entropy solution* to the non-isothermal Cahn-Hilliard model is a triple (u, χ, θ) of sufficient regularity satisfying equations

$$u_t = \Delta\chi \text{ in } L^2(\Omega), \text{ almost everywhere in } (0, T)\,, \tag{2.18}$$

$$\chi\theta = f(u) - \lambda\theta - \Delta u \text{ in } L^2(\Omega), \text{ almost everywhere in } (0, T)\,, \tag{2.19}$$

with the boundary conditions (2.4)-(2.6), the initial condition $u(\cdot,0)=u_0$, and the entropy production inequality:

$$\int_0^T\!\!\int_\Omega \Lambda(\theta)\zeta_t\,dxdt+\int_0^T\!\!\int_\Omega \nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\cdot\nabla\zeta\,dxdt+\int_0^T\!\!\int_\Omega \frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\cdot\nabla\zeta\,dxdt$$
$$\le-\int_0^T\!\!\int_\Omega |\nabla\chi|^2\zeta\,dxdt-\int_0^T\!\!\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\zeta\,dxdt-\int_\Omega \Lambda(\theta_0)\zeta(\cdot,0)\,dx\,,$$
$$\forall\zeta\in\mathcal{C}^\infty(\bar\Omega\times[0,\,T])\text{ such that }\zeta\ge 0,\;\zeta(\cdot,T)=0\,. \tag{2.20}$$

It is worth noting that (2.20) incorporates both the initial condition $\theta(\cdot,0)=\theta_0$ and the no-flux condition (2.7). As for the boundary conditions (2.4)-(2.6), they will be recovered in the sense of traces.

Another notion of solution to our problem can be introduced, that of weak solution. Notice that this notion is somehow “stronger” than the previous one.

Definition 2.2.2. A *weak solution* to the non-isothermal Cahn-Hilliard model is a triple (u,χ,θ) of sufficient regularity satisfying equations (2.18)-(2.19) with the boundary conditions (2.4)-(2.6), the initial condition $u(\cdot,0)=u_0$, and the weak form of the “heat” equation:

$$\int_0^T\int_\Omega Q(\theta)\xi_t\,dx+\int_\Omega Q(\theta_0)\xi(\cdot,0)\,dx-\int_\Omega Q(\theta(\cdot,T))\xi(\cdot,T)\,dx+$$
$$-\int_0^T\int_\Omega \theta(\chi+\lambda)\Delta\chi\xi\,dxdt+\int_0^T\int_\Omega k(\theta)\nabla\frac{1}{\theta}\cdot\nabla\xi\,dxdt=0\,,$$
$$\forall\xi\in\mathcal{C}^\infty(\bar\Omega\times[0,\,T])\,. \tag{2.21}$$

Also in this case, (2.21) incorporates both the initial condition $\theta(\cdot,0)=\theta_0$ and the boundary condition (2.7), whereas (2.4)-(2.6) will be recovered in the sense of traces.

Main existence theorems

Our main results can now be stated as follows:

Theorem 2.2.1 (Existence of entropy solutions)**.** *Under the assumptions stated above, for $\beta\in[0,2)$ in (2.16), the non-isothermal Cahn-Hilliard model admits at least one entropy solution, in the sense of Definition 2.2.1, in the following regularity class:*

$$u\in L^\infty(0,\,T;\,W^{2,\frac43}(\Omega))\cap L^3(0,\,T;\,H^2(\Omega))\cap H^1(0,\,T;\,L^2(\Omega))\,,$$
$$\chi\in L^2(0,\,T;\,H^2(\Omega))\,,\;\chi^2\in L^2(0,\,T;\,H^1(\Omega))\,,$$
$$\theta\in L^\infty(0,\,T;\,L^2(\Omega))\,,\;\theta>0\text{ almost everywhere in }\Omega\times(0,T)\,,$$
$$\log\theta\in L^2(0,\,T;\,L^2(\Omega)),\;\frac{1}{\theta}\in L^2(0,\,T;\,H^1(\Omega))\,,$$
$$\frac{1}{\theta^2}\in L^2(0,\,T;\,H^1(\Omega))\,,\;\frac{1}{\theta^{2-\beta}}\in L^2(0,\,T;\,H^1(\Omega))\,. \tag{2.22}$$

Theorem 2.2.2 (Existence of weak solutions)**.** *Under the assumptions stated above, for $\beta \in (\frac{5}{3}, 2)$ in (2.16), the non-isothermal Cahn-Hilliard model admits at least one weak solution, in the sense of Definition 2.2.2, in the following regularity class:*

$$
\begin{aligned}
& u \in L^\infty(0, T; W^{2,\frac{4}{3}}(\Omega)) \cap L^3(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)), \\
& \chi \in L^2(0, T; H^2(\Omega)), \ \chi^2 \in L^2(0, T; H^1(\Omega)), \\
& \theta \in L^\infty(0, T; L^2(\Omega)) \cap L^q(\Omega \times (0,T)), \ \forall q \in [1, \tfrac{3\beta+1}{3}), \\
& \theta > 0 \ \textit{almost everywhere in } \Omega \times (0,T), \\
& \log\theta \in L^2(0, T; L^2(\Omega)), \ \frac{1}{\theta} \in L^2(0, T; H^1(\Omega)), \\
& \frac{1}{\theta^2} \in L^2(0, T; H^1(\Omega)), \ \frac{1}{\theta^{2-\beta}} \in L^2(0, T; H^1(\Omega)).
\end{aligned} \tag{2.23}
$$

Remark 2.2.3. *Notice that in case we could prove the existence of a sufficiently smooth weak solution (in particular, regular enough in order to integrate back by parts the terms in (2.21)), then it would be possible to show that such a solution also satisfies the "standard" form of the "heat" equation (2.3). Hence, the current notion of weak solution turns out to be compatible with the strong one. On the other hand, in case we could prove the existence of a sufficiently smooth entropy solution (in particular, regular enough in order to integrate back by parts the terms in (2.20)), then the entropy inequality (2.20) would hold as an equality and it would be possible to show that such a solution satisfies (2.11), hence also (2.3) because they are equivalent. Thus, we can conclude that the current notion of entropy solution turns out to be compatible both with the weak and the strong one.*

2.3 A priori estimates

In this section, we will prove some formal a priori estimates holding for a hypothetical triple (u, χ, θ) solving the "strong" formulation of the model, i.e., the system of equations (2.1)-(2.3). Actually, these estimates will follow as direct consequences of the conservation of mass (2.5), the conservation of energy (2.10) and the balance of entropy (2.13), but mainly as results of some technical work (see Subsection 2.3.2). Of course, to make this procedure fully rigorous, one should rather consider a proper regularization or approximation of the "strong" system and prove that it admits at least one solution being sufficiently smooth in order to comply with the estimates. However, the system of equations (2.1)-(2.3) is rather complex and, as a consequence, the related approximation would be particularly long and technical. For all these reasons, we decided to skip this argument and rather proceed formally.

Throughout the section, we will assume the reader to be familiar with the notions and the results presented in Appendix A.2-A.8. However, sometimes we will explicitly recall them in order to emphasize their use.

From now on, in order to simplify the notation, we will denote by $L^p(\Omega)$, instead of $L^p(\Omega; \mathbb{R}^3)$, the space of $\mathbb{R}^3$-valued functions whose components belong to $L^p(\Omega)$.

$\langle \cdot , \cdot \rangle_{X',X}$ will stand for the duality between two Banach spaces X and X', where X' is the dual space of X.
Furthermore, the same letter c (and, sometimes, c_ρ, c_ϵ or c_p when accounting for the dependence on a parameter ρ, ϵ or p) denotes positive constants which may vary from line to line.

2.3.1 Energy and entropy estimates

From the integration with respect to time of the conservation of energy (2.10), using (2.17), we deduce

$$\|\theta\|_{L^\infty(0,T;\,L^2(\Omega))} \leq c\,, \tag{2.24}$$

$$\|\nabla u\|_{L^\infty(0,T;\,L^2(\Omega))} \leq c\,, \tag{2.25}$$

$$\|F(u)\|_{L^\infty(0,T;\,L^1(\Omega))} \leq c\,. \tag{2.26}$$

Using the Poincaré-Wirtinger inequality, from (2.5) and (2.25) it follows that

$$\|u\|_{L^\infty(0,T;\,H^1(\Omega))} \leq c\,. \tag{2.27}$$

Using the continuous embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we then obtain

$$\|u\|_{L^\infty(0,T;\,L^6(\Omega))} \leq c\,. \tag{2.28}$$

Since f is given by (2.15), we can conclude that

$$\|f(u)\|_{L^\infty(0,T;\,L^{\frac{6}{\rho-1}}(\Omega))} \leq c\,, \tag{2.29}$$

where $\frac{6}{\rho-1} \in [\frac{12}{5}, 3]$ for $\rho \in [3, \frac{7}{2}]$.

Integrating the balance of entropy (2.13) with respect to time and using (2.24), (2.27) to control the left hand side, we deduce

$$\|\nabla\chi\|_{L^2(0,T;\,L^2(\Omega))} \leq c\,, \tag{2.30}$$

$$\int_0^T \int_\Omega k(\theta) \Big|\nabla \frac{1}{\theta}\Big|^2 \,dxdt \leq c\,. \tag{2.31}$$

Since k is given by (2.16), from (2.31) it follows that

$$\Big\|\nabla \frac{1}{\theta}\Big\|_{L^2(0,T;\,L^2(\Omega))} \leq c \tag{2.32}$$

and

$$\int_0^T \int_\Omega \theta^{\beta-4} |\nabla\theta|^2 \,dx\,dt = \int_0^T \int_\Omega |\theta^{\frac{\beta}{2}-2} \nabla\theta|^2 \,dx\,dt = \\ = \int_0^T \int_\Omega (\tfrac{\beta}{2} - 1)^{-2} |\nabla \theta^{\frac{\beta}{2}-1}|^2 \,dx\,dt \leq c\,,$$

namely,

$$\|\nabla\theta^{\frac{\beta}{2}-1}\|_{L^2(0,T;\,L^2(\Omega))} \leq c\,. \tag{2.33}$$

2.3.2 Further a priori estimates

Key estimates

First test function for the entropy equation. Firstly, we multiply (2.11) by $-(\frac{\chi^2}{2}+\lambda\chi)$ and we integrate it over Ω, obtaining

$$-\int_\Omega c_V\theta_t\Big(\frac{\chi^2}{2}+\lambda\chi\Big)+\int_\Omega\Big(\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)+\frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\Big)\cdot\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\,dx+$$
$$+\int_\Omega|\nabla\chi|^2\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\,dx+\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\,dx=0\,. \quad (2.34)$$

Taking the partial derivative of (2.2) with respect to time,

$$\chi_t\theta+\chi\theta_t+\lambda\theta_t=-\Delta u_t+f'(u)u_t\,,$$

then multiplying it by χ, we get

$$\Big(\frac{\chi^2}{2}\Big)_t\theta+\chi^2\theta_t+\lambda\chi\theta_t=-\chi\Delta u_t+f'(u)u_t\chi\,. \quad (2.35)$$

On the other hand, multiplying (2.1) by u_t and integrating over Ω, thanks to (2.4), we have

$$\|u_t\|^2_{L^2(\Omega)}=-(\nabla u_t,\nabla\chi)_{L^2(\Omega),\,L^2(\Omega)}\,. \quad (2.36)$$

Integrating (2.35) over Ω and combining it with (2.36), it yields

$$\frac{d}{dt}\int_\Omega\frac{\chi^2}{2}\theta\,dx-\int_\Omega\frac{\chi^2}{2}\theta_t\,dx+\int_\Omega\chi^2\theta_t\,dx+$$
$$+\int_\Omega\lambda\chi\theta_t\,dx+\|u_t\|^2_{L^2(\Omega)}=\int_\Omega f'(u)u_t\chi\,dx\,. \quad (2.37)$$

Finally, multiplying (2.37) by c_V and summing it to (2.34), we obtain

$$\frac{d}{dt}\int_\Omega c_V\frac{\chi^2}{2}\theta\,dx+\int_\Omega\Big(\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)+\frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\Big)\cdot\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\,dx+$$
$$+\,c_V\|u_t\|^2_{L^2(\Omega)}+\int_\Omega|\nabla\chi|^2\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\,dx+\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\,dx$$
$$=c_V\int_\Omega f'(u)u_t\chi\,dx\,. \quad (2.38)$$

Note that $b_1\chi^2-b_2\le\frac{\chi^2}{2}+\lambda\chi$ for $b_1\ge 0$, $b_2>0$ such that $2b_2(1-2b_1)\ge\lambda$. In particular, $-\frac{\lambda}{2}\le\frac{\chi^2}{2}+\lambda\chi$. It follows that

$$\frac{d}{dt}\int_\Omega c_V\frac{\chi^2}{2}\theta\,dx+\int_\Omega\Big(\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)+\frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\Big)\cdot\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\,dx+$$
$$+c_V\|u_t\|^2_{L^2(\Omega)}+b_1\int_\Omega\chi^2|\nabla\chi|^2\,dx\le b_2\|\nabla\chi\|^2_{L^2(\Omega)}+\frac{\lambda}{2}\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\,dx+$$
$$+\,c_V\int_\Omega f'(u)u_t\chi\,dx\,, \quad (2.39)$$

for assigned $b_1, b_2 > 0$ satisfying $2b_2(1-2b_1) \geq \lambda$.
We now control the last term on the right hand side of (2.39). Using (2.1), we can rewrite it as

$$c_V \int_\Omega f'(u) u_t \chi \, dx = c_V \int_\Omega f'(u) \chi \Delta \chi \, dx \,,$$

thus, integrating by parts and using (2.6),

$$c_V \int_\Omega f'(u) u_t \chi \, dx = - c_V \int_\Omega f'(u) |\nabla \chi|^2 dx - c_V \int_\Omega f''(u) \chi \nabla u \cdot \nabla \chi \, dx \,, \tag{2.40}$$

where f' and f'' denote the first and the second derivative of f, respectively. Since f is given by (2.15), then

$$f'(u) = a_1 \rho(\rho-1)|u|^{\rho-2} - 2a_2 \,, \quad f''(u) = a_1 \rho(\rho-1)(\rho-2)\, \mathrm{sgn}(u)|u|^{\rho-3}. \tag{2.41}$$

From (2.40) it then follows that

$$\begin{aligned} c_V \int_\Omega f'(u) u_t \chi \, dx = &- c_V \int_\Omega a_1 \rho(\rho-1)|u|^{\rho-2} |\nabla \chi|^2 dx + c_V \int_\Omega 2a_2 |\nabla \chi|^2 dx + \\ &- c_V \int_\Omega a_1 \rho(\rho-1)(\rho-2)\, \mathrm{sgn}(u)|u|^{\rho-3} \chi \nabla u \cdot \nabla \chi \, dx \,. \end{aligned} \tag{2.42}$$

Observe that, inserting (2.42) into (2.39), we can move the first term on the right hand side of (2.42) to the left hand side of (2.39), hence the only term to control is the last one of (2.42). Using Hölder's inequality, we deduce

$$\begin{aligned} -c_V \int_\Omega a_1 \rho(\rho-1)(\rho-2)\, \mathrm{sgn}(u)|u|^{\rho-3} \chi \nabla u \cdot \nabla \chi \, dx &\leq c_\rho \int_\Omega |u|^{\rho-3} |\nabla u| |\nabla \chi^2| \, dx \\ &\leq c_\rho \| |u|^{\rho-3} \|_{L^{\frac{6}{\rho-3}}(\Omega)} \|\nabla u\|_{L^{\frac{6}{6-\rho}}(\Omega)} \|\nabla \chi^2\|_{L^2(\Omega)} \\ &\leq c_\rho \|u\|^{\rho-3}_{L^6(\Omega)} \|\nabla u\|_{L^{\frac{6}{6-\rho}}(\Omega)} \|\nabla \chi^2\|_{L^2(\Omega)} \,. \end{aligned} \tag{2.43}$$

Using the Gagliardo-Nirenberg interpolation inequality, we obtain

$$\|\nabla u\|_{L^{\frac{6}{6-\rho}}(\Omega)} \leq c \left[\|D^2 u\|_{L^{\frac{4}{3}}(\Omega)} + \|u\|_{L^6(\Omega)} \right] ,$$

whence, thanks to classical elliptic regularity results (cf. Appendix A.8),

$$\|\nabla u\|_{L^{\frac{6}{6-\rho}}(\Omega)} \leq c \left[\|\Delta u\|_{L^{\frac{4}{3}}(\Omega)} + \|u\|_{L^6(\Omega)} \right] , \tag{2.44}$$

where, according to (2.15), $\rho \in [3, \frac{7}{2}]$, hence $\frac{6}{6-\rho} \in [2, \frac{12}{5}]$.
From (2.43) and (2.44) it follows

$$\begin{aligned} & - c_V \int_\Omega a_1 \rho(\rho-1)(\rho-2)\, \mathrm{sgn}(u)|u|^{\rho-3} \chi \nabla u \cdot \nabla \chi \, dx \\ & \leq c_\rho \|u\|^{\rho-3}_{L^6(\Omega)} \|\nabla \chi^2\|_{L^2(\Omega)} \left[\|\Delta u\|_{L^{\frac{4}{3}}(\Omega)} + \|u\|_{L^6(\Omega)} \right] \\ & \leq c_\rho \|\nabla \chi^2\|_{L^2(\Omega)} \left[\|\Delta u\|_{L^{\frac{4}{3}}(\Omega)} + 1 \right] \leq \sigma \|\nabla \chi^2\|^2_{L^2(\Omega)} + c_\sigma c_\rho \|\Delta u\|^2_{L^{\frac{4}{3}}(\Omega)} \right] , \end{aligned} \tag{2.45}$$

where in the last line we used (2.28) and then Young's inequality for $\sigma > 0$ small enough to satisfy $b_1 - 4\sigma > 0$ for b_1 as assigned in (2.39).
Finally, let us control the last term in (2.45). Due to (2.2) and Young's inequality,

$$\|\Delta u\|^2_{L^{\frac{4}{3}}(\Omega)} \le c\big[\|\theta\|^2_{L^{\frac{4}{3}}(\Omega)} + \|f(u)\|^2_{L^{\frac{4}{3}}(\Omega)} + \|\chi\theta\|^2_{L^{\frac{4}{3}}(\Omega)}\big]\,,$$

whence, using Hölder's inequality, we deduce

$$\begin{aligned}\|\Delta u\|^2_{L^{\frac{4}{3}}(\Omega)} &\le c\big[\|\theta\|^2_{L^{\frac{4}{3}}(\Omega)} + \|f(u)\|^2_{L^{\frac{4}{3}}(\Omega)} + \|\chi\theta^{\frac{1}{2}}\|^2_{L^2(\Omega)}\|\theta^{\frac{1}{2}}\|^2_{L^4(\Omega)}\big]\\ &\le c\big[\|\theta\|^2_{L^{\frac{4}{3}}(\Omega)} + \|f(u)\|^2_{L^{\frac{4}{3}}(\Omega)} + \|\chi^2\theta\|_{L^1(\Omega)}\|\theta\|_{L^2(\Omega)}\big]\\ &\le c\big[1 + \|\chi^2\theta\|_{L^1(\Omega)}\big]\,, \end{aligned} \tag{2.46}$$

where in the last inequality we used (2.24) and (2.29).
Combining together (2.39), (2.42), (2.45) and (2.46), it yields

$$\begin{aligned}&\frac{d}{dt}\int_\Omega c_V\frac{\chi^2}{2}\theta\,dx + \int_\Omega\Big(\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big) + \frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\Big)\cdot\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\,dx +\\ &+ c_V\|u_t\|^2_{L^2(\Omega)} + (b_1-4\sigma)\Big\|\nabla\frac{\chi^2}{2}\Big\|^2_{L^2(\Omega)} + c_V\int_\Omega a_1\rho(\rho-1)|u|^{\rho-2}|\nabla\chi|^2\,dx\\ &\le (b_2+2c_Va_2)\|\nabla\chi\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\,dx + c_\sigma c_\rho\big[1+\|\chi^2\theta\|_{L^1(\Omega)}\big]\,. \end{aligned} \tag{2.47}$$

Remark 2.3.1. *Notice that we assumed $\rho \ge 3$ in (2.15) in order to have positive exponents in the expressions for f' and f'' given by (2.41), hence in (2.42). As for the hypothesis $\rho \le \frac{7}{2}$, it is needed in order to obtain (2.44) through the application of the Gagliardo-Nirenberg interpolation inequality. Otherwise, we would obtain $\|\Delta u\|_{L^p(\Omega)}$ on the right hand side of (2.44), where $p > \frac{4}{3}$ for $\rho > \frac{7}{2}$. As a consequence, we would have $\|\chi^2\theta\|_{L^r(\Omega)}$ with $r > 1$ on the right hand side of (2.46). Thus, we could not proceed as we will do in the following, i.e., applying Gronwall's Lemma in order to close the estimates.*

Second test function for the entropy equation. Next, we multiply (2.11) by $-g(\theta)$, where g is a positive function of θ such that its gradient is equal to $\frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}$. Recalling that k is given by (2.16), we have

$$\begin{aligned}\frac{k(\theta)}{\theta}\nabla\frac{1}{\theta} = \Big(\frac{k_0}{\theta} + k_1\theta^{\beta-1}\Big)\nabla\frac{1}{\theta} = \nabla\frac{k_0}{2\theta^2} + \nabla\frac{k_1}{(2-\beta)\theta^{2-\beta}} =\\ = \nabla\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\,, \end{aligned} \tag{2.48}$$

hence g is given by

$$g(\theta) = \frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\,.$$

If $\beta \in [0,1) \cup (1,2)$, multiplying (2.11) by $-(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}})$ and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega c_V\Big(\frac{k_0}{2\theta} - \frac{k_1\theta^{\beta-1}}{(2-\beta)(\beta-1)}\Big)dx + \int_\Omega \nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big)\cdot\nabla\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)dx \\
&+ \int_\Omega \frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\cdot\nabla\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)dx + \int_\Omega\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)|\nabla\chi|^2\,dx \\
&+ \int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2\,dx = 0\,.
\end{aligned} \tag{2.49}$$

Otherwise, if $\beta = 1$, multiplying (2.11) by $-(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta})$ and integrating over Ω, we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega c_V\Big(\frac{k_0}{2\theta} - k_1\log\theta\Big)\,dx + \int_\Omega \nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big)\cdot\nabla\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)\,dx \\
&+ \int_\Omega \frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\cdot\nabla\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)\,dx + \int_\Omega\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)|\nabla\chi|^2\,dx \\
&+ \int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2\,dx = 0\,.
\end{aligned} \tag{2.50}$$

Sum of the first and the second test function for the entropy equation. Summing (2.47) and (2.49), for $\beta \in [0,1) \cup (1,2)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega c_V\Big(\frac{\chi^2}{2}\theta + \frac{k_0}{2\theta} - \frac{k_1\theta^{\beta-1}}{(2-\beta)(\beta-1)}\Big)\,dx + c_V\|u_t\|^2_{L^2(\Omega)} \\
&+ \int_\Omega\Big(\nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big) + \frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\Big)\cdot\Big(\nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big) + \nabla\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big)dx \\
&+ \int_\Omega\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)|\nabla\chi|^2\,dx + \int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2dx \\
&+ (b_1 - 4\sigma)\Big\|\nabla\frac{\chi^2}{2}\Big\|^2_{L^2(\Omega)} + c_V\int_\Omega a_1\rho(\rho-1)|u|^{\rho-2}|\nabla\chi|^2\,dx \\
&\le (b_2 + 2c_V a_2)\|\nabla\chi\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2dx + c_\sigma c_\rho\big[1 + \|\chi^2\theta\|_{L^1(\Omega)}\big]\,,
\end{aligned}$$

while for $\beta = 1$, summing (2.47) and (2.50),

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega c_V\Big(\frac{\chi^2}{2}\theta + \frac{k_0}{2\theta} - k_1\log\theta\Big)\,dx + c_V\|u_t\|^2_{L^2(\Omega)} \\
&+ \int_\Omega\Big(\nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big) + \frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\Big)\cdot\Big(\nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big) + \nabla\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)\Big)\,dx \\
&+ \int_\Omega\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)|\nabla\chi|^2\,dx + \int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2\,dx \\
&+ (b_1 - 4\sigma)\Big\|\nabla\frac{\chi^2}{2}\Big\|^2_{L^2(\Omega)} + c_V\int_\Omega a_1\rho(\rho-1)|u|^{\rho-2}|\nabla\chi|^2\,dx \\
&\le (b_2 + 2c_V a_2)\|\nabla\chi\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\,dx + c_\sigma c_\rho\big[1 + \|\chi^2\theta\|_{L^1(\Omega)}\big]\,.
\end{aligned}$$

Since (2.48) holds true, for $\beta \in [0,1) \cup (1,2)$, it follows that

$$\frac{d}{dt}\int_\Omega c_V\Big(\frac{\chi^2}{2}\theta + \frac{k_0}{2\theta} - \frac{k_1\theta^{\beta-1}}{(2-\beta)(\beta-1)}\Big)\,dx$$
$$+ c_V\|u_t\|^2_{L^2(\Omega)} + \Big\|\nabla\Big(\frac{\chi^2}{2} + \lambda\chi + \frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|^2_{L^2(\Omega)}$$
$$+\int_\Omega\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)|\nabla\chi|^2\,dx + \int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2 dx$$
$$+ (b_1 - 4\sigma)\Big\|\nabla\frac{\chi^2}{2}\Big\|^2_{L^2(\Omega)} + c_V\int_\Omega a_1\rho(\rho-1)|u|^{\rho-2}|\nabla\chi|^2\,dx$$
$$\leq (b_2 + 2c_Va_2)\|\nabla\chi\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2 dx + c_\sigma c_\rho\big[1 + \|\chi^2\theta\|_{L^1(\Omega)}\big], \quad (2.51)$$

whereas, if $\beta = 1$,

$$\frac{d}{dt}\int_\Omega c_V\Big(\frac{\chi^2}{2}\theta + \frac{k_0}{2\theta} - k_1\log\theta\Big)\,dx + c_V\|u_t\|^2_{L^2(\Omega)}$$
$$+ \Big\|\nabla\Big(\frac{\chi^2}{2} + \lambda\chi + \frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)\Big\|^2_{L^2(\Omega)}$$
$$+ \int_\Omega\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)|\nabla\chi|^2\,dx + \int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2 dx$$
$$+ (b_1 - 4\sigma)\Big\|\nabla\frac{\chi^2}{2}\Big\|^2_{L^2(\Omega)} + c_V\int_\Omega a_1\rho(\rho-1)|u|^{\rho-2}|\nabla\chi|^2\,dx$$
$$\leq (b_2 + 2c_Va_2)\|\nabla\chi\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2 dx + c_\sigma c_\rho\big[1 + \|\chi^2\theta\|_{L^1(\Omega)}\big]. \quad (2.52)$$

Let us consider the cases $\beta \in [0,1)$, $\beta = 1$ and $\beta \in (1,2)$ separately.

- $\beta \in [0,1)$: Integrating (2.51) with respect to time, we obtain

$$\int_\Omega c_V\Big(\frac{\chi^2\theta}{2}(\tau) + \frac{k_0}{2\theta}(\tau) + \frac{k_1}{(2-\beta)(1-\beta)\theta^{1-\beta}}(\tau)\Big)\,dx +$$
$$+ c_V\int_0^\tau\|u_t\|^2_{L^2(\Omega)}dt + \int_0^\tau\Big\|\nabla\Big(\frac{\chi^2}{2} + \lambda\chi + \frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|^2_{L^2(\Omega)}dt +$$
$$+ \int_0^\tau\int_\Omega\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)|\nabla\chi|^2\,dxdt +$$
$$+ \int_0^\tau\int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2 dxdt +$$
$$+ (b_1 - 4\sigma)\int_0^\tau\Big\|\nabla\frac{\chi^2}{2}\Big\|^2_{L^2(\Omega)}dt + c_V\int_0^\tau\int_\Omega a_1\rho(\rho-1)|u|^{\rho-2}|\nabla\chi|^2\,dxdt$$
$$\leq \int_\Omega c_V\Big(\frac{\chi_0^2\theta_0}{2} + \frac{k_0}{2\theta_0} + \frac{k_1}{(2-\beta)(1-\beta)\theta_0^{1-\beta}}\Big)\,dx +$$
$$+ (b_2 + 2c_Va_2)\|\nabla\chi\|^2_{L^2(0,T;\,L^2(\Omega))} + \frac{\lambda}{2}\int_0^T\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2 dxdt +$$
$$+ c_\sigma c_\rho T + c_\sigma c_\rho\int_0^\tau\|\chi^2\theta\|_{L^1(\Omega)}dt, \qquad \forall\tau \in [0,T]. \quad (2.53)$$

Remark 2.3.2. *Recall that the a priori estimates deduced in this section should be satisfied by sufficiently smooth solutions to a proper regularization or approximation of the "strong" system (2.1)-(2.2)-(2.3). For this reason, (2.53) and other estimates in the following are supposed to hold true* $\forall \tau \in [0,T]$*, instead of almost everywhere in* $(0,T)$*. However, passing to the limit, thus recovering an entropy or weak solution* (u,χ,θ)*, they may be satisfied only almost everywhere in* $(0,T)$*.*

Observe that from (2.2) we can deduce that $\chi_0 = \frac{f(u_0)-\Delta u_0}{\theta_0} - \lambda$. Using the hypotheses on the initial data provided by (2.17), we can conclude that

$$\chi_0 \theta_0^{\frac{1}{2}} \in L^2(\Omega)\,. \tag{2.54}$$

From (2.54), (2.17), (2.30) and (2.31) it follows that all the terms apart from the last one on the right hand side of (2.53) are bounded. On the other hand, all the left hand side term of (2.53) are positive, hence we deduce

$$\frac{c_V}{2}\|\chi^2\theta(\tau)\|_{L^1(\Omega)} \le c + c_\sigma c_\rho T + c_\sigma c_\rho \int_0^\tau \|\chi^2\theta\|_{L^1(\Omega)} dt\,, \quad \forall \tau \in [0,T]\,.$$

Gronwall's Lemma then yields

$$\frac{c_V}{2}\|\chi^2\theta(\tau)\|_{L^1(\Omega)} \le (c + c_\sigma c_\rho T) e^{\frac{2c_\sigma c_\rho}{c_V}\tau}\,, \quad \forall \tau \in [0,T]\,,$$

whence

$$\int_0^\tau \|\chi^2\theta\|_{L^1(\Omega)} dt \le c(\sigma,\rho,T)\,, \quad \forall \tau \in [0,T]\,,$$

where $c(\sigma,\rho,T)$ is a constant depending on the parameters σ, ρ and T.
It follows that (2.53) can be rewritten as

$$\begin{aligned}
&\int_\Omega c_V\Big(\frac{\chi^2\theta}{2}(\tau) + \frac{k_0}{2\theta}(\tau) + \frac{k_1}{(2-\beta)(1-\beta)\theta^{1-\beta}}(\tau)\Big)\,dx + \\
&+ c_V\int_0^\tau \|u_t\|^2_{L^2(\Omega)} dt + \int_0^\tau \Big\|\nabla\Big(\frac{\chi^2}{2} + \lambda\chi + \frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|^2_{L^2(\Omega)} dt + \\
&+ \int_0^\tau\int_\Omega \Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)|\nabla\chi|^2\,dxdt + \\
&+ \int_0^\tau\int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2 dxdt + \\
&+ (b_1 - 4\sigma)\int_0^\tau \Big\|\nabla\frac{\chi^2}{2}\Big\|^2_{L^2(\Omega)} dt + c_V\int_0^\tau\int_\Omega a_1\rho(\rho-1)|u|^{\rho-2}|\nabla\chi|^2\,dxdt \\
&\le c(\sigma,\rho,T)\,, \qquad \forall \tau \in [0,T]\,.
\end{aligned} \tag{2.55}$$

- $\beta = 1$: Integrating (2.52) with respect to time, we obtain

$$
\begin{aligned}
&\int_\Omega c_V\Big(\frac{\chi^2\theta}{2}(\tau) + \frac{k_0}{2\theta}(\tau)\Big)\,dx + \\
&+ c_V\int_0^\tau \|u_t\|^2_{L^2(\Omega)}dt + \int_0^\tau \Big\|\nabla\Big(\frac{\chi^2}{2} + \lambda\chi + \frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)\Big\|^2_{L^2(\Omega)}dt + \\
&+ \int_0^\tau\int_\Omega\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)|\nabla\chi|^2\,dxdt + \int_0^\tau\int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2 dxdt + \\
&+ (b_1 - 4\sigma)\int_0^\tau \Big\|\nabla\frac{\chi^2}{2}\Big\|^2_{L^2(\Omega)}dt + c_V\int_0^\tau\int_\Omega a_1\rho(\rho-1)|u|^{\rho-2}|\nabla\chi|^2\,dxdt \\
&\le \int_\Omega c_V k_1\log\theta(\tau)\,dx + \int_\Omega c_V\Big(\frac{\chi_0^2\theta_0}{2} + \frac{k_0}{2\theta_0} - k_1\log\theta_0\Big)\,dx + \\
&+ (b_2 + 2c_V a_2)\|\nabla\chi\|^2_{L^2(0,T;\,L^2(\Omega))} + \frac{\lambda}{2}\int_0^T\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\,dxdt + \\
&+ c_\sigma c_\rho T + c_\sigma c_\rho\int_0^\tau \|\chi^2\theta\|_{L^1(\Omega)}dt\,, \qquad \forall\tau\in[0,T]\,.
\end{aligned}
$$

Observe that, since $\log s \le s\,,\ \forall s\in\mathbb{R}^+$, using (2.24), we can deduce

$$
\int_\Omega \log\theta(\tau)\,dx \le \int_\Omega \theta(\tau)\,dx \le c\,,\quad \forall\tau\in[0,T]\,.
$$

On the other hand, since $|\log s| \le \frac{1}{s} + s\,,\ \forall s\in\mathbb{R}^+$,

$$
-\int_\Omega c_V k_1\log\theta_0\,dx \le \int_\Omega c_V k_1|\log\theta_0|\,dx \le \int_\Omega c_V k_1\Big(\frac{1}{\theta_0} + \theta_0\Big)\,dx\,.
$$

It follows that

$$
\begin{aligned}
&\int_\Omega c_V\Big(\frac{\chi^2\theta}{2}(\tau) + \frac{k_0}{2\theta}(\tau)\Big)\,dx + \\
&+ c_V\int_0^\tau \|u_t\|^2_{L^2(\Omega)}dt + \int_0^\tau \Big\|\nabla\Big(\frac{\chi^2}{2} + \lambda\chi + \frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)\Big\|^2_{L^2(\Omega)}dt + \\
&+ \int_0^\tau\int_\Omega\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)|\nabla\chi|^2\,dxdt + \int_0^\tau\int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{\theta}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2 dxdt + \\
&+ (b_1 - 4\sigma)\int_0^\tau \Big\|\nabla\frac{\chi^2}{2}\Big\|^2_{L^2(\Omega)}dt + c_V\int_0^\tau\int_\Omega a_1\rho(\rho-1)|u|^{\rho-2}|\nabla\chi|^2\,dxdt \\
&\le c + \int_\Omega c_V\Big(\frac{\chi_0^2\theta_0}{2} + \frac{k_0}{2\theta_0} + \frac{k_1}{\theta_0} + k_1\theta_0\Big)\,dx + \\
&+ (b_2 + 2c_V a_2)\|\nabla\chi\|^2_{L^2(0,T;\,L^2(\Omega))} + \frac{\lambda}{2}\int_0^T\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\,dxdt + \\
&+ c_\sigma c_\rho T + c_\sigma c_\rho\int_0^\tau \|\chi^2\theta\|_{L^1(\Omega)}dt\,, \qquad \forall\tau\in[0,T]\,.
\end{aligned}
$$

Proceeding analogously as done in the case $\beta\in[0,1)$, the regularities on the initial data (2.54) and (2.17), together with (2.30) and (2.31), and Gronwall's

Lemma allow us to conclude that

$$\begin{aligned}
&\int_\Omega c_V\Big(\frac{\chi^2\theta}{2}(\tau)+\frac{k_0}{2\theta}(\tau)\Big)\,dx+\\
&+c_V\int_0^\tau\|u_t\|^2_{L^2(\Omega)}dt+\int_0^\tau\Big\|\nabla\Big(\frac{\chi^2}{2}+\lambda\chi+\frac{k_0}{2\theta^2}+\frac{k_1}{\theta}\Big)\Big\|^2_{L^2(\Omega)}dt+\\
&+\int_0^\tau\int_\Omega\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{\theta}\Big)|\nabla\chi|^2\,dxdt+\int_0^\tau\int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{\theta}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2dxdt+\\
&+(b_1-4\sigma)\int_0^\tau\Big\|\nabla\frac{\chi^2}{2}\Big\|^2_{L^2(\Omega)}dt+c_V\int_0^\tau\int_\Omega a_1\rho(\rho-1)|u|^{\rho-2}|\nabla\chi|^2\,dxdt\\
&\le c(\sigma,\rho,T)\,,\qquad\forall\tau\in[0,T]\,. \qquad (2.56)
\end{aligned}$$

- $\beta\in(1,2)$: Integrating (2.51) with respect to time, we obtain

$$\begin{aligned}
&\int_\Omega c_V\Big(\frac{\chi^2\theta}{2}(\tau)+\frac{k_0}{2\theta}(\tau)\Big)\,dx+\int_\Omega\frac{k_1\theta_0^{\beta-1}}{(2-\beta)(\beta-1)}\,dx+\\
&+c_V\int_0^\tau\|u_t\|^2_{L^2(\Omega)}dt+\int_0^\tau\Big\|\nabla\Big(\frac{\chi^2}{2}+\lambda\chi+\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|^2_{L^2(\Omega)}dt+\\
&+\int_0^\tau\int_\Omega\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)|\nabla\chi|^2\,dxdt+\\
&+\int_0^\tau\int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2dxdt+\\
&+(b_1-4\sigma)\int_0^\tau\Big\|\nabla\frac{\chi^2}{2}\Big\|^2_{L^2(\Omega)}dt+c_V\int_0^\tau\int_\Omega a_1\rho(\rho-1)|u|^{\rho-2}|\nabla\chi|^2\,dxdt\\
&\le\int_\Omega\frac{k_1\theta^{\beta-1}}{(2-\beta)(\beta-1)}(\tau)\,dx+\int_\Omega c_V\Big(\frac{\chi_0^2\theta_0}{2}+\frac{k_0}{2\theta_0}\Big)\,dx+\\
&+(b_2+2c_Va_2)\|\nabla\chi\|^2_{L^2(0,T;\,L^2(\Omega))}+\frac{\lambda}{2}\int_0^T\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2dxdt+\\
&+c_\sigma c_\rho T+c_\sigma c_\rho\int_0^\tau\|\chi^2\theta\|_{L^1(\Omega)}dt\,,\qquad\forall\tau\in[0,T]\,,
\end{aligned}$$

whence, using (2.24),

$$\begin{aligned}
&\int_\Omega c_V\Big(\frac{\chi^2\theta}{2}(\tau)+\frac{k_0}{2\theta}(\tau)\Big)\,dx+\int_\Omega\frac{k_1\theta_0^{\beta-1}}{(2-\beta)(\beta-1)}\,dx+\\
&+c_V\int_0^\tau\|u_t\|^2_{L^2(\Omega)}dt+\int_0^\tau\Big\|\nabla\Big(\frac{\chi^2}{2}+\lambda\chi+\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|^2_{L^2(\Omega)}dt+\\
&+\int_0^\tau\int_\Omega\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)|\nabla\chi|^2\,dxdt+\\
&+\int_0^\tau\int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2dxdt+\\
&+(b_1-4\sigma)\int_0^\tau\Big\|\nabla\frac{\chi^2}{2}\Big\|^2_{L^2(\Omega)}dt+c_V\int_0^\tau\int_\Omega a_1\rho(\rho-1)|u|^{\rho-2}|\nabla\chi|^2\,dxdt
\end{aligned}$$

$$\le c + \int_\Omega c_V\Big(\frac{\chi_0^2\theta_0}{2} + \frac{k_0}{2\theta_0}\Big)\,dx \, + (b_2 + 2c_V a_2)\|\nabla\chi\|^2_{L^2(0,T;\,L^2(\Omega))} +$$
$$+ \frac{\lambda}{2}\int_0^T\!\!\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2 dxdt \, + c_\sigma c_\rho T + c_\sigma c_\rho \int_0^\tau \|\chi^2\theta\|_{L^1(\Omega)}dt\,,\ \ \forall\tau\in[0,T].$$

Proceeding analogously as done in the previous cases, the regularities on the initial data (2.54) and (2.17), together with (2.30) and (2.31), and Gronwall's Lemma imply

$$\int_\Omega c_V\Big(\frac{\chi^2\theta}{2}(\tau) + \frac{k_0}{2\theta}(\tau)\Big)\,dx \, + \int_\Omega \frac{k_1\theta_0^{\beta-1}}{(2-\beta)(\beta-1)}\,dx +$$
$$+ c_V\int_0^\tau \|u_t\|^2_{L^2(\Omega)}dt + \int_0^\tau \Big\|\nabla\Big(\frac{\chi^2}{2} + \lambda\chi + \frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|^2_{L^2(\Omega)}dt +$$
$$+ \int_0^\tau\!\!\int_\Omega \Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)|\nabla\chi|^2\,dxdt +$$
$$+ \int_0^\tau\!\!\int_\Omega k(\theta)\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big|\nabla\frac{1}{\theta}\Big|^2 dxdt +$$
$$+ (b_1 - 4\sigma)\int_0^\tau \Big\|\nabla\frac{\chi^2}{2}\Big\|^2_{L^2(\Omega)}dt + c_V\int_0^\tau\!\!\int_\Omega a_1\rho(\rho-1)|u|^{\rho-2}|\nabla\chi|^2\,dxdt$$
$$\le c(\sigma,\rho,T)\,, \qquad \forall\tau\in[0,T]\,. \tag{2.57}$$

From (2.55), as well as from (2.56) and from (2.57), we can deduce that, in case that $\beta\in[0,2)$,

$$\|\chi^2\theta\|_{L^\infty(0,T;\,L^1(\Omega))} \le c\,, \tag{2.58}$$
$$\Big\|\frac{1}{\theta}\Big\|_{L^\infty(0,T;\,L^1(\Omega))} \le c\,, \tag{2.59}$$
$$\|u_t\|_{L^2(0,T;\,L^2(\Omega))} \le c\,, \tag{2.60}$$
$$\Big\|\nabla\Big(\frac{\chi^2}{2} + \lambda\chi + \frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|_{L^2(0,T;\,L^2(\Omega))} \le c\,, \tag{2.61}$$
$$\|\nabla\chi^2\|_{L^2(0,T;\,L^2(\Omega))} \le c\,, \tag{2.62}$$
$$\Big\|\frac{\nabla\chi}{\theta}\Big\|_{L^2(0,T;\,L^2(\Omega))} \le c\,, \tag{2.63}$$
$$\Big\|\frac{\nabla\chi}{\theta^{1-\frac{\beta}{2}}}\Big\|_{L^2(0,T;\,L^2(\Omega))} \le c\,, \tag{2.64}$$

and, recalling that k is given by (2.16),

$$\Big\|\nabla\frac{1}{\theta^2}\Big\|_{L^2(0,T;\,L^2(\Omega))} \le c\,, \tag{2.65}$$
$$\Big\|\nabla\frac{1}{\theta^{2-\frac{\beta}{2}}}\Big\|_{L^2(0,T;\,L^2(\Omega))} \le c\,, \tag{2.66}$$
$$\Big\|\nabla\frac{1}{\theta^{2-\beta}}\Big\|_{L^2(0,T;\,L^2(\Omega))} \le c\,. \tag{2.67}$$

Note that (2.65) and (2.67) yield

$$\Big\|\nabla\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|_{L^2(0,T;\,L^2(\Omega))}\le c\,. \tag{2.68}$$

On the other hand, using (2.68), from (2.61) it follows that

$$\Big\|\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\Big\|_{L^2(0,T;\,L^2(\Omega))}\le c\,. \tag{2.69}$$

Consequences

Higher regularity for χ. Firstly, observe that $\chi=\chi\theta^{\frac12}\theta^{-\frac12}$, then from (2.58) and (2.59), using Hölder's inequality, we deduce

$$\|\chi\|_{L^\infty(0,T;\,L^1(\Omega))}\le c\,. \tag{2.70}$$

At this point, (2.30) and (2.70) together with the Poincaré-Wirtinger inequality yield

$$\|\chi\|_{L^2(0,T;\,H^1(\Omega))}\le c\,. \tag{2.71}$$

Furthermore, from (2.60), using (2.1), we obtain

$$\|\Delta\chi\|_{L^2(0,T;\,L^2(\Omega))}\le c\,. \tag{2.72}$$

By classical regularity results for elliptic equations (cf. Appendix A.8), (2.71) and (2.72) imply

$$\|\chi\|_{L^2(0,T;\,H^2(\Omega))}\le c\,, \tag{2.73}$$

whence, using the continuous embedding $H^2(\Omega)\hookrightarrow\mathcal{C}(\bar\Omega)$,

$$\|\chi\|_{L^2(0,T;\mathcal{C}(\bar\Omega))}\le c\,, \tag{2.74}$$

while, thanks to the continuous embedding $H^2(\Omega)\hookrightarrow W^{1,6}(\Omega)$, we infer

$$\|\chi\|_{L^2(0,T;\,W^{1,6}(\Omega))}\le c\,. \tag{2.75}$$

Let us now estimate $\|\chi\|_{L^4(0,T;\,L^2(\Omega))}$. Using standard interpolation, from (2.70) and (2.74) we deduce

$$\begin{aligned}\|\chi\|^4_{L^4(0,T;\,L^2(\Omega))}&\le\int_0^T\big(\|\chi\|^{\frac12}_{L^1(\Omega)}\|\chi\|^{\frac12}_{L^\infty(\Omega)}\big)^4dt\\&\le\|\chi\|^2_{L^\infty(0,T;\,L^1(\Omega))}\|\chi\|^2_{L^2(0,T;\,L^\infty(\Omega))}\le c\,,\end{aligned}$$

namely,

$$\|\chi\|_{L^4(0,T;\,L^2(\Omega))}\le c\,,$$

or equivalently,

$$\|\chi^2\|_{L^2(0,T;\,L^1(\Omega))}\le c\,. \tag{2.76}$$

Using the Poincaré-Wirtinger inequality, from (2.62) and (2.76) it follows that

$$\|\chi^2\|_{L^2(0,T;\,H^1(\Omega))}\le c\,. \tag{2.77}$$

Moreover, thanks to the continuous embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$,

$$\|\chi^2\|_{L^2(0,T;\,L^6(\Omega))} \leq c\,, \tag{2.78}$$

or equivalently,

$$\|\chi\|_{L^4(0,T;\,L^{12}(\Omega))} \leq c\,. \tag{2.79}$$

Using the Gagliardo-Nirenberg interpolation inequality and then (2.70), we obtain

$$\|\chi\|_{L^\infty(\Omega)} \leq c\|\mathrm{D}^2\chi\|^{\frac{1}{3}}_{L^2(\Omega)}\|\chi\|^{\frac{2}{3}}_{L^{12}(\Omega)} + c\|\chi\|_{L^1(\Omega)} \leq c\|\mathrm{D}^2\chi\|^{\frac{1}{3}}_{L^2(\Omega)}\|\chi\|^{\frac{2}{3}}_{L^{12}(\Omega)} + c\,,$$

whence

$$\|\chi\|^3_{L^\infty(\Omega)} \leq c\|\mathrm{D}^2\chi\|_{L^2(\Omega)}\|\chi\|^2_{L^{12}(\Omega)} + c \leq c\|\mathrm{D}^2\chi\|^2_{L^2(\Omega)} + c\|\chi\|^4_{L^{12}(\Omega)} + c\,,$$

where in the last inequality we also used Young's inequality. Using regularities given by (2.73) and (2.79), we can conclude that

$$\|\chi\|_{L^3(0,T;\,L^\infty(\Omega))} \leq c\,. \tag{2.80}$$

Higher regularity for u. Let us consider (2.2), namely,

$$\Delta u = f(u) - \theta - \chi\theta\,.$$

Noting that $\chi\theta$ can be written as $\chi\theta^{\frac{1}{2}}\theta^{\frac{1}{2}}$, from (2.58) and (2.24), using Hölder's inequality, it follows that

$$\|\chi\theta\|_{L^\infty(0,T;\,L^{4/3}(\Omega))} \leq \|\chi\theta^{\frac{1}{2}}\|_{L^\infty(0,T;\,L^2(\Omega))}\|\theta^{\frac{1}{2}}\|_{L^\infty(0,T;\,L^4(\Omega))} \leq c\,. \tag{2.81}$$

On the other hand, (2.29) and (2.24) hold true. Hence, we obtain

$$\|\Delta u\|_{L^\infty(0,T;\,L^{\frac{4}{3}}(\Omega))} \leq c\,. \tag{2.82}$$

Combining (2.82) with (2.27), by classical elliptic regularity results (cf. Appendix A.8), we deduce

$$\|u\|_{L^\infty(0,T;\,W^{2,\frac{4}{3}}(\Omega))} \leq c\,. \tag{2.83}$$

Furthermore, from (2.80) together with (2.24) it follows that

$$\|\chi\theta\|_{L^3(0,T;\,L^2(\Omega))} \leq c\,, \tag{2.84}$$

whence, once more by comparision,

$$\|\Delta u\|_{L^3(0,T;\,L^2(\Omega))} \leq c\,. \tag{2.85}$$

Finally, taking into account (2.27), by classical elliptic regularity results (cf. Appendix A.8), we obtain

$$\|u\|_{L^3(0,T;\,H^2(\Omega))} \leq c\,. \tag{2.86}$$

Higher regularity for powers of θ. Firstly, note that (2.24) and (2.72) imply

$$\|\theta\Delta\chi\|_{L^2(0,T;\,L^1(\Omega))} \le c\,. \tag{2.87}$$

Using (2.80) and (2.87), we deduce that

$$\|\theta\chi\Delta\chi\|_{L^{\frac{6}{5}}(0,T;\,L^1(\Omega))} \le c\,. \tag{2.88}$$

Combining (2.87) and (2.88), we can conclude that

$$\|\theta(\chi+\lambda)\Delta\chi\|_{L^{\frac{6}{5}}(0,T;\,L^1(\Omega))} \le c\,, \tag{2.89}$$

in particular,

$$\|\theta(\chi+\lambda)\Delta\chi\|_{L^1(0,T;\,L^1(\Omega))} \le c\,. \tag{2.90}$$

On the other hand, (2.72) and (2.80) imply

$$\|\chi\Delta\chi\|_{L^{\frac{6}{5}}(0,T;\,L^2(\Omega))} \le c\,, \tag{2.91}$$

whence

$$\|(\chi+\lambda)\Delta\chi\|_{L^1(0,T;\,L^1(\Omega))} \le c\,. \tag{2.92}$$

Let us now multiply (2.3) by $-\theta^{-\epsilon}$, where $\epsilon \in (0,1)$ is such that $\beta \neq 1+\epsilon$, and integrate it over Ω, obtaining

$$-\frac{c_V}{2-\epsilon}\int_\Omega (\theta^{2-\epsilon})_t\,dx\;-\int_\Omega \theta^{1-\epsilon}(\chi+\lambda)\Delta\chi\,dx\;+\int_\Omega k(\theta)\nabla\frac{1}{\theta}\cdot\nabla\theta^{-\epsilon}\,dx = 0\,,$$

which can be rewritten as

$$\int_\Omega k(\theta)\epsilon\theta^{1-\epsilon}\Big|\nabla\frac{1}{\theta}\Big|^2\,dx = \frac{c_V}{2-\epsilon}\frac{d}{dt}\int_\Omega \theta^{2-\epsilon}\,dx\;+\int_\Omega \theta^{1-\epsilon}(\chi+\lambda)\Delta\chi\,dx\,. \tag{2.93}$$

Integrating (2.93) with respect to time,

$$\begin{aligned}&\frac{c_V}{2-\epsilon}\int_\Omega \theta_0^{2-\epsilon}\,dx\;+\int_0^T\!\!\int_\Omega k(\theta)\epsilon\theta^{1-\epsilon}\Big|\nabla\frac{1}{\theta}\Big|^2\,dxdt = \\ &\frac{c_V}{2-\epsilon}\int_\Omega \theta^{2-\epsilon}(T)\,dx\;+\int_0^T\!\!\int_\Omega \theta^{1-\epsilon}(\chi+\lambda)\Delta\chi\,dxdt\,.\end{aligned}$$

whence it follows that

$$\begin{aligned}&\int_0^T\!\!\int_\Omega k(\theta)\epsilon\theta^{1-\epsilon}\Big|\nabla\frac{1}{\theta}\Big|^2\,dxdt\\ &\le \frac{c_V}{2-\epsilon}\int_\Omega (1+\theta^2(T))\,dx\;+\int_0^T\!\!\int_\Omega |(1+\theta)(\chi+\lambda)\Delta\chi|\,dxdt\\ &\le \frac{c_V}{2-\epsilon}\int_\Omega (1+\theta^2(T))\,dx\;+\int_0^T\!\!\int_\Omega |(\chi+\lambda)\Delta\chi|\,dxdt\;+\int_\Omega |\theta(\chi+\lambda)\Delta\chi|\,dxdt\,.\end{aligned}$$

Using (2.24), (2.90) and (2.92), we deduce

$$\int_0^T\!\!\int_\Omega k(\theta)\epsilon\theta^{1-\epsilon}\Big|\nabla\frac{1}{\theta}\Big|^2\,dxdt \le c\,. \tag{2.94}$$

Since k is given by (2.16), (2.94) becomes

$$\int_0^T\!\!\int_\Omega \epsilon(k_0\theta^{1-\epsilon} + k_1\theta^{1+\beta-\epsilon})\Big|\nabla\frac{1}{\theta}\Big|^2\,dxdt =$$
$$= \int_0^T\!\!\int_\Omega \epsilon(k_0\theta^{-3-\epsilon} + k_1\theta^{\beta-3-\epsilon})|\nabla\theta|^2\,dxdt \le c\,,$$

whence

$$\int_0^T\!\!\int_\Omega \theta^{\beta-3-\epsilon}|\nabla\theta|^2\,dxdt = \frac{4}{(\beta-1-\epsilon)^2}\int_0^T\!\!\int_\Omega |\nabla\theta^{\frac{\beta-1-\epsilon}{2}}|^2\,dxdt \le c\,,$$

which is

$$\|\nabla\theta^{\frac{\beta-1-\epsilon}{2}}\|_{L^2(0,T;\,L^2(\Omega))} \le c_\epsilon\,, \tag{2.95}$$

where $c_\epsilon \equiv |\beta-1-\epsilon|c$.
Using the Poincaré-Wirtinger inequality, from (2.32) and (2.59) it follows that

$$\Big\|\frac{1}{\theta}\Big\|_{L^2(0,T;\,H^1(\Omega))} \le c\,, \tag{2.96}$$

whence, using the continuous embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$,

$$\Big\|\frac{1}{\theta}\Big\|_{L^2(0,T;\,L^6(\Omega))} \le c\,. \tag{2.97}$$

Using once more the Poincaré-Wirtinger inequality and (2.32), from (2.65) and (2.67) we can deduce that

$$\Big\|\frac{1}{\theta^2}\Big\|_{L^2(0,T;\,H^1(\Omega))} \le c \tag{2.98}$$

and that

$$\Big\|\frac{1}{\theta^{2-\beta}}\Big\|_{L^2(0,T;\,H^1(\Omega))} \le c\,, \tag{2.99}$$

respectively. Moreover, using the continuous emebedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, (2.98) and (2.99) yield

$$\Big\|\frac{1}{\theta}\Big\|_{L^4(0,T;\,L^{12}(\Omega))} = \Big\|\frac{1}{\theta^2}\Big\|^{\frac{1}{2}}_{L^2(0,T;\,L^6(\Omega))} \le c \tag{2.100}$$

and

$$\Big\|\frac{1}{\theta^{2-\beta}}\Big\|_{L^2(0,T;\,L^6(\Omega))} \le c\,, \tag{2.101}$$

respectively.

Consequences for $\beta \in [0,1)$

Consider the case when $\beta \in [0,1)$ in (2.16). Note that $\nabla\theta^{\beta-1} = (1-\beta)\theta^\beta\nabla\frac{1}{\theta}$. Using Hölder's inequality together with (2.24) and (2.32), we obtain

$$\|\nabla\theta^{\beta-1}\|_{L^2(0,T;\,L^{\frac{2}{\beta+1}}(\Omega))} \le (1-\beta)\|\theta^\beta\|_{L^\infty(0,T;\,L^{\frac{2}{\beta}}(\Omega))}\Big\|\nabla\frac{1}{\theta}\Big\|_{L^2(0,T;\,L^2(\Omega))} \le c\,. \tag{2.102}$$

Since $\beta \in [0,1)$, (2.59) and (2.102) together with the Poincaré-Wirtinger inequality yield

$$\|\theta^{\beta-1}\|_{L^2(0,T;\,W^{1,\frac{2}{\beta+1}}(\Omega))} \le c\,. \tag{2.103}$$

Let $v \in W^{1,p}(\Omega)$, $p > 3$. Multiplying (2.11) by $\frac{1}{\theta^{2-\beta}}v$ and integrating over Ω, we obtain

$$\frac{c_V}{\beta-1}\langle(\theta^{\beta-1})_t, v\rangle_{(W^{1,p})'(\Omega),\,W^{1,p}(\Omega)} = \int_\Omega \nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\cdot\nabla\Big(\frac{1}{\theta^{2-\beta}}v\Big)\,dx + \\ + \int_\Omega \frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\cdot\nabla\Big(\frac{1}{\theta^{2-\beta}}v\Big)\,dx + \int_\Omega \frac{|\nabla\chi|^2}{\theta^{2-\beta}}v\,dx + \int_\Omega \frac{k(\theta)}{\theta^{2-\beta}}\Big|\nabla\frac{1}{\theta}\Big|^2 v\,dx\,,$$

namely,

$$\frac{c_V}{\beta-1}\langle(\theta^{\beta-1})_t, v\rangle_{(W^{1,p})'(\Omega),\,W^{1,p}(\Omega)} \\ = \int_\Omega \nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\cdot\Big(v\nabla\frac{1}{\theta^{2-\beta}}+\frac{1}{\theta^{2-\beta}}\nabla v\Big)\,dx + \\ + \int_\Omega \frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\cdot\Big(\Big(1+\frac{1}{2-\beta}\Big)v\nabla\frac{1}{\theta^{2-\beta}}+\frac{1}{\theta^{2-\beta}}\nabla v\Big)\,dx + \int_\Omega \frac{|\nabla\chi|^2}{\theta^{2-\beta}}v\,dx\,. \tag{2.104}$$

Firstly, consider the first right hand side term of (2.104). Using Hölder's inequality, we can deduce

$$\int_\Omega \Big|\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\cdot\Big(v\nabla\frac{1}{\theta^{2-\beta}}+\frac{1}{\theta^{2-\beta}}\nabla v\Big)\Big|\,dx \\ \le \Big\|\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\Big\|_{L^2(\Omega)}\Big(\Big\|\nabla\frac{1}{\theta^{2-\beta}}\Big\|_{L^2(\Omega)}\|v\|_{L^\infty(\Omega)}+\Big\|\frac{1}{\theta^{2-\beta}}\Big\|_{L^6(\Omega)}\|\nabla v\|_{L^3(\Omega)}\Big) \\ \le c_p\Big\|\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\Big\|_{L^2(\Omega)}\Big\|\frac{1}{\theta^{2-\beta}}\Big\|_{H^1(\Omega)}\|v\|_{W^{1,p}(\Omega)}\,, \tag{2.105}$$

where in the last inequality we used the continuous embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\bar\Omega)$, which holds true for $p > 3$, and we denoted by c_p an embedding constant depending on $p > 3$ and possibly exploding as $p \searrow 3$.
As for the second right hand side term of (2.104), using (2.48), we can rewrite it as

$$\int_\Omega \frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\cdot\Big(\frac{3-\beta}{2-\beta}v\nabla\frac{1}{\theta^{2-\beta}}+\frac{1}{\theta^{2-\beta}}\nabla v\Big)\,dx \;= \\ = \int_\Omega \nabla\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\cdot\Big(\frac{3-\beta}{2-\beta}v\nabla\frac{1}{\theta^{2-\beta}}+\frac{1}{\theta^{2-\beta}}\nabla v\Big)\,dx\,.$$

Using Hölder's inequality, we obtain

$$\int_\Omega \Big|\frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\cdot\Big(\frac{3-\beta}{2-\beta}v\nabla\frac{1}{\theta^{2-\beta}}+\frac{1}{\theta^{2-\beta}}\nabla v\Big)\Big|\,dx \\ \le \Big\|\nabla\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|_{L^2(\Omega)}\Big(\frac{3-\beta}{2-\beta}\Big\|\nabla\frac{1}{\theta^{2-\beta}}\Big\|_{L^2(\Omega)}\|v\|_{L^\infty(\Omega)}+ \\ +\Big\|\frac{1}{\theta^{2-\beta}}\Big\|_{L^6(\Omega)}\|\nabla v\|_{L^3(\Omega)}\Big) \\ \le c_p\Big\|\nabla\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|_{L^2(\Omega)}\Big\|\frac{1}{\theta^{2-\beta}}\Big\|_{H^1(\Omega)}\|v\|_{W^{1,p}(\Omega)} \tag{2.106}$$

where in the last inequality we used once more Sobolev's embeddings.
Finally, let us consider the third right hand side term of (2.104). Hölder's inequality and $H^1(\Omega) \hookrightarrow L^6(\Omega)$ yield

$$\int_\Omega \Big| \frac{|\nabla\chi|^2}{\theta^{2-\beta}} v \Big| \, dx \le \Big\| \frac{\nabla\chi}{\theta^{1-\frac{\beta}{2}}} \Big\|^2_{L^2(\Omega)} \|v\|_{L^\infty(\Omega)} \le c_p \Big\| \frac{\nabla\chi}{\theta^{1-\frac{\beta}{2}}} \Big\|^2_{L^2(\Omega)} \|v\|_{W^{1,p}(\Omega)} \,. \tag{2.107}$$

From (2.104), using (2.105), (2.106) and (2.107), we can deduce that

$$\begin{aligned} \frac{c_V}{\beta-1} |\langle(\theta^{\beta-1})_t, v\rangle_{(W^{1,p})'(\Omega), W^{1,p}(\Omega)}| \le c_p \Big(\Big\| \nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big)\Big\|_{L^2(\Omega)} \Big\| \frac{1}{\theta^{2-\beta}} \Big\|_{H^1(\Omega)} + \\ + \Big\| \nabla\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|_{L^2(\Omega)} \Big\| \frac{1}{\theta^{2-\beta}} \Big\|_{H^1(\Omega)} + \\ + \Big\| \frac{\nabla\chi}{\theta^{1-\frac{\beta}{2}}} \Big\|^2_{L^2(\Omega)} \Big) \|v\|_{W^{1,p}(\Omega)} \,, \end{aligned}$$

which implies

$$\begin{aligned} &\frac{c_V}{\beta-1} \|(\theta^{\beta-1})_t\|_{(W^{1,p})'(\Omega)} = \sup_{\substack{v\in W^{1,p}(\Omega) \\ v\neq 0}} \frac{c_V}{\beta-1} \frac{|\langle(\theta^{\beta-1})_t, v\rangle_{(W^{1,p}(\Omega))', W^{1,p}(\Omega)}|}{\|v\|_{W^{1,p}(\Omega)}} \\ &\le c_p \Big(\Big\| \nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big)\Big\|_{L^2(\Omega)} \Big\| \frac{1}{\theta^{2-\beta}} \Big\|_{H^1(\Omega)} + \\ &\quad + \Big\| \nabla\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|_{L^2(\Omega)} \Big\| \frac{1}{\theta^{2-\beta}} \Big\|_{H^1(\Omega)} + \Big\| \frac{\nabla\chi}{\theta^{1-\frac{\beta}{2}}} \Big\|^2_{L^2(\Omega)} \Big) \,. \end{aligned} \tag{2.108}$$

Integrating (2.108) with respect to time and using Hölder's inequality, we obtain

$$\begin{aligned} &\frac{c_V}{\beta-1} \int_0^T \|(\theta^{\beta-1})_t\|_{(W^{1,p})'(\Omega)} \, dt \\ &\le c_p \Big(\Big\| \nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big)\Big\|_{L^2(0,T;\,L^2(\Omega))} \Big\| \frac{1}{\theta^{2-\beta}} \Big\|_{L^2(0,T;\,H^1(\Omega))} + \\ &\quad + \Big\| \nabla\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|_{L^2(0,T;\,L^2(\Omega))} \Big\| \frac{1}{\theta^{2-\beta}} \Big\|_{L^2(0,T;\,H^1(\Omega))} + \\ &\quad + \Big\| \frac{\nabla\chi}{\theta^{1-\frac{\beta}{2}}} \Big\|^2_{L^2(0,T;\,L^2(\Omega))} \Big) \,. \end{aligned}$$

Taking into account (2.68), (2.69), (2.99) and (2.64), we can conclude that

$$\|(\theta^{\beta-1})_t\|_{L^1(0,T;\,(W^{1,p}(\Omega))')} \le c_p \,, \quad \text{for every } p > 3 \,. \tag{2.109}$$

Consequences for $\beta = 1$

Consider the case when $\beta = 1$ in (2.16). Note that $\nabla \log\theta = \theta\nabla\frac{1}{\theta}$. Using Hölder's inequality together with (2.24) and (2.32), we obtain

$$\|\nabla \log\theta\|_{L^2(0,T;\,L^1(\Omega))} \le \|\theta\|_{L^\infty(0,T;\,L^2(\Omega))} \Big\| \nabla\frac{1}{\theta} \Big\|_{L^2(0,T;\,L^2(\Omega))} \le c \,. \tag{2.110}$$

On the other hand, since $|\log s| \leq s + \frac{1}{s}$, $\forall s \in \mathbb{R}^+$, then

$$\|\log\theta\|_{L^\infty(0,T;\,L^1(\Omega))} \leq \|\theta\|_{L^\infty(0,T;\,L^1(\Omega))} + \Big\|\frac{1}{\theta}\Big\|_{L^\infty(0,T;\,L^1(\Omega))} \leq c\,, \tag{2.111}$$

where in the last inequality we used (2.24) and (2.59). Using the Poincaré-Wirtinger inequality, (2.110) and (2.111) yield

$$\|\log\theta\|_{L^2(0,T;\,W^{1,1}(\Omega))} \leq c\,. \tag{2.112}$$

Let $v \in W^{1,p}(\Omega)$, $p > 3$. Multiplying (2.11) by $\frac{1}{\theta}v$ and integrating over Ω, we obtain

$$c_V\langle(\log\theta)_t, v\rangle_{(W^{1,p})'(\Omega),\,W^{1,p}(\Omega)} = \int_\Omega \nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big)\cdot\nabla\Big(\frac{1}{\theta}v\Big)\,dx + \\ + \int_\Omega \frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\cdot\nabla\Big(\frac{1}{\theta}v\Big)\,dx + \int_\Omega \frac{|\nabla\chi|^2}{\theta}v\,dx + \int_\Omega \frac{k(\theta)}{\theta}\Big|\nabla\frac{1}{\theta}\Big|^2 v\,dx\,,$$

namely,

$$c_V\langle(\log\theta)_t, v\rangle_{(W^{1,p})'(\Omega),\,W^{1,p}(\Omega)} = \int_\Omega \nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big)\cdot\Big(v\nabla\frac{1}{\theta} + \frac{1}{\theta}\nabla v\Big)\,dx + \\ + \int_\Omega \frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\cdot\Big(2v\nabla\frac{1}{\theta} + \frac{1}{\theta}\nabla v\Big)\,dx + \int_\Omega \frac{|\nabla\chi|^2}{\theta}v\,dx\,. \tag{2.113}$$

Note that the right hand side of (2.113) is equal to that of (2.104) with $\beta = 1$. Hence, proceeding analogously as done in the case $\beta \in [0, 1)$, it is not difficult to conclude that

$$\|(\log\theta)_t\|_{L^1(0,T;\,(W^{1,p})'(\Omega))} \leq c_p\,, \quad \text{for every } p > 3\,, \tag{2.114}$$

where c_p is an embedding constant depending on $p > 3$ possibly exploding as $p \searrow 3$.

Consequences for $\beta \in (1, 2)$

Consider the case when $\beta \in (1, 2)$ in (2.16). Let $\epsilon > 0$ be such that $\beta > 1+\epsilon$. Then, noting that $\frac{4}{\beta-1-\epsilon} > 2$, from (2.24) we deduce

$$\|\theta^{\frac{\beta-1-\epsilon}{2}}\|_{L^2(0,T;\,L^2(\Omega))} \leq c\|\theta^{\frac{\beta-1-\epsilon}{2}}\|_{L^\infty(0,T;\,L^{\frac{4}{\beta-1-\epsilon}}(\Omega))} \leq c_\epsilon\,. \tag{2.115}$$

From (2.95) and (2.115) it follows that

$$\|\theta^{\frac{\beta-1-\epsilon}{2}}\|_{L^2(0,T;\,H^1(\Omega))} \leq c_\epsilon\,, \tag{2.116}$$

whence, using the continuous embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$\|\theta^{\frac{\beta-1-\epsilon}{2}}\|_{L^2(0,T;\,L^6(\Omega))} \leq c_\epsilon\,. \tag{2.117}$$

Using (2.115) and (2.117) together with the continuous embedding

$$L^\infty(0,\,T;\,L^{\frac{4}{\beta-1-\epsilon}}(\Omega)) \cap L^2(0,\,T;\,L^6(\Omega)) \hookrightarrow L^q(\Omega\times(0,\,T)),$$

where $q = \frac{2}{3}\frac{3\beta+1-3\epsilon}{\beta-1-\epsilon}$, we can conclude that

$$\|\theta\|_{L^{\bar q}(\Omega\times(0,T))} \le c, \quad \text{where } \bar q = \frac{3\beta+1-3\epsilon}{3}\,. \tag{2.118}$$

Observe that

$$\bar q > 2 \iff \beta > \frac{5}{3} + \epsilon\,. \tag{2.119}$$

Thus, for $\beta > \frac{5}{3}$ we can choose a suitable $\epsilon > 0$ such that $\beta > \frac{5}{3} + \epsilon$ and then we obtain an additional regularity for θ that will play a crucial role in Subsection 2.4.1. In particular, consider $k(\theta)\nabla\frac{1}{\theta}$ which can be rewritten as $\theta\frac{k(\theta)}{\theta}\nabla\frac{1}{\theta} = \theta\nabla\Big(\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)$ due to (2.48). For $\beta > \frac{5}{3}$, we can choose a suitable $\epsilon > 0$ such that $\beta > \frac{5}{3} + \epsilon$, then (2.118) and (2.68) together with Hölder's inequality yield

$$\Big\|k(\theta)\nabla\frac{1}{\theta}\Big\|_{L^{\bar r}(\Omega\times(0,T))} \le c\,, \text{ where } \bar r = \frac{2\bar q}{2+\bar q} \in \Big(1, \frac{14}{13}\Big) \text{ depends only on } \beta\,. \tag{2.120}$$

Let $\epsilon > 0$ still be such that $\beta > 1+\epsilon$. From (2.24) we also deduce that

$$\|\theta^{\frac{3-\beta+\epsilon}{2}}\|_{L^\infty(0,T;\,L^{\frac{4}{3-\beta+\epsilon}}(\Omega))} \le c_\epsilon\,. \tag{2.121}$$

Observe that $\nabla\theta = \frac{2}{\beta-1-\epsilon}\theta^{\frac{3-\beta+\epsilon}{2}}\nabla\theta^{\frac{\beta-1-\epsilon}{2}}$. Using Hölder's inequality together with (2.95) and (2.121), we obtain

$$\begin{aligned}&\|\nabla\theta\|_{L^2(0,T;\,L^{\frac{4}{5-\beta+\epsilon}}(\Omega))}\\ &\le \tfrac{2}{\beta-1-\epsilon}\|\theta^{\frac{3-\beta+\epsilon}{2}}\|_{L^\infty(0,T;\,L^{\frac{4}{3-\beta+\epsilon}}(\Omega))}\|\nabla\theta^{\frac{\beta-1-\epsilon}{2}}\|_{L^2(0,T;\,L^2(\Omega))}\\ &\le c_\epsilon\,.\end{aligned} \tag{2.122}$$

Note that, since $\epsilon > 0$ is such that $\beta > 1+\epsilon$ and $\beta < 2$, then $1 < \frac{4}{5-\beta+\epsilon} < \frac{4}{3}$. Hence, using (2.24) and (2.122) together with the Poincaré-Wirtinger inequality, we can conclude that

$$\|\theta\|_{L^2(0,T;\,W^{1,\frac{4}{5-\beta+\epsilon}}(\Omega))} \le c_\epsilon\,. \tag{2.123}$$

Testing (2.11) by $v \in W^{1,p}(\Omega)$, $p > 3$, we obtain

$$\begin{aligned}c_V\langle\theta_t, v\rangle_{(W^{1,p})'(\Omega),\,W^{1,p}(\Omega)} &= \int_\Omega \nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big)\cdot\nabla v\,dx + \int_\Omega \frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\cdot\nabla v\,dx +\\ &+ \int_\Omega |\nabla\chi|^2 v\,dx + \int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2 v\,dx\,.\end{aligned} \tag{2.124}$$

Using Hölder's inequality and (2.48), we deduce

$$\begin{aligned}\int_\Omega \Big|\nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big)\cdot\nabla v\Big|\,dx &\le \Big\|\nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big)\Big\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}\\ &\le c_p\Big\|\nabla\Big(\frac{\chi^2}{2} + \lambda\chi\Big)\Big\|_{L^2(\Omega)}\|v\|_{W^{1,p}(\Omega)}\,,\end{aligned}$$

$$\int_\Omega \Big|\frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\cdot\nabla v\Big|\,dx \le \Big\|\nabla\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}$$
$$\le c_p\Big\|\nabla\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|_{L^2(\Omega)}\|v\|_{W^{1,p}(\Omega)}\,,$$

$$\int_\Omega ||\nabla\chi|^2 v|\,dx \le \||\nabla\chi|^2\|_{L^1(\Omega)}\|v\|_{L^\infty(\Omega)} = \|\nabla\chi\|^2_{L^2(\Omega)}\|v\|_{L^\infty(\Omega)}$$
$$\le c_p\|\nabla\chi\|^2_{L^2(\Omega)}\|v\|_{W^{1,p}(\Omega)}\,,$$

$$\int_\Omega \Big|k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2 v\Big|\,dx \le \|v\|_{L^\infty(\Omega)}\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\,dx$$
$$\le c_p\|v\|_{W^{1,p}(\Omega)}\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\,dx\,,$$

where in the last inequalities we used once more the continuous embeddings $H^1(\Omega)\hookrightarrow L^6(\Omega)$ and $W^{1,p}(\Omega)\hookrightarrow\mathcal{C}(\bar\Omega)$, which holds true for $p>3$, and we denoted by c_p an embedding constant depending on $p>3$ and possibly exploding as $p\searrow 3$. From (2.124) it then follows that

$$c_V|\langle\theta_t,v\rangle_{(W^{1,p})'(\Omega),W^{1,p}(\Omega)}| \le c_p\|v\|_{W^{1,p}(\Omega)}\Big(\Big\|\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\Big\|_{L^2(\Omega)}+$$
$$+\Big\|\nabla\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|_{L^2(\Omega)}+\|\nabla\chi\|^2_{L^2(\Omega)}+\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\,dx\Big)\,,$$

which implies

$$c_V\|\theta_t\|_{(W^{1,p})'(\Omega)} = \sup_{\substack{v\in W^{1,p}(\Omega)\\ v\neq 0}}\frac{c_V|\langle\theta_t,v\rangle_{(W^{1,p}(\Omega))',W^{1,p}(\Omega)}|}{\|v\|_{W^{1,p}(\Omega)}}$$
$$\le c_p\Big(\Big\|\nabla\Big(\frac{\chi^2}{2}+\lambda\chi\Big)\Big\|_{L^2(\Omega)}+\Big\|\nabla\Big(\frac{k_0}{2\theta^2}+\frac{k_1}{(2-\beta)\theta^{2-\beta}}\Big)\Big\|_{L^2(\Omega)}+$$
$$+\|\nabla\chi\|^2_{L^2(\Omega)}+\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\,dx\Big)\,. \tag{2.125}$$

Integrating (2.125) with respect to time and using the regularities given by (2.69), (2.68), (2.30) and (2.31), we can conclude that

$$\|\theta_t\|_{L^1(0,T;(W^{1,p})'(\Omega))}\le c_p\,,\quad\text{for every } p>3\,. \tag{2.126}$$

2.4 Weak sequential stability

In this section, we assume to have a sequence $\{(u_n,\chi_n,\theta_n)\}_n$ of solutions satisfying a proper approximation of the "strong" system of equations (2.1)-(2.2)-(2.3). This family is assumed to comply with the a priori bounds proved in Section 2.3 uniformly with respect to $n\in\mathbb{N}$. Our aim is showing, by weak compactness arguments, that at least a subsequence converges in a suitable way to an entropy solution to our problem, i.e., to a limit triple (u, χ, θ) satisfying the statement given in Definition 2.2.1. Furthermore, in the subcase when $\beta \in (\frac{5}{3}, 2)$ in (2.16), we will be able to show the convergence to a weak solution, according to Definition 2.2.2. Actually, to further simplify the notation, we intend that all the convergence relations appearing in the following are to be considered up to the extraction of (non-relabelled) subsequences.

Throughout the section, we will assume the reader to be familiar with the notions and the results presented in Appendix A.2-A.12. However, sometimes we will explicitly recall them in order to emphasize their use.
As for the notation, we will denote the weak convergence by $\rightharpoonup$ and the weak star one by $\overset{*}{\rightharpoonup}$.

Collecting the bounds proved in the previous section, we deduce the following convergence relations. In particular, from (2.83), (2.86) and (2.60) we infer

$$u_n \overset{*}{\rightharpoonup} u \ \text{ in } L^\infty(0, T; W^{2,\frac{4}{3}}(\Omega)) \cap L^3(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))\,, \tag{2.127}$$

whereas, from (2.73) and (2.79),

$$\chi_n \rightharpoonup \chi \ \text{ in } L^2(0, T; H^2(\Omega)) \cap L^4(0, T; L^{12}(\Omega))\,, \tag{2.128}$$

and from (2.24)

$$\theta_n \overset{*}{\rightharpoonup} \theta \ \text{ in } L^\infty(0, T; L^2(\Omega)). \tag{2.129}$$

Noting that

$$W^{2,\frac{4}{3}}(\Omega) \subset\subset L^q(\Omega) \hookrightarrow L^1(\Omega)\,, \quad 1 < q < 12\,,$$

(2.127) together with the Aubin-Lions-Simon Lemma yield

$$u_n \to u \ \text{ strongly in } \mathcal{C}([0, T]; L^q(\Omega)), \ \ 1 < q < 12\,. \tag{2.130}$$

From (2.130) we can deduce a convergence relation for $\{f(u_n)\}_n$, where f is given by (2.15). Indeed, observe that $f(u_n) - f(u)$ can be rewritten as

$$f(u_n) - f(u) = \Big(\int_0^1 f'(su_n + (1-s)u)\, ds \Big)(u_n - u)\,,$$

where $f'(su_n + (1-s)u) = a_1\rho(\rho-1)|su_n + (1-s)u|^{\rho-2} - 2a_2$, $a_1, a_2 > 0$, $\rho \in [3, \frac{7}{2}]$, by (2.41). Using Hölder's inequality, we obtain

$$\|f(u_n) - f(u)\|_{L^r(\Omega)} \le \Big\| \int_0^1 f'(su_n + (1-s)u)\, ds \Big\|_{L^s(\Omega)} \|u_n - u\|_{L^q(\Omega)}\,,$$

where r is such that $\frac{1}{r} = \frac{1}{s} + \frac{1}{q}$ for $s \le \frac{12}{\rho-2}$ and $q < 12$. Thus, $r < \frac{12}{\rho-1}$. It follows that

$$\|f(u_n) - f(u)\|_{L^r(\Omega)} \le \Big(\int_0^1 \|f'(su_n + (1-s)u)\|_{L^s(\Omega)} ds \Big) \|u_n - u\|_{L^q(\Omega)}\,,$$

namely,

$$\begin{aligned}
&\|f(u_n)-f(u)\|_{L^r(\Omega)} \\
&\le \Big(\int_0^1 \|a_1\rho(\rho-1)|su_n+(1-s)u|^{\rho-2}-2a_2\|_{L^s(\Omega)}ds\Big)\|u_n-u\|_{L^q(\Omega)}\,.
\end{aligned}$$

Due to (2.83) and the Sobolev embedding $W^{2,\frac{4}{3}}(\Omega)\hookrightarrow L^q(\Omega)$, $q\le 12$, $\{|u_n|^{\rho-2}\}_n$ and $|u|^{\rho-2}$ are uniformly bounded in $L^\infty((0,T);\,L^s(\Omega))$, $s\le\frac{12}{\rho-2}$. Thus, we obtain

$$\|f(u_n)-f(u)\|_{L^r(\Omega)}\le c_\rho\|u_n-u\|_{L^q(\Omega)}\,,$$

whence, using the convergence relation (2.130), we can conclude that

$$f(u_n)\to f(u)\ \text{ strongly in } \mathcal{C}([0,T];\,L^r(\Omega)),\ \ 1\le r<\frac{12}{\rho-1}\,, \tag{2.131}$$

where $\frac{12}{\rho-1}\in[\frac{24}{5},6]$ for $\rho\in[3,\frac{7}{2}]$.
Furthermore, since

$$W^{2,\frac{4}{3}}(\Omega)\subset\subset W^{1,q^*}(\Omega)\hookrightarrow L^1(\Omega)\,,\ \ 1\le q^*<\frac{12}{5}\,,$$

and

$$H^2(\Omega)\subset\subset W^{1,r^*}(\Omega)\hookrightarrow L^1(\Omega)\,,\ \ 1\le r^*<6\,,$$

then we can use again (2.127) and the Aubin-Lions-Simon Lemma to conclude that

$$u_n\to u\ \text{ strongly in } \mathcal{C}([0,T];\,W^{1,q^*}(\Omega)),\ \ 1\le q^*<\frac{12}{5}\,, \tag{2.132}$$

and that

$$u_n\to u\ \text{ strongly in } L^3([0,T];\,W^{1,r^*}(\Omega)),\ \ 1\le r^*<6\,. \tag{2.133}$$

We now show that $\{\frac{1}{\theta_n}\}_n$ converges to $\frac{1}{\theta}$ almost everywhere in $\Omega\times(0,T)$.
To this aim, let us consider the cases $0<\beta<1$, $\beta=1$ and $1<\beta<2$ separately.

- $0<\beta<1:$ From (2.103) and (2.109) we deduce that $\{\theta_n^{\beta-1}\}_n$ and $\{(\theta_n^{\beta-1})_t\}_n$ are uniformly bounded in $L^2(0,T;\,W^{1,\frac{2}{\beta+1}}(\Omega))$ and in $L^1(0,T;\,(W^{1,p})'(\Omega))$, for any $p>3$, respectively. Noting that

  $$W^{1,\frac{2}{\beta+1}}(\Omega)\subset\subset L^{s^*}(\Omega)\hookrightarrow (W^{1,p})'(\Omega)\,,\ 1\le s^*<\frac{6}{3\beta+1},\ p>3\,,$$

  and using the Aubin-Lions-Simon Lemma, we obtain

  $$\theta_n^{\beta-1}\to\eta\ \text{ strongly in } L^2(0,T;\,L^{s^*}(\Omega)),\ \ 1\le s^*<\frac{6}{3\beta+1}\,.$$

  In particular, $\theta_n^{\beta-1}\to\eta$ almost everywhere in $\Omega\times(0,T)$, thus $\theta_n\to\eta^{\frac{1}{\beta-1}}$ almost everywhere in $\Omega\times(0,T)$. Since (2.24) holds true, we deduce that $\theta_n\rightharpoonup$

$\eta^{\frac{1}{\beta-1}}$ weakly in $L^p(0, T; L^2(\Omega))$, $p < +\infty$, (cf. Appendix A.10). On the other hand, we have convergence relation (2.129). Hence, due to the uniqueness of the weak limit, we can conclude that $\theta = \eta^{\frac{1}{\beta-1}}$ almost everywhere in $\Omega \times (0, T)$. It then follows that

$$\theta_n^{\beta-1} \to \theta^{\beta-1} \text{ strongly in } L^2(0, T; L^{s^*}(\Omega)), \ 1 \le s^* < \frac{6}{3\beta+1}. \tag{2.134}$$

Thus, $\theta_n^{\beta-1} \to \theta^{\beta-1}$ almost everywhere in $\Omega \times (0, T)$, whence $\theta_n \to \theta$ and $\frac{1}{\theta}_n \to \frac{1}{\theta}$ almost everywhere in $\Omega \times (0, T)$.

- $\beta = 1$: From (2.112) and (2.114) we deduce that $\{\log \theta_n\}_n$ and $\{(\log \theta_n)_t\}_n$ are uniformly bounded in $L^2(0, T; W^{1,1}(\Omega))$ and in $L^1(0, T; (W^{1,p})'(\Omega))$, for any $p > 3$, respectively. Noting that

$$W^{1,1}(\Omega) \subset\subset L^{s^*}(\Omega) \hookrightarrow (W^{1,p})'(\Omega), \ 1 \le s^* < \frac{3}{2}, \ p > 3,$$

and using the Aubin-Lions-Simon Lemma, proceeding as done in the previous case, we can conclude that

$$\log \theta_n \to \log \theta \text{ strongly in } L^2(0, T; L^{s^*}(\Omega)), \ 1 \le s^* < \frac{3}{2}. \tag{2.135}$$

It follows that $\log \theta_n \to \log \theta$ almost everywhere in $\Omega \times (0, T)$, whence $\theta_n \to \theta$ and $\frac{1}{\theta}_n \to \frac{1}{\theta}$ almost everywhere in $\Omega \times (0, T)$.

- $1 < \beta < 2$: From (2.123) and (2.126) we deduce that, for any arbitrary small $\epsilon > 0$ such that $\beta > 1 + \epsilon$, $\{\theta_n\}_n$ and $\{(\theta_n)_t\}_n$ are uniformly bounded in $L^2(0, T; W^{1,\frac{4}{5-\beta+\epsilon}}(\Omega))$ and in $L^1(0, T; (W^{1,p})'(\Omega))$, for any $p > 3$, respectively. Noting that

$$W^{1,\frac{4}{5-\beta+\epsilon}}(\Omega) \subset\subset L^{s^*}(\Omega) \hookrightarrow (W^{1,p})'(\Omega), \ 1 \le s^* < \frac{12}{11 - 3\beta + 3\epsilon}, \ p > 3,$$

and using Aubin-Lions-Simon Lemma, taking ϵ small enough, we can conclude that

$$\theta_n \to \theta \text{ strongly in } L^2(0, T; L^{s^*}(\Omega)), \ 1 \le s^* < \frac{12}{11 - 3\beta}. \tag{2.136}$$

As above, it follows that $\theta_n \to \theta$ almost everywhere in $\Omega \times (0, T)$. Thus, $\frac{1}{\theta}_n \to \frac{1}{\theta}$ almost everywhere in $\Omega \times (0, T)$.

Next, from (2.100) we deduce that $\{\frac{1}{\theta_n}\}_n$ is uniformly bounded in $L^4(0, T; L^{12}(\Omega))$. Since $\frac{1}{\theta}_n \to \frac{1}{\theta}$ almost everywhere in $\Omega \times (0, T)$, we can conclude that (cf. Appendix A.10)

$$\frac{1}{\theta_n} \to \frac{1}{\theta} \text{ strongly in } L^s(0, T; L^q(\Omega)), \ 1 \le s < 4, \ 1 \le q < 12. \tag{2.137}$$

From (2.96) we deduce that $\{\frac{1}{\theta_n}\}_n$ is uniformly bounded in $L^2(0, T; H^1(\Omega))$, hence it weakly converges to a function ζ in $L^2(0, T; H^1(\Omega))$. Since $L^2(0, T; H^1(\Omega)) \hookrightarrow$

$L^s(0, T; L^q(\Omega))$, $1 \le s \le 2$, $1 \le q \le 6$, from (2.137) and the uniqueness of the weak limit it follows that $\zeta = \frac{1}{\theta}$ almost everywhere in $\Omega \times (0, T)$. Since ∇ can be considered as a continuous linear operator from $L^2(0, T; H^1(\Omega))$ to $L^2(0, T; L^2(\Omega))$, then we can conclude that

$$\nabla \frac{1}{\theta_n} \rightharpoonup \nabla \frac{1}{\theta} \text{ in } L^2(0, T; L^2(\Omega))\,. \tag{2.138}$$

From (2.101) we deduce that $\{\frac{1}{\theta_n^{2-\beta}}\}_n$ is uniformly bounded in $L^2(0, T; L^6(\Omega))$. Since $\frac{1}{\theta}_n \to \frac{1}{\theta}$ almost everywhere in $\Omega \times (0, T)$, then $\frac{1}{\theta_n^{2-\beta}} \to \frac{1}{\theta^{2-\beta}}$ almost everywhere in $\Omega \times (0, T)$. Thus, we obtain (cf. Appendix A.10)

$$\frac{1}{\theta_n^{2-\beta}} \rightharpoonup \frac{1}{\theta^{2-\beta}} \text{ in } L^2(0, T; L^6(\Omega))\,.$$

From (2.99) we deduce that $\{\frac{1}{\theta_n^{2-\beta}}\}_n$ is uniformly bounded in $L^2(0, T; H^1(\Omega))$, hence it weakly converges in $L^2(0, T; H^1(\Omega))$. Proceeding as done above, we can conclude that

$$\nabla \frac{1}{\theta_n^{2-\beta}} \rightharpoonup \nabla \frac{1}{\theta^{2-\beta}} \text{ in } L^2(0, T; L^2(\Omega))\,. \tag{2.139}$$

Analogously, if we consider $\{\frac{1}{\theta_n^2}\}_n$, from (2.100) and the almost everywhere convergence we deduce

$$\frac{1}{\theta_n^2} \rightharpoonup \frac{1}{\theta^2} \text{ in } L^2(0, T; L^6(\Omega))\,,$$

then, thanks to (2.98), we can conlude that

$$\nabla \frac{1}{\theta_n^2} \rightharpoonup \nabla \frac{1}{\theta^2} \text{ in } L^2(0, T; L^2(\Omega))\,. \tag{2.140}$$

Note that in Section 2.3, we implicitly assumed the temperature θ to be (almost everywhere) positive. This fact is used in several estimates which, otherwise, would not make sense. Positivity of $\{\theta_n\}_n$ should be shown, indeed, at the n-level, i.e., for the hypothetical regularized problem which we decided not to detail here. We cannot give here a proof of this fact, since this would require to provide the details of the regularization. However, we can at least show that, if $\{\theta_n\}_n$ is almost everywhere positive, and satisfies the estimates given in Section 2.3, then positivity is preserved in the limit. To see this, we first notice that, for $\beta = 1$ we have convergence relation (2.135), hence the integrability of $\log \theta$ allows us to conclude that $\theta > 0$ almost everywhere in $\Omega \times (0, T)$. As for the cases $\beta \in [0, 1)$ or $\beta \in (1, 2)$, note that $|\log s| \le s + \frac{1}{s}$, $\forall s \in \mathbb{R}^+$. Thus, (2.24) and (2.96) imply $\|\log \theta_n\|_{L^2(0, T; L^2(\Omega))} \le c, \forall n \in \mathbb{N}$. Since we showed that $\theta_n \to \theta$ almost everywhere in $\Omega \times (0, T)$, we obtain (cf. Appendix A.10)

$$\log \theta_n \to \log \theta \text{ strongly in } L^r(0, T; L^r(\Omega)),\;\; r < 2\,.$$

Once again, due to the integrability of $\log \theta$, we can conclude that $\theta > 0$ almost everywhere in $\Omega \times (0, T)$.

In order to simplify the notation, from now on we will denote exponents such as $p-\delta$ and $p+\delta$, for a properly chosen small $\delta>0$, by p^- and p^+, respectively. Let us now consider the difference between χ_n and χ_m, $\forall n,m\in\mathbb{N}$, which is given by (2.2), namely,

$$\begin{aligned}\chi_n-\chi_m=&-\frac{\Delta\left(u_n-u_m\right)}{\theta_n}-\Delta u_m\Big(\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big)+\\&+\frac{f\left(u_n\right)-f\left(u_m\right)}{\theta_n}+f\left(u_m\right)\Big(\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big)\,.\end{aligned}\tag{2.141}$$

Testing (2.141) by $v\in W^{1,p}(\Omega)$, $p>3$, and integrating by parts the second term on the right hand side, we obtain

$$\begin{aligned}\int_\Omega(\chi_n-\chi_m)v\,dx=&\int_\Omega\nabla(u_n-u_m)\Big(v\nabla\frac{1}{\theta_n}+\frac{1}{\theta_n}\nabla v\Big)\,dx+\\&-\int_\Omega\Delta u_m\Big(\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big)v\,dx+\int_\Omega\frac{f(u_n)-f(u_m)}{\theta_n}v\,dx+\\&+\int_\Omega f(u_m)\Big(\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big)v\,dx\,.\end{aligned}\tag{2.142}$$

Let us consider the first term on the right hand side of (2.142). Using Hölder's inequality, from (2.96) we deduce

$$\begin{aligned}&\int_\Omega\Big|\nabla(u_n-u_m)\Big(v\nabla\frac{1}{\theta_n}+\frac{1}{\theta_n}\nabla v\Big)\Big|\,dx\\&\le\|\nabla(u_n-u_m)\|_{L^{6^-}(\Omega)}\Big(\Big\|v\nabla\frac{1}{\theta_n}\Big\|_{L^{\frac{6}{5}^+}(\Omega)}+\Big\|\frac{1}{\theta_n}\nabla v\Big\|_{L^{\frac{6}{5}^+}(\Omega)}\Big)\\&\le\|\nabla(u_n-u_m)\|_{L^{6^-}(\Omega)}\Big(\Big\|\nabla\frac{1}{\theta_n}\Big\|_{L^2(\Omega)}\|v\|_{L^{3^+}(\Omega)}+\Big\|\frac{1}{\theta_n}\Big\|_{L^2(\Omega)}\|\nabla v\|_{L^{3^+}(\Omega)}\Big)\\&\le c_p\|\nabla(u_n-u_m)\|_{L^{6^-}(\Omega)}\Big\|\frac{1}{\theta_n}\Big\|_{H^1(\Omega)}\|v\|_{W^{1,p}(\Omega)}\,,\end{aligned}\tag{2.143}$$

where we denoted by c_p an embedding constant depending on $p>3$ and possibly exploding as $p\searrow 3$.
As for the second term on the right hand side of (2.142), using Hölder inequality, it follows that

$$\begin{aligned}\int_0^T\!\!\int_\Omega\Big|\Delta u_m\left(\frac{1}{\theta_n}-\frac{1}{\theta_m}\right)v\Big|\,dxdt&\le\|\Delta u_n\|_{L^2(\Omega)}\Big\|\Big(\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big)v\Big\|_{L^2(\Omega)}\\&\le\|\Delta u_n\|_{L^2(\Omega)}\Big\|\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big\|_{L^2(\Omega)}\|v\|_{L^\infty(\Omega)}\\&\le c_p\|\Delta u_n\|_{L^2(\Omega)}\Big\|\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big\|_{L^2(\Omega)}\|v\|_{W^{1,p}(\Omega)}\,,\end{aligned}\tag{2.144}$$

where in the last inequality we used the continuous embedding $W^{1,p}(\Omega)\hookrightarrow\mathcal{C}(\bar\Omega)$, holding for any $p>3$.

Let us consider the third term on the right hand side of (2.142). Using Hölder's inequality together with the continuous embedding above, we obtain

$$\begin{aligned}\int_\Omega \Big|\frac{f(u_n)-f(u_m)}{\theta_n} v\Big|\, dx &\le \|f(u_n)-f(u_m)\|_{L^2(\Omega)} \Big\|\frac{1}{\theta_n} v\Big\|_{L^2(\Omega)} \\ &\le \|f(u_n)-f(u_m)\|_{L^2(\Omega)} \Big\|\frac{1}{\theta_n}\Big\|_{L^2(\Omega)} \|v\|_{L^\infty(\Omega)} \\ &\le c_p \|f(u_n)-f(u_m)\|_{L^2(\Omega)} \Big\|\frac{1}{\theta_n}\Big\|_{L^2(\Omega)} \|v\|_{W^{1,p}(\Omega)}\,. \end{aligned} \tag{2.145}$$

Lastly, we consider the last term on the right hand side of (2.142). Using Hölder's inequality, we deduce

$$\begin{aligned}\int_\Omega \Big| f(u_m)\left(\frac{1}{\theta_n}-\frac{1}{\theta_m}\right) v\Big|\, dx &\le \Big\|\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big\|_{L^2(\Omega)} \|f(u_m)\, v\|_{L^2(\Omega)} \\ &\le \Big\|\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big\|_{L^2(\Omega)} \|f(u_m)\|_{L^2(\Omega)} \|v\|_{L^\infty(\Omega)} \\ &\le c_p \Big\|\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big\|_{L^2(\Omega)} \|f(u_m)\|_{L^2(\Omega)} \|v\|_{W^{1,p}(\Omega)}\,, \end{aligned} \tag{2.146}$$

where in the last inequality we used once more Sobolev's embedding.
Collecting (2.143),(2.144),(2.145), and (2.146), from (2.142) it follows that

$$\begin{aligned} &|\langle \chi_n-\chi_m, v\rangle_{(W^{1,p})'(\Omega),\, W^{1,p}(\Omega)}| = \Big|\int_\Omega (\chi_n-\chi_m) v\, dx\Big| \\ &\le c_p \|v\|_{W^{1,p}(\Omega)} \Big(\|\nabla(u_n-u_m)\|_{L^{6^-}(\Omega)} \Big\|\frac{1}{\theta_n}\Big\|_{H^1(\Omega)} + \|\Delta u_n\|_{L^2(\Omega)} \Big\|\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big\|_{L^2(\Omega)} \\ &+ \|f(u_n)-f(u_m)\|_{L^2(\Omega)} \Big\|\frac{1}{\theta_n}\Big\|_{L^2(\Omega)} + \Big\|\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big\|_{L^2(\Omega)} \|f(u_m)\|_{L^2(\Omega)}\Big),\ p>3, \end{aligned}$$

which implies

$$\begin{aligned} &\|\chi_n-\chi_m\|_{(W^{1,p})'(\Omega)} = \sup_{\substack{v\in W^{1,p}(\Omega)\\ v\neq 0}} \frac{|\langle \chi_n-\chi_m, v\rangle_{(W^{1,p})'(\Omega),\, W^{1,p}(\Omega)}|}{\|v\|_{W^{1,p}(\Omega)}} \\ &\le c_p \Big(\|\nabla(u_n-u_m)\|_{L^{6^-}(\Omega)} \Big\|\frac{1}{\theta_n}\Big\|_{H^1(\Omega)} + \|\Delta u_n\|_{L^2(\Omega)} \Big\|\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big\|_{L^2(\Omega)} \\ &+ \|f(u_n)-f(u_m)\|_{L^2(\Omega)} \Big\|\frac{1}{\theta_n}\Big\|_{L^2(\Omega)} + \Big\|\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big\|_{L^2(\Omega)} \|f(u_m)\|_{L^2(\Omega)}\Big)\,. \end{aligned} \tag{2.147}$$

Integrating (2.147) with respect to time and using Hölder's inequality, we obtain

$$\begin{aligned} \int_0^T \|\chi_n-\chi_m\|_{(W^{1,p})'(\Omega)}\, dt &\le c_p \Big(\|\nabla(u_n-u_m)\|_{L^2(0,T;\, L^{6^-}(\Omega))} \Big\|\frac{1}{\theta_n}\Big\|_{L^2(0,T;\, H^1(\Omega))} + \\ &+ \|\Delta u_n\|_{L^3(0,T;\, L^2(\Omega))} \Big\|\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big\|_{L^{\frac{3}{2}}(0,T;\, L^2(\Omega))} + \\ &+ \|f(u_n)-f(u_m)\|_{L^\infty(0,T;\, L^2(\Omega))} \Big\|\frac{1}{\theta_n}\Big\|_{L^1(0,T;\, L^2(\Omega))} + \\ &+ \Big\|\frac{1}{\theta_n}-\frac{1}{\theta_m}\Big\|_{L^1(0,T;\, L^2(\Omega))} \|f(u_m)\|_{L^\infty(0,T;\, L^2(\Omega))}\Big)\,. \end{aligned}$$

Thus, using (2.29), (2.85) and (2.96), it follows that

$$\int_0^T \|\chi_n - \chi_m\|_{(W^{1,p})'(\Omega)}\, dt \le c_p\Big(\|\nabla(u_n - u_m)\|_{L^2(0,T;\, L^{6^-}(\Omega))} + \\ + \Big\|\frac{1}{\theta_n} - \frac{1}{\theta_m}\Big\|_{L^{\frac{3}{2}}(0,T;\, L^2(\Omega))} + \\ + \|f(u_n) - f(u_m)\|_{L^\infty(0,T;\, L^2(\Omega))} + \\ + \Big\|\frac{1}{\theta_n} - \frac{1}{\theta_m}\Big\|_{L^1(0,T;\, L^2(\Omega))}\Big)\,. \tag{2.148}$$

Passing to the limit in (2.148), convergence relations (2.131), (2.133) and (2.137) yield

$$\|\chi_n - \chi_m\|_{L^1(0,T;\,(W^{1,p})'(\Omega))} \to 0\,, \quad \text{for every } p > 3\,. \tag{2.149}$$

Let us consider $\|\chi_n - \chi_m\|_{L^1(0,T;\, L^2(\Omega))}$, which can be rewritten as

$$\|\chi_n - \chi_m\|_{L^1(0,T;\, L^2(\Omega))} = \int_0^T \langle \chi_n - \chi_m, \chi_n - \chi_m\rangle^{\frac{1}{2}}_{(H^2)'(\Omega),\, H^2(\Omega)}\, dt\,.$$

Thus,

$$\begin{aligned}\|\chi_n - \chi_m\|_{L^1(0,T;\, L^2(\Omega))} &\le \int_0^T \|\chi_n - \chi_m\|^{\frac{1}{2}}_{(H^2)'(\Omega)}\, \|\chi_n - \chi_m\|^{\frac{1}{2}}_{H^2(\Omega)}\, dt \\ &\le \Big(\int_0^T \|\chi_n - \chi_m\|_{(H^2)'(\Omega)}\Big)^{\frac{1}{2}} \Big(\int_0^T \|\chi_n - \chi_m\|_{H^2(\Omega)}\, dt\Big)^{\frac{1}{2}} \\ &\le \|\chi_n - \chi_m\|^{\frac{1}{2}}_{L^1(0,T;(H^2)'(\Omega))} \big(\|\chi_n\|_{L^1(0,T;\, H^2(\Omega))} + \|\chi_m\|_{L^1(0,T;\, H^2(\Omega))}\big)^{\frac{1}{2}}\end{aligned} \tag{2.150}$$

where in the second inequality we used Hölder's inequality. Note that $H^2(\Omega) \hookrightarrow W^{1,p}(\Omega)$ for $p \le 6$, hence $(W^{1,p})'(\Omega) \hookrightarrow (H^2)'(\Omega)$ for $p \le 6$. Using (2.73) and (2.149), from (2.150) we deduce

$$\|\chi_n - \chi_m\|_{L^1(0,T;\, L^2(\Omega))} \to 0\,. \tag{2.151}$$

By completeness, it follows the existence of ξ such that $\chi_n \to \xi$ strongly in $L^1(0, T;\, L^2(\Omega))$. In view of (2.128), by the uniqueness of the weak limit, we deduce that $\xi = \chi$ almost everywhere in $\Omega \times (0, T)$. Hence, $\chi_n \to \chi$ strongly in $L^1(0, T;\, L^2(\Omega))$, which implies in particular that $\chi_n \to \chi$ almost everywhere in $\Omega \times (0, T)$.
Since $\{\chi_n\}_n$ is uniformly bounded in $L^3(0, T;\, L^\infty(\Omega))$ due to (2.80), then we deduce (cf. Appendix A.10)

$$\chi_n \to \chi \text{ strongly in } L^{r_1}(0, T;\, L^{r_2}(\Omega)),\ r_1 < 3,\ r_2 < +\infty. \tag{2.152}$$

Combining (2.129) together with (2.152), we can conclude that

$$\chi_n\theta_n \rightharpoonup \chi\theta \ \text{ in } L^{r_1}(0, T;\, L^{r_2}(\Omega)),\ r_1 < 3\,,\ r_2 < 2\,. \tag{2.153}$$

Since $\chi_n \to \chi$ almost everywhere in $\Omega \times (0, T)$, then $\chi_n^2 \to \chi^2$ almost everywhere in $\Omega \times (0, T)$. Thus, from (2.78) we deduce that (cf. Appendix A.10)

$$\chi_n^2 \to \chi^2 \text{ strongly in } L^{q_1}(0, T; L^{q_2}(\Omega)), \; q_1 < 2, \, q_2 < 6 \,. \tag{2.154}$$

On the other hand, from (2.77) it follows that $\{\chi_n^2\}_n$ is uniformly bounded in $L^2(0, T; H^1(\Omega))$, hence it weakly converges in $L^2(0, T; H^1(\Omega))$. Due to the uniqueness of the weak limit, we can conclude that

$$\chi_n^2 \rightharpoonup \chi^2 \text{ in } L^2(0, T; H^1(\Omega)).$$

Since ∇ can be considered as a continuous linear operator from $L^2(0, T; H^1(\Omega))$ to $L^2(0, T; L^2(\Omega))$, then we obtain

$$\nabla\chi_n^2 \rightharpoonup \nabla\chi^2 \text{ in } L^2(0, T; L^2(\Omega)). \tag{2.155}$$

2.4.1 Subcase $\beta \in (\frac{5}{3}, 2)$

As already observed, for $\beta \in (\frac{5}{3}, 2)$ in (2.16) we have additional regularity for θ. In particular, we refer to (2.118), which implies (2.120). In this subsection, we derive some convergence relations which directly follow from (2.118) and thus hold only for $\beta \in (\frac{5}{3}, 2)$. These will be fundamental in order to pass to the limit in the weak form of the "heat" equation (2.21) and thus conclude about the existence of a weak solution in the sense of Definition 2.2.2, as stated in Theorem 2.2.2.

Let $v \in W^{1,p}(\Omega)$, $p = \frac{2\bar{q}}{\bar{q}-2}$, where $\bar{q}$ is given by (2.118) with $\epsilon > 0$ taken so small that $\beta > \frac{5}{3} + \epsilon$. Thus, for $5/3 < \beta < 2$, $p \in (14, +\infty)$. Testing (2.3) by v, we obtain

$$\langle Q(\theta_n)_t, v\rangle_{(W^{1,p})'(\Omega), W^{1,p}(\Omega)} = -\int_\Omega \theta_n(\chi_n + \lambda)\Delta\chi_n v \, dx + \int_\Omega k(\theta_n)\nabla\frac{1}{\theta_n}\cdot\nabla v \, dx. \tag{2.156}$$

Integrating (2.156) with respect to time between arbitrary $\tau, t \in [0, T]$, $\tau < t$, we deduce

$$\begin{aligned}&\langle Q(\theta_n(t)), v\rangle_{(W^{1,p})'(\Omega), W^{1,p}(\Omega)} - \langle Q(\theta_n(\tau)), v\rangle_{(W^{1,p})'(\Omega), W^{1,p}(\Omega)} = \\ &-\int_\tau^t \int_\Omega \theta_n(\chi_n + \lambda)\Delta\chi_n v \, dxdt + \int_\tau^t \int_\Omega k(\theta_n)\nabla\frac{1}{\theta_n}\cdot\nabla v \, dxdt \,,\end{aligned}$$

whence, using Hölder's inequality,

$$\begin{aligned}&|\langle Q(\theta_n(t)), v\rangle_{(W^{1,p})'(\Omega), W^{1,p}(\Omega)} - \langle Q(\theta_n(\tau)), v\rangle_{(W^{1,p})'(\Omega), W^{1,p}(\Omega)}| \\ &\leq \int_\tau^t \left(\|\theta_n(\chi_n + \lambda)\Delta\chi_n\|_{L^1(\Omega)}\|v\|_{L^\infty(\Omega)} + \left\|k(\theta_n)\nabla\frac{1}{\theta_n}\right\|_{L^{\bar{r}}(\Omega)}\|\nabla v\|_{L^p(\Omega)}\right)dt \,,\end{aligned} \tag{2.157}$$

where $\bar{r} \in (1, \frac{14}{13})$ is the conjugate exponent to $p \in (14, +\infty)$. Namely, $\bar{r}$ has the same expression as in (2.120). Using the continuous embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\bar{\Omega})$, $p \in$

$(14, +\infty)$, from (2.157) it follows that

$$\|Q(\theta_n(t)) - Q(\theta_n(\tau))\|_{(W^{1,p})'(\Omega)} =$$
$$= \sup_{\substack{v \in W^{1,p}(\Omega) \\ v \neq 0}} \frac{|\langle Q(\theta_n(t)), v\rangle_{(W^{1,p})'(\Omega), W^{1,p}(\Omega)} - \langle Q(\theta_n(\tau)), v\rangle_{(W^{1,p})'(\Omega), W^{1,p}(\Omega)}|}{\|v\|_{W^{1,p}(\Omega)}}$$
$$\leq c \int_\tau^t \Big(\|\theta_n(\chi_n + \lambda)\Delta\chi_n\|_{L^1(\Omega)} + \Big\| k(\theta_n) \nabla \frac{1}{\theta_n} \Big\|_{L^{\bar r}(\Omega)} \Big) dt \,.$$

Noting that $\bar r > 1$ and using Hölder's inequality, we obtain

$$\|Q(\theta_n(t)) - Q(\theta_n(\tau))\|_{(W^{1,p})'(\Omega)}$$
$$\leq c \|\theta_n(\chi_n + \lambda)\Delta\chi_n\|_{L^{\bar r}(0,T;\, L^1(\Omega))} \|1\|_{L^p(\tau,t)} + c \Big\| k(\theta_n) \nabla \frac{1}{\theta_n} \Big\|_{L^{\bar r}((0,T)\times\Omega)} \|1\|_{L^p(\tau,t)} \,. \tag{2.158}$$

Since $\bar r < \frac{6}{5}$, from (2.158) it follows that

$$\|Q(\theta_n(t)) - Q(\theta_n(\tau))\|_{(W^{1,p})'(\Omega)}$$
$$\leq c \|\theta_n(\chi_n + \lambda)\Delta\chi_n\|_{L^{\frac{6}{5}}(0,T;\, L^1(\Omega))} \|1\|_{L^p(s,t)} + c \Big\| k(\theta_n) \nabla \frac{1}{\theta_n} \Big\|_{L^{\bar r}((0,T)\times\Omega)} \|1\|_{L^p(s,t)}$$
$$\leq c|t - \tau|^{\frac{1}{p}} \,, \quad t, \tau \in [0, T] \,, \tag{2.159}$$

where in the last inequality we used (2.89) and (2.120). From (2.159) we infer

$$\|Q(\theta_n)\|_{\mathcal{C}^{0,\alpha}([0,T];\,(W^{1,p})'(\Omega))} =$$
$$= \|Q(\theta_n)\|_{\mathcal{C}^0([0,T];\,(W^{1,p})'(\Omega))} + \sup_{\substack{t,\tau \in [0,T] \\ t \neq \tau}} \frac{\|Q(\theta_n(t)) - Q(\theta_n(\tau))\|_{(W^{1,p})'(\Omega)}}{|t - \tau|^\alpha} \leq c \,, \tag{2.160}$$

where $\alpha = \frac{1}{p} \in (0, \frac{1}{14})$. Notice that $\{Q(\theta_n)\}_n \subset \mathcal{C}([0, T];\, L^2(\Omega))$ is implicitly assumed, thanks to (2.24) and Remark 2.3.2.
Observe that from (2.159) it follows that the sequence $\{Q(\theta_n)\}_n$ is equicontinuous with values in $(W^{1,p})'(\Omega)$.
On the other hand, consider any Banach space X such that $(W^{1,p})'(\Omega) \subset\subset X$. For instance, $X \equiv (H^3)'(\Omega)$. Indeed, from the compact embedding $H^3(\Omega) \subset\subset W^{2,3}(\Omega) \subset\subset W^{1,q}(\Omega)$, $q \in [1, +\infty)$, we deduce $H^3(\Omega) \subset\subset W^{1,p}(\Omega)$, $p \in (14, +\infty)$. Thus, $(W^{1,p})'(\Omega) \subset\subset (H^3)'(\Omega)$, $p \in (14, +\infty)$ (cf. Appendix A.9).
In general, having $(W^{1,p})'(\Omega) \subset\subset X$, then from (2.160) we obtain that $\{Q(\theta_n)\}_n$ is pointwise relatively compact in X, i.e., $\{Q(\theta_n)\}_n$ is relatively compact in X, $\forall t \in [0, T]$.
Using the Ascoli-Arzelà Theorem, we conclude that $\{Q(\theta_n)\}_n$ is relatively compact in $\mathcal{C}^0([0, T];\, X)$. It then follows that there exists $\zeta \in \mathcal{C}^0([0, T]; X)$ such that

$$Q(\theta_n) \to \zeta \text{ strongly in } \mathcal{C}^0([0, T]; X) \,. \tag{2.161}$$

On the other hand, for $\beta \in (\frac{5}{3}, 2)$, from (2.136) we can deduce that

$$\theta_n^2 \to \theta^2 \text{ strongly in } L^1(0, T; L^1(\Omega)). \tag{2.162}$$

Recalling that $Q(\theta_n) = \frac{c_V}{2}\theta_n^2$ and combining (2.161) with (2.162), we can conclude that $\zeta = \frac{c_V}{2}\theta^2$ almost everywhere in $\Omega \times (0, T)$. In particular, $\zeta \in \mathcal{C}^0([0,T]; X)$ is a representative of $\frac{c_V}{2}\theta^2 \in L^1((0,T); L^1(\Omega))$ in the distributional sense. It then follows that

$$Q(\theta_n) \to Q(\theta) \text{ strongly in } \mathcal{C}^0([0,T]; X). \tag{2.163}$$

As a consequence, we have

$$Q(\theta_n(t)) \to Q(\theta(t)) \text{ strongly in } X, \quad \forall t \in [0,T]. \tag{2.164}$$

Combining (2.24) with (2.118), we obtain

$$\theta_n \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(0, T; L^2(\Omega)) \cap L^{\bar{q}}(\Omega \times (0,T)), \tag{2.165}$$

where $\bar{q} = \frac{3\beta+1-3\epsilon}{3} > 2$ for $\epsilon > 0$ such that $\beta > \frac{5}{3} + \epsilon$. Using standard interpolation, from (2.165) we deduce that $\forall q_1 \in (\bar{q}, +\infty)$ $\exists q_2 = q_2(q_1) > 2$ such that

$$\theta_n \rightharpoonup \theta \text{ in } L^{q_1}(0, T; L^{q_2}(\Omega)). \tag{2.166}$$

More precisely, $q_1 = \frac{\bar{q}}{\alpha}$, while $q_2 = \frac{2\bar{q}}{(1-\alpha)\bar{q}+2\alpha} = \frac{2\alpha q_1}{(1-\alpha)\alpha q_1 + 2\alpha}$, where $\alpha \in (0,1)$ is an interpolation coefficient. Being $\bar{q} > 2$, then $q_2 > 2$, $\forall q_1 \in (\bar{q}, +\infty)$. Since we showed that $\theta_n \to \theta$ almost everywhere in $\Omega \times (0, T)$, then it follows that (cf. Appendix A.10)

$$\theta_n \to \theta \text{ strongly in } L^{p_1}(0, T; L^{p_2}(\Omega)), \; p_1 < q_1, \; p_2 < q_2. \tag{2.167}$$

In particular,

$$\theta_n \to \theta \text{ strongly in } L^{6^+}(0, T; L^{2^+}(\Omega)). \tag{2.168}$$

We now show that $\{\theta_n \chi_n\}_n$ strongly converges to $\theta\chi$ in $L^2(0, T; L^2(\Omega))$. Using Young's inequality, we obtain

$$\begin{aligned} \|\theta_n\chi_n - \theta\chi\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \|\theta_n\chi_n - \theta_n\chi + \theta_n\chi - \theta\chi\|^2_{L^2(\Omega)} dt \\ &\leq 2\int_0^T \|\theta_n\chi_n - \theta_n\chi\|^2_{L^2(\Omega)} dt + 2\int_0^T \|\theta_n\chi - \theta\chi\|^2_{L^2(\Omega)} dt. \end{aligned} \tag{2.169}$$

Let us consider the first term on the right hand side of (2.169). Hölder's inequality yields

$$\int_0^T \|\theta_n\chi_n - \theta_n\chi\|^2_{L^2(\Omega)} dt \leq \|\theta_n\|_{L^{6^+}(0,T;L^{2^+}(\Omega))} \|\chi_n - \chi\|_{L^{3^-}(0,T;L^\delta(\Omega))}, \tag{2.170}$$

where $\delta < +\infty$ is such that $\frac{1}{2^+} + \frac{1}{\delta} = \frac{1}{2}$. As for the second term on the right hand side of (2.169), proceeding analogously, we obtain

$$\int_0^T \|\theta_n \chi - \theta\chi\|^2_{L^2(\Omega)} dt \leq \|\theta_n - \theta\|_{L^{6+}(0,T;\,L^{2^+}(\Omega))} \|\chi\|_{L^{3-}(0,T;\,L^{\delta}(\Omega))}. \tag{2.171}$$

Combining (2.169) with (2.170) and (2.171), then using convergence properties given by (2.152) and (2.168), we can conclude that

$$\theta_n \chi_n \to \theta\chi \text{ strongly in } L^2(0,\,T;\,L^2(\Omega))\,. \tag{2.172}$$

At last, in order to deduce a convergence relation for $\{k(\theta_n)\nabla\frac{1}{\theta_n}\}_n$, note that

$$k(\theta_n)\nabla\frac{1}{\theta_n} = \theta_n \frac{k(\theta_n)}{\theta_n}\nabla\frac{1}{\theta_n} = \theta_n \nabla\Big(\frac{k_0}{2\theta_n^2} + \frac{k_1}{(2-\beta)\theta_n^{2-\beta}}\Big)\,,$$

where in the last inequality we used (2.48). Let us set $\frac{1}{s_1} \equiv \frac{1}{p_1} + \frac{1}{2}$ and $\frac{1}{s_2} \equiv \frac{1}{p_2} + \frac{1}{2}$, respectively. Being p_1 and p_2 in (2.165) such that $p_1 < q_1$ and $p_2 < q_2$ with $q_1, q_2 > 2$, then there exist p_1, p_2 satisfying $2 \leq p_1 < q_1$ and $2 \leq p_2 < q_2$. Thus, $s_1, s_2 \geq 1$. Combining (2.140) and (2.139) with (2.165), we obtain

$$k(\theta_n)\nabla\frac{1}{\theta_n} \rightharpoonup k(\theta)\nabla\frac{1}{\theta} \text{ in } L^{s_1}(0,\,T;\,L^{s_2}(\Omega))\,, \text{ for some } s_1, s_2 \geq 1\,. \tag{2.173}$$

2.4.2 Limit of the non-isothermal Cahn-Hilliard model

We assumed that, for every $n \in \mathbb{N}$, the approximate solution (u_n, χ_n, θ_n) fulfils the "strong" system of equations (2.1)-(2.2)-(2.3) (actually, its hypothetical approximation). Then, multiplying by suitable test functions and integrating by parts, it readily follows that $\{(u_n, \chi_n, \theta_n)\}_n$ also complies with Definitions 2.2.1 and 2.2.2. Then, we can see what happens as we let $n \nearrow \infty$.

Limit of the Cahn-Hilliard system

We assumed that (2.1) and (2.3) are satisfied by the approximate solution (u_n, χ_n, θ_n), $\forall n \in \mathbb{N}$, namely,

$$(u_n)_t = \Delta\chi_n \text{ almost everywhere in } \Omega \times (0,\,T)\,, \tag{2.174}$$
$$\chi_n\theta_n = f(u_n) - \lambda\theta_n - \Delta u_n \text{ almost everywhere in } \Omega \times (0,\,T)\,, \tag{2.175}$$

with the initial condition $u_n(\cdot, 0) = u_{n,0}$, the boundary conditions $\nabla\chi_n \cdot \nu = 0$ and $\nabla u_n \cdot \nu = 0$, where ν is the unit outer normal to $\partial\Omega$. Since we have the convergence relations provided by (2.127) and by (2.128), taking the limit in (2.174), we directly obtain (2.18).
As for equation (2.175), combining (2.131) with (2.129) and (2.127), it yields

$$f(u_n) - \lambda\theta_n - \Delta u_n \overset{*}{\rightharpoonup} f(u) - \lambda\theta - \Delta u \text{ in } L^\infty(0,\,T;\,L^{\frac{4}{3}}(\Omega)) \cap L^3(0,\,T;\,L^2(\Omega))\,.$$

On the other hand, we have convergence relation given by (2.153). Hence, taking the limit in (2.175), by comparison, we reduce to (2.19).
At last, we recover the initial and the boundary conditions given in Definition 2.2.1. Observe that $u(\cdot, 0) = u_0$ directly follows from the convergence relation (2.130). As for the boundary conditions, we appeal to the theory of traces in Sobolev spaces (cf. Appendix A.12). Let $T_1 : H^2(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ be a trace operator which maps a function of class $H^2(\Omega)$ to the trace of the normal component of its gradient. Since T_1 is linear and continuous, from (2.127) and (2.128) we deduce that

$$T_1\chi_n \rightharpoonup T_1\chi \text{ and } T_1u_n \rightharpoonup T_1u \text{ in } H^{\frac{1}{2}}(\partial\Omega)\,, \text{ almost everywhere in } (0,\,T)\,.$$

It follows that the boundary conditions (1.13) and (1.14) are recovered at least in the sense of traces.

Limit of the balance of entropy equation

We assumed that (2.11) is satisfied by the approximate solution (u_n, χ_n, θ_n), $\forall n \in \mathbb{N}$, then we can test it by $\zeta \in \mathcal{C}^\infty(\bar{\Omega} \times [0,\,T])$ such that $\zeta \geq 0$, $\zeta(\cdot, T) = 0$. Then, integrating by parts, we obtain

$$\begin{aligned}&\int_0^T\!\!\int_\Omega \Lambda(\theta_n)\zeta_t\,dxdt + \int_0^T\!\!\int_\Omega \nabla\Big(\frac{\chi_n^2}{2} + \lambda\chi_n\Big)\cdot\nabla\zeta\,dxdt + \int_0^T\!\!\int_\Omega \frac{k(\theta_n)}{\theta_n}\nabla\frac{1}{\theta_n}\cdot\nabla\zeta\,dxdt\\ &= -\int_0^T\!\!\int_\Omega |\nabla\chi_n|^2\zeta\,dxdt - \int_0^T\!\!\int_\Omega k(\theta_n)\Big|\nabla\frac{1}{\theta_n}\Big|^2\zeta\,dxdt - \int_\Omega \Lambda(\theta_n(\cdot,0))\zeta(\cdot,0)\,dx\,.\end{aligned} \tag{2.176}$$

Our aim is taking the supremum limit in (2.176). Firstly, recall that $\Lambda(\theta_n) = c_V\theta_n$, $c_V > 0$, and that $\frac{k(\theta_n)}{\theta_n}\nabla\frac{1}{\theta_n} = \nabla\frac{k_0}{2\theta^2} + \nabla\frac{k_1}{(2-\beta)\theta^{2-\beta}}$, $k_0, k_1 > 0$, as shown in (2.48). Hence, using convergence relations (2.129), (2.155), (2.128), (2.140) and (2.139), the first row of (2.176) passes to the desired limit. Indeed, we recover the first row of (2.20) not only as a supremum limit, but as a true limit. As for the first two terms in the second row of (2.176), we apply a useful lower semicontinuity result due to A.D. Ioffe, that is known as Ioffe's Theorem - see Theorem A.11.1 in Appendix A.11. Referring to the notation used in Theorem A.11.1, $Q \equiv \Omega \times (0,\,T)$, while $f : Q \times \mathbb{R}^+ \times \mathbb{R}^3 \to [0,\,+\infty]$ is such that $(x,\,t) \times w \times v \mapsto w|v|^2$. Observe that f is measurable and non-negative. Moreover, $f((x,t), w, \cdot)$ is lower semi-continuous on $\mathbb{R}^+ \times \mathbb{R}^3$, $\forall (x,t) \in Q$, and $f((x,t),\cdot,\cdot)$ is convex on $\mathbb{R}^3$, $\forall ((x,t),w) \in Q \times \mathbb{R}^+$. Let us now consider the first two terms in the second row of (2.176) separately. In particular, referring to the first term, $w_n \equiv \zeta$ and $v_n \equiv \nabla\chi_n$, $\forall n \in \mathbb{N}$. The almost everywhere convergence of $\{w_n\}_n$ in Q is then obvious, and the weak one of $\{\nabla\chi_n\}_n$ to $\nabla\chi$ in $L^1(Q)$ easily follows from (2.128). Thus, we can conclude that

$$\liminf_{n\to+\infty}\int_0^T\!\!\int_\Omega |\nabla\chi_n|^2\zeta\,dxdt \geq \int_0^T\!\!\int_\Omega |\nabla\chi|^2\zeta\,dxdt\,. \tag{2.177}$$

As for the second term, $w_n \equiv \zeta k(\theta_n)$ and $v_n \equiv \nabla\frac{1}{\theta_n}$, $\forall n \in \mathbb{N}$. Since we showed at the beginning of Section 2.4 that $\theta_n \to \theta$ almost everywhere in $\Omega \times (0,\,T)$, then $k(\theta_n) \to k(\theta)$ almost everywhere in $\Omega \times (0,\,T)$. That implies the almost everywhere convergence of $\{\zeta k(\theta_n)\}_n$ in Q. On the other hand, convergence relation (2.138) gives us the weak convergence of $\{\nabla\frac{1}{\theta_n}\}_n$ to $\nabla\frac{1}{\theta}$ in $L^1(Q)$. Then, it follows that

$$\liminf_{n\to+\infty}\int_0^T\int_\Omega k(\theta_n)\Big|\nabla\frac{1}{\theta_n}\Big|^2\zeta\,dxdt \geq \int_0^T\int_\Omega k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2\zeta\,dxdt\,. \tag{2.178}$$

At last, assuming that $\theta_n(\cdot,0)$ converges properly to θ_0, the last term in the second row of (2.176) passes to the desired supremum limit and we recover (2.20). Observe that the inequality sign in (2.20) is due to the application of Ioffe's Theorem, and in particular to relations (2.177) and (2.178).

Limit of the "heat" equation

In order to pass to the limit in the "heat" equation, we need the additional information for $\{\theta_n\}_n$ holding for $\frac{5}{3}<\beta<2$, as already stated. To this aim, we assume $\frac{5}{3}<\beta<2$.
Since the approximate solutions (u_n,χ_n,θ_n) fulfils (2.3) (actually, its hypothetical approximation) strongly, $\forall n\in\mathbb{N}$, then we can test it by $\xi\in\mathcal{C}^\infty(\bar\Omega\times[0,T])$ and, integrating by parts, we obtain

$$\begin{aligned}\int_0^T\int_\Omega Q(\theta_n)\xi_t\,dx+\int_\Omega Q(\theta_n(\cdot,0))\xi(\cdot,0)\,dx-\int_\Omega Q(\theta_n(\cdot,T))\xi(\cdot,T)\,dx+\\ -\int_0^T\int_\Omega\theta_n(\chi_n+\lambda)\Delta\chi_n\xi\,dxdt+\int_0^T\int_\Omega k(\theta_n)\nabla\frac{1}{\theta_n}\cdot\nabla\xi\,dxdt=0\,.\end{aligned} \tag{2.179}$$

Taking the limit in (2.179), using convergence relations (2.165) and (2.164), the first row of (2.179) passes to the desired limit, i.e., we recover the first row of (2.19). Next, we consider the second row of (2.179), which is that requiring the additional regularity provided by (2.118) and (2.120). Indeed, in order to pass to the limit in the first term we use the strong convergences provided by (2.172) and (2.167) combined with the weak one by (2.128). As for the second term, convergence relation (2.173) allows us to conclude. Thus, we recover (2.19).

2.5 An open problem

In this section we try to weaken a hypothesis made on the double-well potential (2.14). In particular, we would like to generalize Theorems 2.2.1 and 2.2.2 to fourth order polynomial potentials such as (1.6). We will see that, in this case, some a priori bounds obtained in Subsection 2.3.2 do not hold true anymore. However, we will propose a strategy to overcome this problem.

Consider a polynomial double-well potential of the following form

$$F(u)=a_1|u|^\rho-a_2u^2-a_3,\quad \rho\in[3,4]\,, \tag{2.180}$$

where a_1, $a_2>0$. Notice that the expression (2.180) includes fourth order potentials such as (1.6), or more generally (1.8). In this case, the derivative of F reads

$$f(u)=a_1\rho\,\mathrm{sgn}(u)|u|^{\rho-1}-2a_2u\,,\quad \rho\in[3,4]\,, \tag{2.181}$$

where sgn represents the sign function and a_1, $a_2 > 0$.
If we assume a double-well potential of the form (2.180), instead of (2.14), then (2.44) does not hold true anymore. As already notice in Remark 2.3.1, in this case, the Gagliardo-Nirenberg interpolation inequality yields the L^p norm of the Laplacian of u with $p > \frac{4}{3}$ for $\rho > \frac{7}{2}$ on the right hand side (2.44). As a consequence, we obtain $\|\chi^2\theta\|_{L^r(\Omega)}$ with $r > 1$ on the right hand side of (2.46). Thus, we cannot proceed anymore as done in Subsection 2.3.2, i.e., we cannot apply Gronwall's Lemma in order to close the estimates. This suggests that we should adopt different techniques to recover the estimates collected in Subsection 2.3.2. In particular, we should provide an alternative method to control the last term on the right hand side of (2.39), namely $c_V \int_\Omega f'(u)u_t\chi\, dx$. To this aim, we can try to proceed as follows.

From (2.181) we deduce that $f'(u) = a_1\rho(\rho-1)|u|^{\rho-2} - 2a_2$, $\rho \in [3,4]$, a_1, $a_2 > 0$. It follows that

$$\begin{aligned}\int_\Omega f'(u)u_t\chi\, dx &\le c_\rho \int_\Omega (|u|^{\rho-2}+1)|u_t\chi|\, dx\\ &\le c_\rho(\||u|^{\rho-2}\chi\|_{L^2(\Omega)} + \|\chi\|_{L^2(\Omega)})\|u_t\|_{L^2(\Omega)}\\ &\le c_\rho(\||u|^{\rho-2}\|_{L^3(\Omega)}\|\chi\|_{L^6(\Omega)} + \|\chi\|_{L^2(\Omega)})\|u_t\|_{L^2(\Omega)}\\ &\le c_\rho(\|u\|^{\rho-2}_{L^{3(\rho-2)}(\Omega)} + c)\|\chi\|_{L^6(\Omega)}\|u_t\|_{L^2(\Omega)}\end{aligned}$$

where we used Hölder's inequality twice and c_ρ denotes a positive constant depending on $\rho \in [3,4]$ which may vary from line to line. Notice that $3(\rho-2) \in [3,6]$ for $\rho \in [3,4]$. Thus, using (2.28) together with Young's inequality, we obtain

$$c_V \int_\Omega f'(u)u_t\chi\, dx \le \frac{c_V}{2}\|u_t\|^2_{L^2(\Omega)} + c_\rho\|\chi\|^2_{L^6(\Omega)},$$

whence,

$$c_V \int_\Omega f'(u)u_t\chi\, dx \le \frac{c_V}{2}\|u_t\|^2_{L^2(\Omega)} + c_\rho\|\chi\|^2_{H^1(\Omega)}, \tag{2.182}$$

due to the continuous embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$.
Observe that we can move the first term on the right hand side of (2.182) to the left hand side of (2.39). As for the second one, we should find a way to control it. First of all, we would like to estimate the average of χ over Ω, i.e. $\langle\chi\rangle$, which is given by (2.2), namely $\langle\chi\rangle = \langle\frac{f(u)}{\theta}\rangle - \langle\frac{\Delta u}{\theta}\rangle - 1$. Integrating by parts, using the boundary condition (2.6) and Hölder's inequality, we obtain

$$|\langle\chi\rangle| \le \|f(u)\|_{L^2(\Omega)}\Big\|\frac{1}{\theta}\Big\|_{L^2(\Omega)} + c\|\nabla u\|_{L^2(\Omega)}\Big\|\nabla\frac{1}{\theta}\Big\|_{L^2(\Omega)} + c,$$

where f is given by (2.181). Notice that, for $\rho \in [3,4]$, $\frac{6}{\rho-1} \in [2,3]$ in (2.29). Thus, using (2.27) and (2.29), we deduce

$$|\langle\chi\rangle| \le c_\rho\Big(\Big\|\frac{1}{\theta}\Big\|_{L^2(\Omega)} + \Big\|\nabla\frac{1}{\theta}\Big\|_{L^2(\Omega)} + 1\Big).$$

The Poincaré-Wirtinger inequality then yields

$$\|\chi\|_{L^2(\Omega)} \leq c_\rho \Big(\|\nabla\chi\|_{L^2(\Omega)} + \Big\|\frac{1}{\theta}\Big\|_{L^2(\Omega)} + \Big\|\nabla\frac{1}{\theta}\Big\|_{L^2(\Omega)} + 1 \Big) ,$$

whence

$$\|\chi\|_{H^1(\Omega)} \leq c_\rho \Big(\|\nabla\chi\|_{L^2(\Omega)} + \Big\|\frac{1}{\theta}\Big\|_{L^2(\Omega)} + \Big\|\nabla\frac{1}{\theta}\Big\|_{L^2(\Omega)} + 1 \Big) . \tag{2.183}$$

In order to recover the estimates collected in Subsection 2.3.2, we should now square (2.183), insert it into (2.182), combine what we obtain together with (2.39), and then integrate with respect to time. Observe that this procedure fails because the so-called energy and entropy estimates do not provide any upper bound for $\|\frac{1}{\theta}\|_{L^2(0,T;\,L^2(\Omega))}$ (see Subsection 2.3.1). However, if we were able to prove that the spatial average of $\frac{1}{\theta}$, i.e. $\langle\frac{1}{\theta}\rangle$, is somehow bounded by the $L^2(\Omega)$ norm of $\nabla\frac{1}{\theta}$ and by that of θ, then the Poincaré-Wirtinger inequality together with (2.32) and (2.24) would allow us to conclude that $\|\frac{1}{\theta}\|_{L^2(0,T;\,L^2(\Omega))} \leq c$. Thus, we would recover the estimates collected in Subsection 2.3.2. Then, we could proceed analogously as done in the previous sections. As a consequence, we would be able generalize Theorems 2.2.1 and 2.2.2 to fourth order polynomial potentials given by (2.180).

The problem is that we do not know how to obtain an estimate for $\langle\frac{1}{\theta}\rangle$ depending on $L^2(\Omega)$ norm of $\nabla\frac{1}{\theta}$ and on that of θ.
For the sake of simplicity, let us ignore the dependence on the time variable $t \in (0,T)$. In one-spatial dimension, i.e., for $\Omega \equiv (a,b) \subset \mathbb{R}$, the following result holds true.

Proposition 2.5.1. *Let $v \in W^{1,1}(a,b)$ such that $v > 0$ almost everywhere in (a,b). Let $\langle\,\cdot\,\rangle$ denote the average over (a,b) and let*

$$M := \langle v\rangle > 0\,, \quad R := \Big\langle \frac{1}{v} \Big\rangle > 0 \quad \textit{and} \quad G := \int_a^b |v'(x)|\,dx\,,$$

where v' is the spatial derivative of v. Then,

$$M \leq G + \frac{1}{R}\,.$$

Proof. For the sake of contradiction, suppose $M > G + \frac{1}{R}$. Since $v \in W^{1,1}(a,b)$ can be represented by a continuous function, then let $\xi \in [a,b]$ such that $v(\xi) = M$. Thus, for every $y \in (a,b)$, it holds

$$v(y) = v(\xi) + \int_\xi^y v'(x)\,dx \geq M - \int_\xi^x |v'(x)|\,dx = M - G\,,$$

where $M - G > 0$ because $M > G + \frac{1}{R}$. Then, we can deduce that

$$\frac{1}{v(y)} \leq \frac{1}{M-G}\,, \quad \forall y \in [a,b]\,,$$

whence

$$R \leq \frac{1}{M-G}\,.$$

It follows that

$$M \leq G + \frac{1}{R}\,.$$

Since we have a contradiction, the statement follows. □

Setting $v \equiv \frac{1}{\theta}$, since $\theta > 0$ almost everywhere in $\Omega \equiv (a,b)$ (see Section 2.4), then Proposition 2.5.1 together with (2.32) yield the desired estimate for $\langle \frac{1}{\theta} \rangle$.
If we were able to generalize Proposition 2.5.1 to strictly positive function v belonging to $W^{1,1}(\Omega)$, where Ω is a regular subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, then we would be able to conclude and thus generalize Theorems 2.2.1 and 2.2.2 to fourth order polynomial potentials given by (2.180). However, the generalization of Proposition 2.5.1 to higher spatial dimension is not immediate and it may be particularly technical. Indeed, for $\Omega \subset \mathbb{R}^n$, $n > 1$, $v \in W^{1,1}(\Omega)$ does not imply the existence of a continuous representative and the proof may involve the use of the so-called Lebesgue points (cf. Appendix A.4-A.5).

2.6 A tentative approximation of the “strong” system

We already observed that the system of equations (2.1)–(2.3) is rather complex and, as a consequence, the related approximation could be particularly long and technical. For all these reasons, in the previous sections, we decided to skip this argument and rather proceed formally. In this section we give some highlights regarding a possible approximation of the “strong” system (2.1)–(2.3) that one could try to develop.

Recall that, in order to make the procedure carried out in Section 2.3 fully rigorous, one should rather consider a proper regularization or approximation of the “strong” system and prove that it admits at least one solution being sufficiently smooth in order to comply with the estimates. To this aim, we propose as a possible approximation of (2.1)–(2.3) the following system of equations

$$u_t = m\Delta\chi + \sigma\frac{|\Delta\chi|^{p-2}\Delta\chi}{\theta} - \gamma\,\mathrm{sgn}(\chi)|\chi|^q\,, \tag{2.184}$$

$$\chi\theta = -\alpha\Delta u - \lambda\theta + f(u)\,, \tag{2.185}$$

$$(Q(\theta))_t + m\theta\Delta\chi(\chi+\lambda) + \mathrm{div}\left(k(\theta)\nabla\frac{1}{\theta}\right) + \varepsilon\theta^s - \delta\frac{1}{\theta^r} = 0\,, \tag{2.186}$$

where sgn denotes the sign function, for (small) $\sigma, \gamma, \varepsilon, \delta \in (0,1)$ and for properly fixed (large) $p \geq 3$, $q, s, r \geq 1$, possibly depending on each other. The system of equations (2.184)-(2.186) may be endowed with the same boundary conditions as before, and properly regularized initial conditions. In particular, we expect that θ_0 may be truncated both from below and above, where we intend the truncation to be removed passing to the limit.

First of all, as done for the equation (2.3) in Section 2.1, we can multiply (2.186) by $\frac{1}{\theta}$. Thus, using the chain rule, we obtain an equivalent formulation of (2.186), which is given by

$$(\Lambda(\theta))_t + m\Delta\Big(\frac{\chi^2}{2} + \lambda\chi\Big) + \mathrm{div}\Big(\frac{k(\theta)}{\theta}\nabla\frac{1}{\theta}\Big) + \\ - m|\nabla\chi|^2 - k(\theta)\Big|\nabla\frac{1}{\theta}\Big|^2 + \varepsilon\theta^{s-1} - \delta\frac{1}{\theta^{r+1}} = 0. \qquad (2.187)$$

Observe that, due to the introduction of the regularization terms, the energy and the entropy estimates cannot be recovered separately, as done in Subsection 2.3.1. However, one could try to collect them together with the "key" estimates obtained in Subsection 2.3.2. In other words, one could first multiply (2.184) and (2.185) by $\theta\chi$ and by u_t, respectively, then take the sum of them, obtaining a balance of energy equation for the approximate problem. On the other hand, one could multiply (2.187) by $-(\frac{\chi^2}{2} + \lambda\chi) - (\frac{k_0}{2\theta^2} + \frac{k_1}{(2-\beta)\theta^{2-\beta}})$, use (2.37), then sum the result to the difference between the balance of energy equation and (2.187). A single equation is thus obtained and one could try to recover the estimates collected in Section 2.3 by controlling the non-positive terms on the left hand side, or equivalently, the positive ones on the right hand side. Furthermore, the recovered estimates should be uniform with respect to the parameters $\sigma, \gamma, \varepsilon, \delta$.

The motivations for the introduction of the regularization terms in system (2.184)-(2.186), thus in (2.187), are the following. The term $\sigma\frac{|\Delta\chi|^{p-2}\Delta\chi}{\theta}$ in (2.184) could be helpful to collect further regularity for $\Delta\chi$, hence for χ. In particular, it is rescaled by θ so that θ disappears when we multiply (2.184) by $\theta\chi$ in order to obtain an "approximated" balance of energy equation. Also $-\gamma\,\mathrm{sgn}(\chi)|\chi|^q$ in (2.184) is introduced in order to improve our estimates for χ. As for the regularization terms $\varepsilon\theta^s$ and $-\delta\frac{1}{\theta^r}$ in (2.186), thus $\varepsilon\theta^{s-1}$ and $-\delta\frac{1}{\theta^{r+1}}$ in (2.187), they could be useful in order to obtain higher spatial regularity for θ and $\frac{1}{\theta}$, respectively.
In particular, the solutions χ and θ to the approximated system (2.184)-(2.186) should be regular enough so that (2.186) and (2.187) are equivalent, i.e., both are satisfied almost everywhere in $\Omega \times (0, T)$. Indeed, as observed at the beginning of Subsection 2.4.1, in order to conclude about the existence of weak solutions (see Def. 2.2.2) as well as about that of entropy solutions (see Def. 2.2.1) to the system (2.1)- (2.3), we need more regularity for θ.

One could provide the proof of the existence of solutions to the approximate system (2.184)-(2.186) by means of a fixed point argument of Schauder type (cf. [40]). In particular, one could fix θ in (2.184)-(2.185) and then prove the existence of unique solutions u and χ by some more or less standard techniques such as a time discretization argument or a Faedo–Galerkin approximation (cf. [55, §7.3]). Then, once obtained u and χ, one could prove the existence of a unique solution θ to (2.186) by standard techniques. Notice that, due to the structure of equation (2.186), we expect θ to be positive almost everywhere in $\Omega \times (0, T)$. Finally, applying the Schauder fixed point theorem, one could conclude about the existence of at least one triple (u, χ, θ) satisfying the approximate system (2.184)-(2.186) almost everywhere in $\Omega \times (0, T)$. At least in principle, the regularity class of the approximate solution (u, χ, θ) obtained by the application of the Schauder fixed point theorem could be not sufficient in order for the estimates collected in Section 2.3 to be fully rigorous. However, these regularities would be just the outcome of the fixed point argument, and, hence, they could not be at all optimal. Further regularity of the approximate solution might be standardly proved by working separately on the equations of the approximate system and performing some bootstrap argument. This procedure may require a notable amount of technical work, as mentioned above, including the need for regularizing the initial data.

At last, observe that the approximate solution to (2.184)-(2.186) provided by the fixed point argument may depend on the parameters $\sigma, \gamma, \varepsilon, \delta$. On the other hand, the a priori bounds performed in Section 2.3 should be adapted to the approximate solution so to be uniform with respect to $\sigma, \gamma, \varepsilon, \delta$. Thus, if we consider any sequences of parameters $\{\sigma_n\}_n$, $\{\gamma_n\}_n$, $\{\varepsilon_n\}_n$, $\{\delta_n\}_n \subset (0, 1)$ converging to zero as we let $n \nearrow \infty$ and we denote by $\{(u_n, \chi_n, \theta_n)\}_n$ the corresponding sequence of approximate solutions provided by the fixed point argument, then the estimates obtained in Section 2.3 are uniform with respect to $n \in \mathbb{N}$. As a consequence, they are preserved to the limit as we let $n \nearrow \infty$, i.e., they are satisfied by the limit (u, χ, θ). Notice that, since the parameters $\sigma, \gamma, \varepsilon, \delta$ in (2.184)-(2.186) possibly depend on each other, the sequences $\{\sigma_n\}_n$, $\{\gamma_n\}_n$, $\{\varepsilon_n\}_n$, $\{\delta_n\}_n$ may not pass to the limit in a independent way.

Conclusions

In this work we proved the local-in-time existence for the initial-boundary value problem associated to the entropy formulation and, in a subcase, also to the weak one of a non-isothermal Cahn-Hilliard model proposed by Miranville and Schimperna.
In Chapter 1, we first gave an overview of Cahn-Hilliard models, which describe phase-separation processes in two-phase systems. We presented the phenomenological derivation of the standard isothermal Cahn-Hilliard equation, as originally deduced by Cahn and Hilliard. Afterwards, we introduced Gurtin's two-scale approach which is based on a new balance law for microforces. This led us to some generalizations of Cahn-Hilliard equation.
Since also thermal effects should be taken into account when studying realistic physical systems, then we focused on some non-isothermal Cahn-Hilliard models. In particular, we compared and outlined the main differences between Alt and Pawłow's model and Miranville and Schimperna's one. We saw that Alt and Pawłow deduced their model of non-isothermal phase separation from constituive relations for the mass and heat fluxes and the chemical potential and that the thermodynamic consistency is shown a posteriori. On the other hand, Miranville and Schimperna adopted Gurtin's approach and used the two fundamental laws of Thermodynamics as a starting point to derive their model. Furthermore, they did not made any a priori specification on the mass and heat fluxes and on the chemical potential, deducing a posteriori the admissible expressions for the physical parameters. Therefore, since this procedure seems to allow us to describe the most general class of free energies and of chemical potentials, we proposed their model as an extension to Alt and Pawłow's one to non-isotropic materials and to systems that are far from the equilibrium.
At least to our knowledge, a result regarding the existence of (weak) solutions to Miranville and Schimperna's non-isothermal Cahn-Hilliard model was still lacking. To this end, in Chapter 2, we made some working assumptions on the Ginzburg-Landau free energy and on the heat flux. Then, we looked for some formal a priori estimates holding for hypothetical solutions of the strong formulation of the model, more precisely, to a proper regularization or approximation of it. By compactness argument, we showed that at least a subsequence of approximate solutions converges in a suitable way to an entropy solution to the initial-boundary value problem and, when possible, also to a weak one. This procedure thus led us to obtain two theorems about the existence of entropy and weak solutions to Miranville and Schimperna's non-isothermal Cahn-Hilliard model, respectively. We also noticed

that if a preliminary result could be extended to the three-dimensional case, then our theorems could be generalized to a wider class of polynomial double-well potentials. At last, we gave some highlights regarding a possible approximation of the “strong” system that one could try to develop.

We now recall some already mentioned open problems and we present further possible extensions to this work.

First of all, we still have to develop the details of the approximation of the strong formulation of the model. Here, we decided to skip this part and rather give only the highlights of the procedure because it could be particularly complex, long and technical.

Furthermore, we observed that in order to generalize our local-in-time existence theorems to fourth order polynomial potentials we should extend a preliminary result to the three-dimensional case. This is one of our first future objectives.

Another future objective could be that of recovering the Fourier’s law for the heat flux at least for high temperatures. To this aim, we should develop another strategy to recover suitable a priori estimates.

A physically reasonable change regarding the homogeneous part of a Ginzburg-Landau free energy could be done. In particular, one could adopt a temperature coupling term which is quadratic in the order parameter, instead of linear. As we saw at the end of Chapter 1, that would lead to some changes in the heat equation and thus to some mathematical difficulties in looking for suitable a priori estimates.

Other possible extensions may go in the direction of introducing an anisotropy coefficient in the inhomogeneous (or gradient) part of Ginzburg-Landau free energy.

Otherwise, we could try to extend our results to logarithmic potentials and thus appeal to the theory of monotone operators, in order to define a proper concept of solution.

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My grateful thanks go to my supervisor Giulio for his invaluable help and for guiding me patiently step by step in writing this work. I admit that at first I was very skeptical that we would come to an end. However, I still don't know how, thanks to some "magic", we got something unexpected. I have learned a lot, and for this I am grateful to him.

I also thank Prof. Gianni Gilardi for helping us in proving a proposition and for giving us the hope of obtaining more general results in future.

Appendix A

Mathematical Tools

In this appendix we collect some mathematical tools used throughout the thesis.

A.1 Solution to thermodynamic inequalities

Let F be a smooth function from $\mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^m$, $n, m \in \mathbb{N}, n, m \geq 1$.
Our aim is to find a general solution of the inequality

$$F(X,Y) \cdot Y \leq 0\,, \quad \forall X \in \mathbb{R}^n, \forall Y \in \mathbb{R}^m\,. \tag{A.1}$$

In the following, we present a resolution strategy that is given in [31, Appendix B]. The variable X can be considered as a parameter and we may, without loss of generality, suppress it when convenient.
Suppose that (A.1) holds true, for $\lambda > 0$, $F(\lambda Y) \cdot \lambda Y \leq 0$ and hence $F(\lambda Y) \cdot Y \leq 0$. Letting $\lambda \to 0$, we obtain $F(0) \cdot Y \leq 0$, $\forall Y \in \mathbb{R}^m$, so that $F(0) = 0$. It follows that

$$F(Y) = \Big(\int_0^1 \nabla F(sY)\, ds\Big) Y\,, \quad \forall Y \in \mathbb{R}^m.$$

Setting $-A(Y) \equiv \int_0^1 \nabla F(sY)\, ds$, we obtain $F(Y) = -A(Y)Y$, $\forall Y \in \mathbb{R}^m$.
The general solution F of (A.1) is therefore

$$F(X,Y) = -A(X,Y)Y\,,$$

where $A(X,Y)$, for each $(X,Y) \in \mathbb{R}^n \times \mathbb{R}^m$, is a linear transformation from $\mathbb{R}^m$ into $\mathbb{R}^m$ consistent with the inequality

$$Y \cdot A(X,Y)Y \geq 0\,. \tag{A.2}$$

Because of the dependence of $A(X,Y)$ on Y, the inequality (A.2) is weaker than positive definiteness for $A(X,Y)$. However, when F is quasilinear, i.e., $F(X,Y)$ is linear in Y for each X, then

$$F(X,Y) = -A(X)Y\,, \quad \forall (X,Y) \in \mathbb{R}^n \times \mathbb{R}^m\,,$$

where $A(X)$ is positive semi-definite because of (A.2). More generally,

$$F(X,Y) = -A(X)Y - o(|Y|)\,, \quad \text{as } Y \to 0\,, \ \forall X \in \mathbb{R}^n$$

where $A(X)$ is positive semi-definite.

A.2 Young's inequalities

Lemma A.2.1 (Young's inequality)**.** *Let* $1 < p, q < \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Then,*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}\,, \quad a, b > 0\,.$$

Proof. Write $ab = e^{\ln(ab)}$. The statement directly follows from the convexity of the map $x \mapsto e^x$. □

Corollary A.2.2 (Generalized Young's inequality)**.** *Let* $1 < p, q < \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Then,*

$$ab \le \epsilon a^p + c_\epsilon b^q\,, \quad a, b > 0\,, \epsilon > 0\,,$$

for $c_\epsilon = (\epsilon p)^{-\frac{q}{p}} q^{-1}$*.*

Proof. Write $ab = ((\epsilon p)^{\frac{1}{p}} a)((\epsilon p)^{-\frac{1}{p}} b)$ and apply Young's inequality. □

A.3 Gronwall's lemma

Lemma A.3.1 (Gronwall's Lemma)**.** *Let* $I \subset \mathbb{R}$ *be an interval. Let* β *and* v *be real-valued functions defined on* I*. Assume that* β *and* v *are continuous and that* $\beta \ge 0$*. If there exists* $\alpha \in \mathbb{R}$ *such that*

$$v(\tau) \le \alpha + \int_a^\tau \beta(t) v(t)\, dt\,, \quad \forall \tau \in I\,,\, \tau \ge a\,,$$

then

$$v(\tau) \le \alpha e^{\int_a^\tau \beta(t)\, dt}\,, \quad \forall \tau \in I\,, \tau \ge a\,.$$

Proof. The proof can be found in [60, §1.3]. □

A.4 Lebesgue differentiation theorem

In this section, we show that a summable function is "approximately continuous" at almost every point. More precisely, for almost every point, the value of an summable function is the limit of infinitesimal averages taken about the point.

Let $L^1_{\text{loc}}(\mathbb{R}^n)$ denote the space of locally summable functions on $\mathbb{R}^n$, $n \in \mathbb{N}$.

Definition A.4.1. Let $v \in L^1_{\text{loc}}(\mathbb{R}^n)$, $n \in \mathbb{N}$. A point $x \in \mathbb{R}^n$ is a Lebesgue point of v if

$$\lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} |v(y) - v(x)| \mathrm{d}y = 0\,,$$

where is a ball centered at x with radius $r > 0$, and $|B(x,r)|$ is its Lebesgue measure.

The Lebesgue points of a locally summable function are thus points where the function does not oscillate too much, in an average sense. If the function is continuous, then every point is a Lebesgue point. On the contrary, for a locally summable function, it is far from obvious that there exist Lebesgue points. However, the following remarkable theorem shows that they actually exist.

Theorem A.4.1 (Lebesgue differentiation theorem)**.** *If* $v \in L^1_{loc}(\mathbb{R}^n)$, $n \in \mathbb{N}$, *then almost every* $x \in \mathbb{R}^n$ *is a Lebesgue point of* v*.*

Proof. The proof can be found in [57, §7.6]. □

Observe that Theorem A.4.1 is an analogue, and a generalization, of the fundamental theorem of calculus in higher dimensions.

A.5 Sobolev spaces

The aim of this section is to briefly recall Sobolev spaces. For more details, see [8, §8-9]. Notice that we suppose the reader to be familiar with L^p spaces, their norms, their dual spaces and their main properties, such as reflexivity for $1 \leq p < \infty$ and separability for $1 < p < \infty$. Otherwise, we refer the reader to [8, §3] for the general theory regarding reflexive and separable spaces, and to [8, §4] for L^p spaces and their properties.

Let Ω be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $\mathcal{C}^\infty_c(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω.

Definition A.5.1. Suppose v, $w \in L^1_{\text{loc}}(\Omega)$, and α is a multi-index. We say that w is the α^{th}-weak partial derivative of u, written $D^\alpha v = w$, if

$$\int_\Omega v D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega w \phi \, dx$$

for all test functions $\phi \in \mathcal{C}^\infty_c(\Omega)$

Once recalled the notion of weak partial derivative, we can introduce the definition of Sobolev space.
Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer.

Definition A.5.2. The Sobolev space $W^{k,p}(\Omega)$ consists of all locally summable functions $v : \Omega \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha v$ exists in the weak sense and belongs to $L^p(\Omega)$.

If $p = 2$, we usually write $H^k(\Omega) \equiv W^{k,2}(\Omega)$.

Definition A.5.3. If $v \in W^{k,p}(\Omega)$, we define its norm to be

$$\|v\|_{W^{k,p}(\Omega)} := \begin{cases} \sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^p(\Omega)} \, , \; 1 \leq p < \infty \, , \\ \sum_{|\alpha| \leq k} \operatorname{ess\,sup}_\Omega |D^\alpha v| \, , \; p = \infty \, . \end{cases}$$

Theorem A.5.1. *For each k nonnegative integer, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space for every $1 \leq p \leq \infty$. $W^{k,p}(\Omega)$ is reflexive for $1 \leq p < \infty$ and it is separable for $1 < p < \infty$.*

Proof. The proof can be found in [8, §9.1]. □

Consider the case $k = 1$ and $n = 1$, i.e., $\Omega \equiv (a, b) \subset \mathbb{R}$. The functions in $W^{1,p}(a, b)$ are roughly speaking the primitives of the $L^p(a, b)$ functions. More precisely, we have the following:

Theorem A.5.2. *Let $v \in W^{1,p}(a, b)$ with $1 \leq p \leq \infty$, then there exists a function $\tilde{v} \in \mathcal{C}([a, b])$ such that $v = \tilde{v}$ almost everywhere in (a, b) and*

$$\tilde{v}(x) - \tilde{v}(z) = \int_z^x v'(y)\, dy\,, \quad \forall x, z \in [a, b]\,,$$

where v' is the first derivative of v.

Proof. The proof can be found in [8, §8.2]. □

Let us emphasize the content of Theorem A.5.2. It asserts that every function $v \in W^{1,p}(a, b)$ admits one (and only one) continuous representative on (a, b), i.e., there exists a continuous function on (a, b) that belongs to the equivalence class of v ($v \sim \tilde{v}$ if $v = \tilde{v}$ almost everywhere). When it is useful v can be replaced by its continuous representative. In order to simplify the notation we also write v for its continuous representative. We finally point out that the property "v has a continuous representative" is not the same as "v is continuous almost everywhere".
Furthermore, when $p = 1$ and (a, b) is bounded, the functions of $W^{1,p}(a, b)$ coincide with the absolutely continuous functions $AC([a, b])$. Indeed, they are also characterized by the property: $\forall \epsilon > 0\ \exists \delta > 0$ such that for every finite sequence of disjoint intervals $(a_j, b_j) \subset (a, b)$ such that $\sum_j |b_j - a_j| < \delta$, we have $\sum_j |v(b_j) - v(a_j)| < \epsilon$.

Next, we present some other sort of Sobolev spaces, which comprise functions mapping time into Banach spaces. These are essential in the construction of weak solutions to evolution equations, which are partial differential equations explicitly depending on both time and spatial variables. Before defining them, we should first of all introduce the measure and integration theory for mappings taking value in a Banach space, i.e. the so-called Bochner's integral. However, since the main results in Lebesgue theory of integration can be adapted to the case of Bochner's integral at least when the target space X is separable, we decided to skip this part and we refer the reader to [66, V.5], [38, §8.1] and [22, Appendix E.5].

Let $[0, T] \subset \mathbb{R}$ be a time interval and let X denote a Banach space, with norm $\|\cdot\|_X$.

Definition A.5.4. The space $L^p(0, T; X)$ consists of all (strongly) measurable functions $v : [0, T] \to X$ with

$$\|v\|_{L^p(0,T;X)} := \Big(\int_0^T \|v(t)\|_X^p dt \Big)^{\frac{1}{p}} < \infty$$

for $1 \leq p < \infty$, and

$$\|v\|_{L^\infty(0,\,T;\,X)} := \operatorname*{ess\,sup}_{0\leq t\leq T} \|v(t)\|_X < \infty\,.$$

Proposition A.5.3. *$L^p(0,\,T;\,X)$ is a Banach space.*
If X is separable, then $L^p(0,\,T;\,X)$ is separable for any $1 \leq p < \infty$.
If X is reflexive, then $L^p(0,\,T;\,X)$ is reflexive for any $1 < p < \infty$. In particular, the dual space is given by $(L^p(0,\,T;\,X))' \equiv L^{p'}(0,\,T;\,X')$, where p' is the conjugate exponent to p and X' is the dual space of X.
If X is a Hilbert space, then $L^2(0,\,T;\,X)$ is a Hilbert space.

Proof. The proof can be found in [38, §8.2]. □

Notice that, using the integration theory for Banach-valued functions, it can be proven that

$$L^p(0,\,T;L^p(\Omega)) \cong L^p(\Omega\times(0,\,T))\,,\ 1\leq p<\infty\,. \tag{A.3}$$

This result is due to the separability of L^p spaces for $1 \leq p < \infty$. Indeed, for $p = \infty$,

$$L^\infty(0,\,T;L^\infty(\Omega)) \subsetneq L^\infty(\Omega\times(0,\,T))\,.$$

Definition A.5.5. The space $\mathcal{C}([0,\,T];\,X)$ comprises all continuous functions $v : [0,T] \to X$ with

$$\|v\|_{\mathcal{C}([0,\,T];\,X)} := \max_{0\leq t\leq T} \|v(t)\|_X < \infty\,.$$

By standard techniques, the following proposition can be proven.

Proposition A.5.4. *$\mathcal{C}([0,\,T];\,X)$ is a Banach space.*

As we weaken the definition of spatial partial derivative, we can do the same for the time derivative.

Definition A.5.6. Let $v \in L^1(0,\,T;\,X)$. We say $w \in L^1(0,\,T;\,X)$ is the weak derivative of v, written $v_t = w$, provided

$$\int_0^T \frac{d\phi}{dt}(t)v(t)\,dt = -\int_0^T \phi(t)w(t)\,dt\,,$$

for all scalar test functions $\phi \in \mathcal{C}_c^\infty(0,T)$.

Definition A.5.7. The Sobolev space $W^{1,p}(0,\,T;\,X)$ consists of all functions $v \in L^p(0,\,T;\,X)$ such that u_t exists in the weak sense and belongs to $L^p(0,\,T;\,X)$. Furthermore,

$$\|v\|_{W^{1,p}(0,\,T;\,X)} := \begin{cases} \|v\|_{L^p(0,\,T;\,X)} + \|v_t\|_{L^p(0,\,T;\,X)}\,,\ 1\leq p<\infty\,,\\ \operatorname{ess\,sup}_{0\leq t\leq\infty}(\|v(t)\| + \|v_t(t)\|)\,,\ p=\infty\,.\end{cases}$$

If $p = 2$, we usually write $H^k(0,\,T;\,X) \equiv W^{k,2}(0,\,T;\,X)$.

A.6 Inequalities involving L^p and Sobolev norms

In this section, we recall some useful inequality usually used to deduce properties of some functions.

Theorem A.6.1 (Hölder's inequality)**.** *Let $1 \le p \le \infty$. Assume that $v \in L^p$ and $w \in L^q$, where q is the conjugate exponent to p, i.e., it satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Then, $vw \in L^1$ and*

$$\|vw\|_{L^1} \le \|v\|_{L^p} \|w\|_{L^q} .$$

Proof. The proof can be found in [8, §4.2]. □

Direct consequences of Hölder's inequality are the following extensions:

Corollary A.6.2. *Assume that $v_1, v_2, ..., v_k$ are functions such that $v_i \in L^{p_i}$, $1 \le i \le k$ with $\frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_k} = \frac{1}{p} \le 1$. Then, the product $v = v_1 v_2 ... v_k$ belongs to L^p and*

$$\|v\|_{L^p} \le \|v_1\|_{L^{p_1}} \|v_2\|_{L^{p_2}} \dots \|v_k\|_{L^{p_k}} .$$

Corollary A.6.3 (Interpolation inequality)**.** *If $v \in L^p \cap L^q$ with $1 \le p \le q \le \infty$, then $v \in L^r$ for all r such that $p \le r \le q$ and the interpolation inequality holds:*

$$\|v\|_{L^r} \le \|v\|^{\alpha}_{L^p} \|v\|^{1-\alpha}_{L^q},$$

where $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$, $0 \le \alpha \le 1$.

Theorem A.6.4 (Poincaré-Wirtinger's inequality)**.** *Let Ω be a bounded, connected, Lipschitz, open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. Assume $1 \le p \le \infty$. Then there exists a constant c, depending only on n, p and Ω, such that*

$$\|v - \langle v \rangle\|_{L^p(\Omega)} \le c \|\nabla v\|_{L^p(\Omega)} , \quad \textit{where } \langle v \rangle \equiv \frac{1}{|\Omega|} \int_{\Omega} v \, dx ,$$

for each function $v \in W^{1,p}(\Omega)$.

Proof. The proof can be found in [22, §5.8]. □

The following theorem is due to L. Nirenberg [50] and to E. Gagliardo [25].

Theorem A.6.5 (Gagliardo-Nirenberg inequality)**.** *Let $1 \le q, r \le \infty$, let k be a natural number and let $v \in L^q(\mathbb{R}^n) \cap W^{k,r}(\mathbb{R}^n)$, $n \in \mathbb{N}$. Suppose that a real number α and a natural number j are such that*

$$\frac{1}{p} = \frac{j}{n} + \Big(\frac{1}{r} - \frac{k}{n}\Big)\alpha + \frac{1-\alpha}{q} \quad \textit{and} \quad \frac{j}{k} \le \alpha \le 1 .$$

Then, there exists a constant c depending only on k, n, j, q, r and α such that

$$\|D^j u\|_{L^p(\Omega)} \le c \|D^k u\|^{\alpha}_{L^r(\Omega)} \|u\|^{1-\alpha}_{L^q(\Omega)} , \tag{A.4}$$

with the following exceptional cases:

1. *If $j = 0$, $kr < n$ and $q = \infty$, then it is necessary to make the additional assumption that either v tends to zero at infinity or that v lies in L^s for some finite $s > 0$.*

2. *If $1 < r < \infty$ and $k - j - \frac{n}{r}$ is a non-negative integer, then it is necessary to assume also that $\alpha \neq 1$.*

For a bounded, connected, Lipschitz, open subset $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, and $v \in L^q(\Omega) \cap W^{k,r}(\Omega)$, the result holds with the same hypotheses as above and reads

$$\|D^j u\|_{L^p(\Omega)} \leq c_1 \|D^k u\|^{\alpha}_{L^r(\Omega)} \|u\|^{1-\alpha}_{L^q(\Omega)} + c_2 \|u\|_{L^s(\Omega)} ,$$

where c_1, c_2 are constants depending only on $k, n, j, q, r, \alpha, \Omega$, and $s \in [1, q]$ is arbitrary.

Proof. A sketch of the proof can be found in [50] and in [38, §12.5] □

A.7 Continuous embedding theorems

The embedding characteristics of Sobolev spaces are essential in their use in analysis, especially in the study of differential and integral operators. The most important imbedding results for Sobolev spaces are often gathered together into a single theorem, that is known as the Sobolev embedding theorem. On the other hand, they are of several different types and can require different methods of proof. The core results are due to Sobolev [64, §1.8], but our statement (Theorem A.7.1) also includes refinements due to others, in particular Morrey [49] and Gagliardo [24].

Theorem A.7.1 (Sobolev embedding theorem)**.** *Let Ω be an bounded, Lipschitz, open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $j \geq 0$ and $k \geq 1$ be integers and let $1 \leq p < \infty$. Then, we have the following continuous embeddings:*

1. *If $kp < n$, then*

$$W^{j+k,p}(\Omega) \hookrightarrow W^{j,q}(\Omega) , \quad \text{for } p \leq q \leq \frac{np}{n - kp} .$$

In particular,

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) , \quad \text{for } p \leq q \leq \frac{np}{n - kp} .$$

2. *If $kp = n$, then*

$$W^{j+k,p}(\Omega) \hookrightarrow W^{j,q}(\Omega) , \quad \text{for } p \leq q < \infty .$$

In particular,

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) , \quad \text{for } p \leq q < \infty .$$

3. *If $kp > n$, then*

$$W^{j+k,p}(\Omega) \hookrightarrow W^{j,q}(\Omega) , \quad \text{for } p \leq q \leq \infty .$$

In particular,

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) , \quad \text{for } p \leq q \leq \infty .$$

Furthermore,

1. *If* $kp > n > (k-1)p$*, then*

$$W^{j+k,p}(\Omega) \hookrightarrow \mathcal{C}^{j,\lambda}(\bar{\Omega})\,, \text{ for } 0 < \lambda \le k - \frac{n}{p}\,.$$

2. *If* $(k-1)p = n$*, then*

$$W^{j+k,p}(\Omega) \hookrightarrow \mathcal{C}^{j,\lambda}(\bar{\Omega})\,, \text{ for } 0 < \lambda < 1\,.$$

   In particular, if $n = k-1$ *and* $p = 1$*, it holds for* $\lambda = 1$ *as well.*

Proof. The proof can be found in [1, §4]. □

Next, we present a useful embedding result holding for L^p spaces that comprise functions mapping time into other L^p spaces. Its proof directly follows from an application of the standard interpolation inequality A.6.3.

Proposition A.7.2. *Let* $[0,T] \subset \mathbb{R}$ *be a time interval and let* Ω *be an open subset of* $\mathbb{R}^n$*,* $n \in \mathbb{N}$*.*

$$L^\infty(0,\,T;\,L^r(\Omega)) \cap L^p(0,\,T;\,L^s(\Omega)) \hookrightarrow L^q(\Omega \times (0,\,T))\,,$$

for q *satisfying*

$$(1-\alpha)q = p \quad \text{and} \quad \frac{\alpha}{r} + \frac{1-\alpha}{s} = \frac{1}{q}\,, \quad \text{where } \ 0 \le \alpha \le 1\,.$$

A.8 A regularity theorem for elliptic equations

In this section, one of the regularity results for elliptic equations is given. In particular, we report that holding for Neumann boundary conditions.

Theorem A.8.1 (Regularity for the Neumann problem)**.** *Let* $\Omega \subset \mathbb{R}^n$*,* $n \in \mathbb{N}$*, be an open, bounded subset of class* $\mathcal{C}^2$*. Let* $f \in L^2(\Omega)$ *and* $v \in H^1(\Omega)$ *satisfy*

$$\int_\Omega \nabla v \cdot \nabla \varphi\, dx + \int_\Omega v\varphi\, dx = \int_\Omega f v\, dx\,, \quad \forall \varphi \in H^1(\Omega)\,. \tag{A.5}$$

Then, $v \in H^2(\Omega)$ *and* $\|v\|_{H^2(\Omega)} \le c\|f\|_{L^2(\Omega)}$*, where* c *is a constant depending only on* Ω*.*

Proof. The proof can be found in [8, §9.6] □

Notice that (A.5) is the weak formulation of $-\Delta v + v = f$ in Ω endowed with the Neumann boundary condition $\nabla v \cdot \nu = 0$ on $\partial\Omega$, where ν is the unit outer normal vector to the boundary $\partial\Omega$. Thus, if $v, \Delta v \in L^2(\Omega)$ and $\nabla v \cdot \nu = 0$ on $\partial\Omega$, then Theorem A.8.1 allows us to conclude that $v \in H^2(\Omega)$.

A.9 Compactness theorems

In this section, we present some compactness results, fundamental for our applications of functional analysis to partial differential equations.

First of all, we recall that, for any metric space X with distance d, the notion of compactness and sequential compactness coincide. Thus, if X is compact, any X-valued sequence admits at least a subsequence which converges in X. Similarly, if any $A \subset X$ is compact, any A-valued sequence admits at least a subsequence which converges in A. Otherwise, if the closure of A, but not A, is compact, any A-valued sequence admits at least a subsequence which converges in X, hence not necessarily in A. In this case, we say that A is relatively (sequentially) compact.
We can now give the definition of compact operator.

Definition A.9.1. Suppose X and Y are Banach spaces and B_X is the open unit ball in X. A linear map $T : X \to Y$ is said to be compact if $T(B_X)$ is relatively compact, i.e., its closure is compact in Y.

For any linear bounded operator $T : X \to Y$, where X, Y Banach spaces, there exists a unique linear bounded operator $T' : X' \to Y'$ that satisfies $\langle Tx, y' \rangle = \langle x, T'y' \rangle$, $\forall x \in X$, $y' \in Y'$, and such that its operator norm is equal to that of T (see [58, Theorem 4.10]). Furthermore, the following result holds true:

Lemma A.9.1. *Suppose X and Y are Banach spaces and consider a linear bounded operator $T : X \to Y$. Then, T is compact if and only if T' is compact.*

Proof. The proof can be found in [58, Theorem 4.19]. □

Thus, since a compact embedding is nothing but a continuous embedding with an inclusion map that is not only continuous but also compact, we observe that if X is compactly embedded in Y, i.e., $X \subset\subset Y$, then Y' is compactly embedded in X', i.e., $Y' \subset\subset X'$.

The Ascoli-Arzelà theorem is a fundamental result giving necessary and sufficient conditions to decide whether every sequence of a family of continuous Banach space valued functions defined on a compact metric space has a uniformly convergent subsequence.

Theorem A.9.2 (Ascoli-Arzelà theorem)**.** *Let X be a compact metric space with distance d and let Y be a Banach space with norm $\|\cdot\|_Y$. Let $\mathcal{F} \subset \mathcal{C}(X;Y)$. $\mathcal{F}$ is relatively compact in $\mathcal{C}(X;Y)$ if and only if*

1. *for all $x \in X$, $\mathcal{F}$ is equicontinuous in x, i.e., $\forall \epsilon > 0$ $\exists \delta > 0$ such that, $\forall y \in X : d(x,y) < \delta$, $\|f(x) - f(y)\|_Y \leq \epsilon$, $\forall f \in \mathcal{F}$;*

2. *$\mathcal{F}$ is pointwise relatively compact, i.e., $\forall x \in X$, $\mathcal{F}(x) := \cup_{f \in \mathcal{F}} \{f(x)\}$ is relatively compact in Y.*

Proof. The proof can be found in [60, §5.2]. □

Next, we are concerned with the compactness of the Sobolev continuous embeddings stated in Theorem A.7.1.

Theorem A.9.3 (Sobolev compact embedding theorem)**.** *Let Ω be a bounded, Lipschitz, open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $j \geq 0$ and $k \geq 1$ be integers and let $1 \leq p < \infty$. Then, we have the following compact embeddings:*

1. *If $kp < n$, then*
$$W^{j+k,p}(\Omega) \subset\subset W^{j,q}(\Omega)\,, \quad \textit{for } 1 \leq q < \frac{np}{n-kp}\,.$$
In particular,
$$W^{k,p}(\Omega) \subset\subset L^q(\Omega)\,, \quad \textit{for } 1 \leq q < \frac{np}{n-kp}\,.$$

2. *If $kp = n$, then*
$$W^{j+k,p}(\Omega) \subset\subset W^{j,q}(\Omega)\,, \quad \textit{for } 1 \leq q < \infty\,.$$
In particular,
$$W^{k,p}(\Omega) \subset\subset L^q(\Omega)\,, \quad \textit{for } 1 \leq q < \infty\,.$$

3. *If $kp > n$, then*
$$W^{j+k,p}(\Omega) \subset\subset W^{j,q}(\Omega)\,, \quad \textit{for } 1 \leq q \leq \infty\,.$$
In particular,
$$W^{k,p}(\Omega) \subset\subset L^q(\Omega)\,, \quad \textit{for } 1 \leq q \leq \infty\,.$$

Furthermore,

1. *If $kp > n \geq (k-1)p$, then*
$$W^{j+k,p}(\Omega) \subset\subset \mathcal{C}^{j,\lambda}(\bar{\Omega})\,, \ \textit{for } 0 < \lambda < k - \frac{n}{p}\,.$$

2. *If $kp > n$, then*
$$W^{j+k,p}(\Omega) \subset\subset \mathcal{C}^{j}(\bar{\Omega})\,.$$

Proof. The proof can be found in [1, §6]. □

In the study of nonlinear evolutionary partial differential equations, in order to prove the existence of solutions, typically, one first constructs approximate solutions, then uses some compactness result to show that there is a convergent subsequence of approximate solutions whose limit is a solution. A compactness criterion in the theory of Sobolev spaces of Banach space-valued functions is given by the Aubin–Lions lemma (or theorem).

Theorem A.9.4 (Aubin-Lions-Simon lemma)**.** *Let X_0, X and X_1 be three Banach spaces such that $X_0 \subset\subset X \hookrightarrow X_1$, i.e., X_0 is compactly embedded in X and X is continuously embedded in X_1. For $1 \leq p, q \leq \infty$, let*

$$W = \{u \in L^p(0, T; X_0) \mid u_t \in L^q(0, T; X_1)\}.$$

1. *If $p < \infty$, then the embedding of W into $L^p(0, T; X)$ is compact.*
2. *If $p = \infty$ and $q > 1$, then the embedding of W into $\mathcal{C}([0, T]; X)$ is compact.*

Proof. The proof can be found in [5] and in [63]. □

At last, we provide a result whose corollaries plays a crucial role in deducing convergence of sequences of approximate solutions in the proofs of the existence of solutions to evolution equations. We suppose the reader to be familiar with the theory of weak topologies, reflexive and separable spaces. A standard reference is given by [8, §3].
Let X be a Banach space and let X' be its dual space. Let $\bar{B}_X$ be the closed unit ball of X, while let $\bar{B}_{X'}$ be the closed unit ball of X'.

Theorem A.9.5 (Banach-Alaoglu-Bourbaki theorem)**.** *$\bar{B}_{X'}$ is compact with respect to the weak star topology.*

Proof. The proof can be found in [8, §3.4]. □

Corollary A.9.6. *If X is separable, then $\bar{B}_{X'}$ is sequentially compact with respect to the weak star topology. In other words, if $\{x'_n\}_n \subset X'$ is a bounded sequence, then there exists a subsequence $\{x'_{n_k}\}_{n_k}$ which converges in X' with respect to the weak star topology.*

Corollary A.9.7. *If X is reflexive, then $\bar{B}_X$ is sequentially compact with respect to the weak topology. In other words, if $\{x_n\}_n \subset X$ is a bounded sequence, then there exists a subsequence $\{x_{n_k}\}_{n_k}$ which converges in X with respect to the weak topology.*

A.10 Convergence theorems

In this section, we introduce some convergence theorems which can be found in the Appendix of [7] due to C. Sbordone. We suppose the reader to be familiar with weak convergence in L^p spaces, which can be found in [8, §3-4].
We will denote the weak convergence by $\rightharpoonup$.

Let Ω be a bounded, open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. Recall that a sequence of real-valued function $\{v_n\}_n$ defined on Ω is said to converge almost everywhere on Ω to v if and only if the set $\{x \in \Omega \,|\, v_n(x)$ does not converge to $v(x)\}$ has measure zero.

Proposition A.10.1. *Let $1 \leq p \leq \infty$. Let $\{v_n\}_n$ be a sequence in $L^p(\Omega)$ such that $v_n \to v$ strongly in $L^p(\Omega)$. Then, there exists a subsequence $\{v_{n_k}\}_{n_k}$ such that $v_{n_k} \to v$ almost everywhere in Ω.*

Lemma A.10.2. *Let $1 < p < \infty$ and let $c > 0$. Let $\{v_n\}_n$ be a sequence in $L^p(\Omega)$ such that $v_n \to v$ almost everywhere in Ω and $\|v_n\|_{L^p(\Omega)} \leq c$, $\forall n \in \mathbb{N}$. Then, $v_n \rightharpoonup v$ in $L^p(\Omega)$ and $v \in L^p(\Omega)$.*

Theorem A.10.3. *Let $1 \leq q < p < \infty$. Let $\{v_n\}_n$ be a sequence in $L^p(\Omega)$. If $v_n \rightharpoonup v$ in $L^p(\Omega)$ and $v_n \to v$ almost everywhere in Ω, then $v_n \to v$ strongly in $L^q(\Omega)$.*

Proof. The proof can be found in [7, Appendix]. □

Remark A.10.4. *Notice that, due to (A.3), both the above results hold true also for sequences belonging to L^p spaces of L^p space-valued functions. Furthermore, they can be generalized to L^{p_1} spaces of L^{p_2} space-valued functions for $p_1 \neq p_2$, $1 \leq p_1, p_2 < \infty$.*

At last, we present a result that directly follows from the definition of weak convergence.

Proposition A.10.5. *Let X, Y be Banach spaces and let $\{x_n\}_n \subset X$ be a sequence such that $x_n \rightharpoonup x$ in X. Consider a linear and continuous operator $T : X \to Y$. Then, $Tx_n \rightharpoonup Tx$ in Y.*

A.11 Ioffe's theorem

In this section, we report the statement of a lower semicontinuity result due to A.D. Ioffe [33]. This can be useful in the study of evolution equations in order to recover at least an inequality when passing to the limit in the approximate problem.

Theorem A.11.1 (Ioffe's theorem)**.** *Let $Q \subset \mathbb{R}^d$ be a smooth, bounded, open subset and let $f : Q \times \mathbb{R}^l \times \mathbb{R}^m \to [0, +\infty]$, $d, l, m \in \mathbb{N}$, $d, l, m \geq 1$, be a measurable non-negative function such that:*

$$f(y, \cdot, \cdot) \text{ is lower semicontinuous on } \mathbb{R}^l \times \mathbb{R}^m \text{ for every } y \in Q,$$

$$f(y, w, \cdot) \text{ is convex on } \mathbb{R}^m \text{ for every } (y, w) \in Q \times \mathbb{R}^l.$$

Let also (w_n, v_n), $(w, v) : Q \to \mathbb{R}^l \times \mathbb{R}^m$ be measurable functions such that

$$w_n \to w \text{ almost everywhere in } Q\,, \quad v_n \rightharpoonup v \text{ in } L^1(Q)\,.$$

Then,

$$\liminf_{n \to +\infty} \int_Q f(y, w_n(y), v_n(y))\, dy \geq \int_Q f(y, w(y), v(y))\, dy\,.$$

Proof. The proof can be found in [33]. □

A.12 Theory of traces in Sobolev spaces

In this section, we discuss the possibility of assigning boundary values along $\partial\Omega$ to $v \in W^{2,p}(\Omega)$, where Ω is a bounded, Lipschitz, open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. If $v \in \mathcal{C}(\Omega)$, then clearly v has values on $\partial\Omega$ in the usual sense. The problem is that a typical function $v \in W^{2,p}(\Omega)$ is not in general continuous and, even worse, is only defined almost everywhere in Ω. Since $\partial\Omega$ has n-dimensional Lebesgue measure zero, there is no direct meaning we can give to the expression "v restricted to $\partial\Omega$". The notion of traces resolves this problem.

Theorem A.12.1 (Trace theorem)**.** *Let Ω be a bounded, Lipschitz, open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $1 \leq p < \infty$. Then, there exist surjective, bounded, linear operators*

$$T_0 : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega) \quad \textit{and} \quad T_1 : W^{2,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$$

such that

$$T_0 v = v|_{\partial\Omega}\,, \ \textit{if } v \in W^{1,p}(\Omega) \cap \mathcal{C}(\bar{\Omega})\,,$$

$$T_1 v = \nabla v|_{\partial\Omega} \cdot \nu\,, \ \textit{if } v \in W^{2,p}(\Omega) \cap \mathcal{C}^1(\bar{\Omega})\,,$$

where ν is the unit outer normal vector to the boundary $\partial\Omega$.

Proof. For $p = 2$, the proof can be found in [39]. Otherwise, for $p \neq 2$, we refer to the references in [39]. □

Definition A.12.1. We say that $T_0 v$ is the trace of v on $\partial\Omega$, while $T_1 v$ is the trace of the normal component of ∇v on $\partial\Omega$.

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